Some properties of the Boolean Quadric Polytope

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Abstract

The topic of our research is unconstrained 0-1 quadratic programming. We find some properties of the Boolean Quadric Polytope, which is obtained from the standard linearization of the given 0-1 quadratic function. By using a result obtained by Deza and Laurent on a class of hypermetric inequalities for the Cut Polytope, we find a necessary and sufficient condition for a class of inequalities to be facet defining for the Boolean Quadric Polytope. Furthermore we find a property characterizing the non integral vertices of a class of relaxations of the Boolean Quadric Polytope.

1 Introduction

We present some properties of the Boolean Quadric Polytope which is obtained from the standard linearization of a 0-1 quadratic programming problem

\[
\min f(x) \\
\text{s.t. } x \in \{0, 1\}^n
\]

where

\[
f(x) = q_0 + \sum_{1 \leq j \leq n} q_j x_j + \sum_{1 \leq i < j \leq n} q_{ij} x_j x_j.
\]

The standard linearization of the given quadratic problem is obtained by introducing new variables \(y_{ij}, 1 \leq i < j \leq n\), and imposing the following linear constraints (see, e.g., [7]):

\[
\begin{align*}
-y_{ij} + x_i + x_j & \leq 1 \\
y_{ij} - x_i & \leq 1 \\
y_{ij} - x_j & \leq 1 \\
y_{ij} & \geq 0
\end{align*}
\]

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which, for all binary vectors $x$, enforce the identities $y_{ij} = x_ix_j$, $1 \leq i < j \leq n$. The quadratic problem can then be formulated as a linear integer problem

$$\min_{x} q_0 + \sum_{1 \leq j \leq n} q_jx_j + \sum_{1 \leq i < j \leq n} q_{ij}y_{ij}$$

$$s.t. (1)$$

$$x \in \{0, 1\}^n.$$
Let $C_V \subset \mathbb{R}^{n(n+1)/2}$ be the cone of nonnegative 0-1 quadratic functions of the variables $x_i, i \in V$, and for a given $k \in \{2, \ldots, n\}$, let $C^k_V \subseteq C_V$ be the cone of nonnegative 0-1 quadratic functions of $k$ variables.

If constraints (1) hold, then $L_h(x,y) \geq 0$ is valid for $BQ_V$ if and only if $h(x) \geq 0$ for all binary vectors $x$ ([1]). Based on this fact Boros, Crama and Hammer ([1]) defined a hierarchy of relaxations of $BQ_V$:

$$Q^k_V = \{ (x,y) \in \mathbb{R}^{n(n+1)/2} : L_h(x,y) \geq 0, h \in C^k_V \}, k = 2, \ldots, n$$

such that $Q^n_V = BQ_V$.

For any given set $K \subseteq V$, we define also the cone $C_V[K] \subset \mathbb{R}^{n(n+1)/2}$ of nonnegative 0-1 quadratic functions of the variables in $K$ and the relaxation of $BQ_V$

$$Q_V[K] = \{ (x,y) \in \mathbb{R}^{n(n+1)/2} : L_g(x,y) \geq 0, g \in C_V[K] \}.$$ 

Given a vector $a \in \mathbb{R}^{n(n+1)/2}$, $a = (a_1, \ldots, a_n, a_{12}, \ldots, a_{n-1,n})$, and $K \subset V$ such that $|K| = k$, we define the canonical restriction of $a$ to $K$ as the vector $a^K \in \mathbb{R}^{k(k+1)/2}$ obtained by discarding all components $a_j$ such that $j \in V - K$ and all components $a_{ij}$ such that $i \in V - K$ or $j \in V - K$. Based on the definition of canonical restriction of a vector $a$, we can define:

- the canonical restriction $b^K(x,y)^K \geq 0$ of an inequality $b(x,y) \geq 0$;
- the canonical restriction $P^K$ of a polytope $P$.

**Remark 2.1** The canonical restrictions to $K$ of the elements in $C_V[K]$ are all and only the elements of $C_K$, i.e., they define the Boolean Quadric Polytope $BQ_K$ on the set $K$.

## 3 A property of the vertices of some relaxations of the Boolean Quadric Polytope

In this section we prove a property of the non integral vertices of the relaxations $Q_V[K]$ and $Q^k_V$.

**Theorem 3.1** Let $K$ be a subset of $V$ having cardinality at least two and let $(\bar{x}, \bar{y})$ be a vertex of $Q_V[K]$ such that $\bar{x}_j$ is not integral for some $j \in K$. Then $\exists s \in N - K$ such that $\bar{x}_s$ is not integral.

**Proof.** By Remark 2.1, the canonical restriction to $K$ of $Q_V[K]$ is $BQ_K$ and then all its vertices are integral. It follows that the canonical restriction $(\bar{x}, \bar{y})^K$ of $(\bar{x}, \bar{y})$ is not a vertex of $BQ_K$ since it is not integral. By the Caratheodory’s theorem there exist the vertices $(x^1, y^1), \ldots, (x^p, y^p)$ of $BQ_K$ such that $(\bar{x}, \bar{y})^K = \sum_{h=1,\ldots,p} \alpha_h (x^h, y^h)$, $\alpha_h \geq 0$, $h = 1, \ldots, p$ and $\sum_{h=1,\ldots,p} \alpha_h = 1$.

Suppose that $\bar{x}_s$ is integral for each $s \in V - K$. For $h = 1, \ldots, p$, define the vectors $(\bar{x}^h, \bar{y}^h) \in \mathbb{R}^{n(n+1)/2}$ as follows:

$$\bar{x}^h_j = \begin{cases} x^h_j & j \in K \\ \bar{x}_j & j \in V - K \end{cases}$$
\[ y_{ij}^h = \begin{cases} 
  y_{ij}^h & i \in K, j \in K \\
  \bar{y}_{ij} & i \in V - K, j \in V - K \\
  \min \{ x_{ij}^h, \bar{x}_j \} & i \in K, j \in V - K \\
  \min \{ \bar{x}_i, x_j^h \} & i \in V - K, j \in K 
\end{cases} \]

These vectors are integral and belong to \( BQ_V \) and to \( Q_V[K] \) (since \( BQ_V \subseteq Q_V[K] \)). Moreover:

\[ (\bar{x}, \bar{y}) = \sum_{k=1}^{p} \alpha_k (\bar{x}^h, \bar{y}^h). \]

But this contradicts the hypothesis that \( (\bar{x}, \bar{y}) \) is a vertex of \( Q_V[K] \). Hence there exists \( s \in N - K \) such that \( x_s \) is not integral.

\[ \square \]

**Corollary 3.2** For any non integral vertex \( (\bar{x}, \bar{y}) \) of \( Q_V^k \), at least \( k + 1 \) among the components \( \bar{x}_1, \ldots, \bar{x}_n \) are non integral.

## 4 A Class of facets of the Boolean Quadric Polytope

Let \( P_C(K_N) \) be the cut polytope defined on a complete graph having \( N \) vertices. It has been proved by several authors that there exists a linear bijective transformation mapping the cut polytope \( P_C(K_N) \) onto the Boolean Quadric Polytope \( BQ_V, |V| = N - 1 \), so that any result for \( P_C(K_N) \) can be translated into a result for \( BQ_V \) and vice versa (see e.g. [4]). In particular the following proposition holds.

**Proposition 4.1** The inequality

\[ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_{ij} z_{ij} \leq d \]

is valid (facet defining) for \( P_C(K_N) \) if and only if the inequality

\[ \sum_{i=1}^{N-1} a_{i} x_{i} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} a_{ij} y_{ij} \leq d \]

is valid (facet defining) for \( BQ_V \), \( |V| = N - 1 \), where:

\[ c_{iN} = a_i + \frac{1}{2} \sum_{j=1, j \neq i}^{N-1} a_{ij} \quad 1 \leq i \leq N - 1 \]

(2)

and

\[ c_{ij} = -\frac{1}{2} a_{ij} \quad 1 \leq i < j < N. \]

(3)
The facets of the cut polytope have been extensively studied. Deza and Laun-
rent, 1992 [5], present some classes of facets defining inequalities: Hypermetric,
Cycle, Parachute, Grishukhin, Barahona-Grotschel-Mahajoub, Kelly, Poljak-
Turzik.

Given \( N \) integers \( b_1, \ldots, b_N \) such that \( \sum_{i=1}^{N} b_i = 1 \), the inequality
\[
\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} b_i b_j x_{ij} \leq 0
\]
(4)
is an hypermetric inequality. All hypermetric inequalities are valid for the cut
polytope. If all negative \( b_i \) are equal to \(-1\), the hypermetric inequality is called
linear; if at most one negative coefficient is different from \(-1\) the hypermetric
inequality is called quasilinear. Deza and Laurent, 1992, proved the following
theorem.

**Theorem 4.2** [5] We are given an hypermetric inequality such that:
\[
b_1 \geq b_2 \geq \ldots \geq b_f > 0 > b_{f+1} \geq \ldots \geq b_N.
\]
1) If the inequality is linear, then it is facet defining if and only if, either
\( b = (1, 1, -1) \), or \( b = (1, 1, 1, -1, -1) \) or \( 3 \leq f \leq N - 3 \).
2) If the inequality is quasi-linear, then it is facet defining if and only if, either
\( b = (1, 1, -1) \), or \( b = (1, \ldots, 1, -1, -N + 4) \) or \( 3 \leq f \leq N - 3 \) and \( b_1 + b_2 \leq n - f + 1 + \text{sign}(b_1 - b_f) \).

Given the set \( V = \{1, \ldots, n\} \) and an integral vector \( \rho_1, \ldots, \rho_n \), consider the
pseudo-Boolean quadratic function
\[
g(x) = (\sum_{j=1}^{n} \rho_j x_j)(\sum_{j=1}^{n} \rho_j x_j - 1).
\]
(5)

Since \( g(x) \geq 0 \) for all integral vectors \( x \), the corresponding linear inequal-
ity \( L_g(x, y) \geq 0 \) is valid for the Boolean Quadric Polytope. It can be shown
that these inequalities correspond precisely to hypermetric inequalities. In [3],
a more general class of facet defining inequalities for the Boolean Quadric Poly-
tope has been introduced and sufficient conditions for these inequalities to be
facet defining have been presented. The corresponding inequalities for the cut
polytope include hypermetric and cycle inequalities. In [6], a class of valid in-
equalities for the cut cone including hypermetric and cycle inequalities has been
introduced.

In the following we suppose:
\[
\rho_1 \geq \rho_2 \geq \ldots \geq \rho_p > 0 > \rho_{p+1} \geq \ldots \geq \rho_n
\]
(6)
and
\[
\rho_i = -1, i = p + 1, \ldots, n.
\]
(7)
Boros, Crama and Hammer, 1990 [1], proved that for \( n \geq 3 \), if \( \rho_1 = 1 \) and
\( 2 \leq p \leq n - 1 \) then \( L_g(x, y) \geq 0 \) is facet defining.
Theorem 4.3 If the 0-1 quadratic function $g(x)$ satisfies (5), (6) and (7), the inequality $L_g(x, y) \geq 0$ valid (facet defining) for $BQ_V$ can be transformed into a linear or quasilinear hypermetric inequality valid (facet defining) for $P_{C}(K_{n+1})$.

Vice versa a linear or quasilinear hypermetric inequality valid (facet defining) for $P_{C}(K_{n+1})$ can be transformed into the inequality $L_g(x, y) \geq 0$ valid (facet defining) for $BQ_V$, for some 0-1 quadratic function $g(x)$ satisfying (5), (6) and (7).

Proof. Using the bijection defined in (2) and (3) we obtain a linear or quasi linear hypermetric inequality such that

$$b_i = \rho_i, \quad b_{n+1} = -\sum_{i=1}^{p} \rho_i + n - p + 1. \quad (8)$$

Notice that the number $f$ of positive elements in vector $b$, depends on the value of $b_{n+1}$. In particular if $b_{n+1} > 0$ (i.e. $\sum_{i=1}^{p} \rho_i \leq n - p$) then $f = p + 1$; if $b_{n+1} = 0$ (i.e. if $\sum_{i=1}^{p} \rho_i = n - p + 1$), $f = p$ and the obtained hypermetric inequality is defined on $n$ variables; if $b_{n+1} < 0$ (i.e. if $\sum_{i=1}^{p} \rho_i \geq n - p + 2$), $f = p$.

In a specular way, given a linear or quasilinear hypermetric inequality such that:

$$b_1 \geq b_2 \geq \ldots \geq b_f \geq 0 > b_{f+1} \geq \ldots \geq b_N$$

by using the bijection defined in (2) and (3), we obtain the inequality $L_g(x, y) \geq 0$ such that relations (8) hold. \qed

The following theorem is a consequence of Theorems (4.2) and (4.3).

Theorem 4.4 The inequality $L_g(x, y) \geq 0$ is facet defining for $BQ_V$ if and only if one of the following conditions holds:

1. either $n = 2$, $\rho_1 = 1$ and $p = 1, 2$;
2. or $n = 3, 4$, $\rho_1 = 1$ and $2 \leq p \leq n - 1$;
3. or $n \geq 5$, $\rho_1 = 1$ and $p = n - 1$;
4. or $n \geq 5$, $2 + \max\{0, \text{sign}(\sum_{i=1}^{p} \rho_i - n + p)\} \leq p \leq n - 2$ and $\rho_1 + \rho_2 \leq n - p + \text{sign}(\rho_1 - \rho_p)$.

Proof. Transform $L_g(x, y) \geq 0$ into an hypermetric inequality for $P_{C}(K_{n+1})$ as in the proof of Theorem 4.3 so that $L_g(x, y) \geq 0$ is facet defining for $BQ_V$ if and only if the obtained linear or quasilinear hypermetric inequality is facet defining for $P_{C}(K_{n+1})$.

In order to apply Theorem 4.2 to the obtained hypermetric inequality, we distinguish different cases.

a) $\sum_{i=1}^{p} \rho_i \leq n - p$

In this case $b_{n+1} \geq 1$, the hypermetric inequality is linear and $f = p + 1$, hence it is facet defining if and only if:
\( \bullet n = 2, \rho_1 = 1 \) and \( p = 1; \)
\( \bullet n = 4, \rho_1 = 1 \) and \( p = 2; \)
\( \bullet n \geq 5, 2 \leq p \leq n - 3. \)

b) \( \sum_{i=1}^{p} \rho_i = n - p + 1 \)
In this case \( b_{n+1} = 0, \) the hypermetric inequality is linear and \( f = p \) and it is facet defining if and only if:
\( \bullet n = 3, \rho_1 = 1 \) and \( p = 2; \)
\( \bullet n = 5, \rho_1 = 1 \) and \( p = 3; \)
\( \bullet n \geq 6, 3 \leq p \leq n - 3. \)

c) \( \sum_{i=1}^{p} \rho_i = n - p + 2 \)
In this case \( b_{n+1} = -1, \) the hypermetric inequality is linear and \( f = p \) and it is facet defining if and only if:
\( \bullet n = 2, \rho_1 = 1 \) and \( p = 2; \)
\( \bullet n = 4, \rho_1 = 1 \) and \( p = 3; \)
\( \bullet n \geq 5, 3 \leq p \leq n - 2. \)

d) \( \sum_{i=1}^{p} \rho_i \geq n - p + 3 \)
In this case \( b_{n+1} \leq -2, \) the hypermetric inequality is quasi-linear, \( f = p \) and it is facet defining if and only if:
\( \bullet n \geq 5, \rho_1 = 1 \) and \( p = n - 1; \)
\( \bullet n \geq 5, 3 \leq p \leq n - 2 \) and (*) \( \rho_1 + \rho_2 \leq n - p + \text{sign}(\rho_1 - \rho_p). \)

Notice that condition (*) always holds for linear inequalities such that \( 3 \leq f \leq n - 3. \)

The above conditions can be summarized as follows:

1. either \( n = 2, \rho_1 = 1 \) and \( p = 1, 2; \)
2. or \( n = 3, 4, \rho_1 = 1 \) and \( 1 \leq p \leq n - 1; \)
3. or \( n \geq 5, \rho_1 = 1 \) and \( p = n - 1; \)
4. or \( n \geq 5, 2 + \max\{0, \text{sign}[^{\sum_{i=1}^{p} \rho_i - n + p}] \leq p \leq n - 2 \) and \( \rho_1 + \rho_2 \leq n - p + \text{sign}(\rho_1 - \rho_p). \)

By Theorem 4.3 \( L_g(x, y) \geq 0 \) is facet defining if and only if one among cases 1., ..., 4. holds. \( \square \)
References


