Iterated elastic Brownian motions and fractional diffusion equations

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October 15, 2007

Abstract

Fractional diffusion equations of order $\nu \in (0, 2)$ are examined and solved under different types of boundary conditions. In particular, for the fractional equation on the half-line $[0, +\infty)$ and with an elastic boundary condition at $x = 0$, we are able to provide the general solution in terms of the density of the elastic Brownian motion. This permits us, for equations of order $\nu = \frac{1}{2^n}$, to write the solution as the density of the process obtained by composing the elastic Brownian motion with the $(n-1)$-times iterated Brownian motion. Also the limiting case for $n \to \infty$ is investigated and the explicit form of the solution is expressed in terms of exponential.

Moreover, the fractional diffusion equations on the half-lines $[0, +\infty)$ and $(-\infty, a]$ with additional first order space derivatives are analyzed also under reflecting or absorbing conditions. The solutions in this case lead to composed process where only the driving one is affected from drift, while the role of time is played by iterated Brownian motions.

Key words and phrases: Fractional diffusion equations; Iterated Brownian motions; Mittag-Leffler functions; Elastic Brownian motion.

MSC 2000 subject classifications. Primary: 60E05, 60J65, 60K99; secondary: 33E12.

1 Introduction

Fractional diffusion equations represent extensions of basic equations of mathematical physics (i.e. the heat and wave equations) and, in some sense, they inherit their main qualitative features, which reverberate on the form of their solutions. This kind of equations have been intensively studied since the Eighties: see, for example, Schneider and Weyss (1989), Fujita (1990, I-II), Podlubny (1999), Gorenflo et al. (2000).

Telegraph-type fractional equations have been studied and resolved under different initial or boundary-value conditions by Beghin and Orsingher (2003), Orsingher and Beghin (2004), Saxena et al. (2006), Chen et al (2007), Zhang (2007).

The relationship between initial-value problems for fractional equations and the distribution of processes obtained composing independent Brownian motions (or other processes) has been introduced in Orsingher and Beghin (2004) and subsequently extended and applied in Orsingher and Beghin (2007).

We consider here time-fractional equations on half-lines subject to different kinds of boundary conditions. Also in this case the corresponding stochastic processes can be constructed explicitly by means of well-known processes, as Brownian motion or stable processes.

A particularly interesting case is the fractional diffusion equation

$$\frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \qquad x, t > 0, \tag{1.1}$$

for $0 < \nu < 2$, on the half line $[0, +\infty)$ subject to the elastic boundary condition at $x = 0$

$$u(0,t) + \gamma \frac{\partial u(x,t)}{\partial x}\bigg|_{x=0^+} = 0, \qquad \gamma < 0, \tag{1.2}$$

where

$$\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0^+} = \lim_{\varepsilon \to 0} \frac{u(\varepsilon,t) - u(0,t)}{\varepsilon}$$

and to the initial condition

$$u(x,0) = \delta(x - x_0). \tag{1.3}$$

The fractional derivative appearing in (1.1) must be understood in the Dzherbashyan-Caputo sense, as

$$\frac{\partial^\nu u}{\partial t^\nu}(x,t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{\partial^m u(x,z)}{\partial t^m} \frac{dz}{(t-z)^{1+\nu-m}}, & \text{for } m-1 < \nu < m \\ \frac{\partial^m u(x,z)}{\partial t^m}, & \text{for } \nu = m \end{cases},$$

where $m - 1 = \lfloor \alpha \rfloor$

For $\gamma = 0$ condition (1.2) corresponds to an absorbing barrier, while, for $\gamma \to \infty$ we get a reflecting behavior at $x = 0$. If we consider equations on half-lines with rather general conditions like the elastic one (1.2), we are able to obtain explicit solutions which can be interpreted as distributions of compositions of processes. The role of the guiding process is played by the elastic Brownian motion which adequately represents the heat diffusion on semi-infinite bars, when an adiabatic and reflecting behavior of the heat flow is envisaged.

The explicit law of the elastic Brownian motion was obtained by means of probabilistic arguments in Ito and McKean (1965) while in Gallavotti and McKean (1972) some additional information on its behavior is given.

When the time-derivative is of fractional order a more analytic approach must be used: we write the solution of (1.1) under the conditions (1.2)-(1.3), for $0 < \nu \leq 1$, in terms of the Wright function

$$W_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(\alpha k + \beta)}, \qquad \alpha > -1,\ \beta > 0,\ x \in \mathbb{R},$$

as

$$\begin{aligned}
& u_\nu^{el}(x,t;x_0,0) & (1.4)\\
= \ & \frac{1}{\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}\left(-\frac{y}{\lambda t^\nu}\right)\Big\{\frac{e^{-\frac{(x-x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + \\
& -\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + 2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty}\frac{v e^{-\frac{v^2}{2(2\lambda y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda y)^3}}dv\Big\}dy \\
= \ & \frac{1}{\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}\left(-\frac{y}{\lambda t^\nu}\right)\Big\{\frac{e^{-\frac{(x-x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + \\
& +\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + \frac{2}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty}\frac{e^{-\frac{v^2}{2(2\lambda y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda y)}}dv\Big\}dy.
\end{aligned}$$

The kernel of (1.4) is the distribution $p^{el}(x,t;x_0,0)$ of an elastic Brownian motion B^{el} on $[0,+\infty)$ with starting point at $x = x_0$.

Clearly

$$\begin{aligned}
& p^{el}(x,t;x_0,0)dx & (1.5)\\
= \ & \Pr\left\{ B^{el}(t) \in dx \middle| B(0) = x_0 \right\} \\
= \ & \Pr\left\{ \overline{B}(t) \in dx \middle| B(0) = x_0 \right\} + 2e^{-\frac{(x+x_0)}{\gamma}} dx \int_{x+x_0}^{+\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} dv \\
= \ & \Pr\left\{ \overset{+}{B}(t) \in dx \middle| B(0) = x \right\} + \frac{2}{\gamma} e^{-\frac{(x+x_0)}{\gamma}} dx \int_{x+x_0}^{+\infty} \frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}} dv,
\end{aligned}$$

where $\overline{B}$ denotes the Brownian motion with absorbing barrier and $\overset{+}{B}$ the Brownian motion with reflecting barrier.

The integrals in (1.5) represent the reflecting effect of the elastic boundary condition. It is easy to check that (1.5) for $\gamma \to \infty$ becomes the distribution of the reflecting Brownian motion.

For $\nu = \frac{1}{2^n}$ and $\lambda^2 = 2^{\frac{1}{2^n}-2}$ it is well known that the solution to

$$\begin{cases} \frac{\partial^{\frac{1}{2^n}} u}{\partial t^{\frac{1}{2^n}}} = 2^{\frac{1}{2^n}-2}\frac{\partial^2 u}{\partial x^2} \\ u(x,0) = \delta(x) \end{cases}$$

can be written as

$$u_{\frac{1}{2^n}}(x,t) = \frac{1}{(2t)^{\frac{1}{2^{n+1}}}} W_{-\frac{1}{2^{n+1}},1-\frac{1}{2^{n+1}}}\left(-\frac{2|x|}{(2t)^{\frac{1}{2^{n+1}}}}\right).$$

and it has been proved in Orsingher and Beghin (2007) that it coincides with the distribution of the n-times iterated Brownian motion

$$I_n(t) = B_1(|B_2(...|B_{n+1}(t)|...)|)$$

(where B_j, $j = 1, ..., n+1$ are independent Brownian motions).

Therefore (1.4) can be interpreted as the distribution of

$$\mathcal{T}_n^{el}(t) = B^{el}(|I_{n-1}(t)|), \qquad t > 0$$

or, alternatively, in terms of free iterated Brownian motions emanating from the sources placed at $x = x_0$ and $x = -x_0$ and from the continuum of sources on the half-line $(-\infty, -x_0)$.

In light of (1.4) we can thus write that

$$\begin{aligned} & \Pr\left\{\mathcal{T}_n^{el}(t) \in dx\right\} \qquad (1.6) \\ = \; & \Pr\left\{I_n(t) \in dx |\, B_1(0) = x_0\right\} + \Pr\left\{I_n(t) \in dx |\, B_1(0) = -x_0\right\} \\ & + \frac{2}{\gamma} e^{-\frac{(x+x_0)}{\gamma}} dx \int_{x+x_0}^{+\infty} e^{\frac{v}{\gamma}} \Pr\left\{I_n(t) \in dv |\, B_1(0) = 0\right\}. \end{aligned}$$

Since $\gamma < 0$ the third term of (1.6) represents the contribution of negative sources exerting their action in $(x_0 + x, \infty)$.

The effect of the elastic barrier is played by the sources with exponentially decaying intensity distributed on $(-\infty, -x_0)$ and this is similar to what happens in the case of classical Brownian motion whose role is here played by the n-times iterated Brownian motion.

It is also interesting to note that for $\nu = \frac{1}{2^n}$ and letting $n \to \infty$ the solution (1.4) to equation (1.1) with the elastic boundary condition takes the following simple form

$$\lim_{n\to\infty} u^{el}_{\frac{1}{2^n}}(x, t; x_0, 0) = e^{-2|x-x_0|} + \frac{2\gamma+1}{2\gamma-1} e^{-2|x+x_0|}, \; x, x_0 > 0, \qquad (1.7)$$

which is an asymmetric function, does not integrate to one (because of the partially absorbing nature of the elastic barrier at $x = 0$) and does not depend on t.

Section 3 is devoted to different types of iterated processes (or compositions of processes) constructed by means of the elastic Brownian motion and their mutual relationships. In particular, we are able to show that the distribution of $B_1^{el}(B_2^{el}(t))$, $t > 0$ (with starting point at $x_0 = 0$) is the solution of

$$\frac{\partial^{1/2} u}{\partial t^{1/2}} = 2^{\frac{1}{2^2}-2} \frac{\partial^2 u}{\partial x^2}$$

subject to the non-homogeneous boundary-value condition

$$u(0, t) + \gamma_1 \left.\frac{\partial u(x, t)}{\partial x}\right|_{x=0^+} + \frac{2\gamma_1}{\gamma_2} E_{\frac{1}{2},1}\left(\frac{\sqrt{t}}{\sqrt{2}\gamma_2}\right) = 0$$

for $\gamma_1, \gamma_2 < 0$.

The Mittag-Leffler function appears also in the relationship between the distributions of $B_1^{el}(B_2^{el}(t))$ and $B_1^{el}(|B_2(t)|)$, which reads

$$\begin{aligned} & \Pr\left\{B_1^{el}(B_2^{el}(t)) \in dx\right\} \\ = & \frac{1}{2\gamma_2^2} \int_0^t \Pr\left\{B_2^{el}(s) > 0\right\} \Pr\left\{B_1^{el}(|B_2(t-s)|) \in dx\right\} ds, \end{aligned}$$

for $x > 0$, where

$$\Pr\left\{B_2^{el}(s) > 0\right\} = \frac{1}{2\gamma_2^2 s} E_{\frac{1}{2},0}\left(\frac{1}{\gamma_2}\sqrt{\frac{s}{2}}\right),$$

for $s > 0$ and $\gamma_2 < 0$. An analogous relationship holds also between $B_1^{el}(B_2^{el}(...B_{n+1}^{el}(t)...))$ and $B_1^{el}(B_2^{el}(...|B_{n+1}(t)|...))$.

A key role for the analysis of $B_1^{el}(B_2^{el}(...B_{n+1}^{el}(t)...))$, $t > 0$ is played by the Laplace transform

$$\begin{aligned} & \int_0^{+\infty} e^{-\eta t} \Pr\left\{B_1^{el}(B_2^{el}(...(B_{n+1}^{el}(t))...)) \in dx\right\} dt \qquad (1.8) \\ = & \frac{dx 2^{n+1} e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}}{\prod_{j=1}^{n+1}\left(2^{1-\frac{1}{2^{n-j+2}}} \eta^{\frac{1}{2^{n-j+2}}} - \frac{1}{\gamma_j}\right)}, \quad \eta > 0.. \end{aligned}$$

As $\gamma_1, \gamma_2, ...\gamma_{n+1} \to \infty$ we get from (1.8) that

$$\begin{aligned} & \int_0^{+\infty} e^{-\eta t} \Pr\left\{|B_1(|B_2(...|B_{n+1}(t)|...)|)| \in dx\right\} dt \qquad (1.9) \\ = & \frac{dx e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}}{\left(\frac{\eta}{2}\right)^{1-\frac{1}{2^{n+1}}}} \end{aligned}$$

and this shows that the absolute value of the n-times iterated Brownian motion $|I_n(t)|$ is just a particular case of the iterated elastic Brownian motion. Moreover from (1.9) it is clear that it converges in distribution, for $n \to \infty$, to an exponential r.v. with parameter 2 (compare with formula (3.12) of Orsingher and Beghin (2007)).

In section 4 we analyze the fractional diffusion equation with drift μ

$$\frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial u}{\partial x}, \qquad x, t > 0, \qquad (1.10)$$

for $0 < \nu < 2$, either without any barrier or subject to the reflecting condition

$$\lambda^2 \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0^+} - \mu u(0,t) = 0. \qquad (1.11)$$

We also study the case where equation (1.10) is subject to an absorbing condition

$$u(0,t) = 0. \tag{1.12}$$

Moreover, the case of equation (1.10) on the half-line $(-\infty, a]$, for $a > 0$, and reflecting or absorbing conditions at point $x = a$ is examined.

The solution to (1.10) without restricting barriers coincides with the distribution of a process of the form

$$\mathcal{T}_\nu^\mu(t) = B^{\mu/\lambda}\left(|\mathcal{T}_{2\nu}(t)|\right), \qquad t > 0,$$

where $B^{\mu/\lambda}$ is a Brownian motion with drift μ/λ independent from the process $\mathcal{T}_{2\nu}$, which is not affected by the drift.

2 Fractional diffusion equations subject to elastic boundaries: general results

In this section we consider the time-fractional diffusion equation (1.1) on the half line $[0, +\infty)$ subject to the elastic boundary condition (1.2) and the initial condition (1.3). It is well-known (see Weyss (1986), Fujita (1990), Orsingher and Beghin (2007)) that the solutions $u_\nu(x,t)$ to this kind of equations are non-negative for $0 < \nu \leq 2$ and can be interpreted as probability distributions since $\int_{-\infty}^{+\infty} u_\nu(x,t)dx = 1$.

Moreover their explicit form is given as

$$u_\nu(x,t) = \frac{1}{2\lambda t^{\nu/2}} W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x|}{\lambda t^{\nu/2}}\right). \tag{2.1}$$

For the case $0 < \nu \leq 1$ we have the following general result where the solutions are expressed in terms of the transition density of the elastic Brownian motion $B^{el}(t), t > 0$ running on the half-line $(0, +\infty)$, which reads, for $x, x_0 > 0$,

$$\begin{aligned} & p^{el}(x,t;x_0,0) \\ = \quad & \frac{e^{-\frac{(x-x_0)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x+x_0)^2}{2t}}}{\sqrt{2\pi t}} + 2e^{-\frac{(x+x_0)}{\gamma}} \int_{x+x_0}^{+\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} dv. \end{aligned} \tag{2.2}$$

Theorem 2.1

The solution to the Cauchy problem

$$\begin{cases} \frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2}, & x, t > 0 \\ u(0,t) + \gamma \frac{\partial u(x,t)}{\partial x}\Big|_{x=0^+} = 0 & \gamma < 0 \\ u(x,0) = \delta(x - x_0) \end{cases} \tag{2.3}$$

for $0 < \nu \leq 1$, is given by

$$\begin{aligned} & u_\nu^{el}(x,t;x_0,0) \qquad\qquad (2.4)\\ = \; & \frac{1}{\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}(-\frac{y}{\lambda t^\nu})p^{el}(x,2\lambda y;x_0,0)dy \\ = \; & \int_0^{+\infty} \widehat{u}_{2\nu}(y,t)p^{el}(x,2\lambda y;x_0,0)dy \end{aligned}$$

where by $\widehat{u}_{2\nu}(y,t)$ we denote

$$\widehat{u}_{2\nu}(y,t) = \begin{cases} 2u_{2\nu}(y,t), & y>0 \\ 0, & y<0 \end{cases} \qquad (2.5)$$

and $u_{2\nu}(y,t)$ is the solution to

$$\begin{cases} \frac{\partial^{2\nu}u}{\partial t^{2\nu}} = \lambda^2 \frac{\partial^2 u}{\partial y^2} \\ u(y,0) = \delta(y) \end{cases}, \qquad y \in \mathbb{R},\ t>0. \qquad (2.6)$$

Proof The general solution to (2.3) can be obtained by means of the method of separation of variables, i.e. by assuming that $u(x,t) = y(x)s(t)$. This leads to the ordinary differential equations

$$\frac{1}{\lambda^2}\frac{d^\nu s(t)}{dt^\nu}\frac{1}{s(t)} = \frac{d^2y(x)}{dx^2}\frac{1}{y(x)} = -\beta^2, \qquad \beta \in \mathbb{R}. \qquad (2.7)$$

By resolving (2.7) and taking into account the linearity of (1.1) we have the general solution of (2.3) in the form

$$u_\nu(x,t;x_0,0) = \int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2t^\nu)\left\{A(\beta)e^{i\beta x} + B(\beta)e^{-i\beta x}\right\}d\beta, \qquad (2.8)$$

where

$$E_{\nu,1}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\nu k+1)}, \qquad \nu>0,$$

is the Mittag-Leffler function. For the linear ordinary fractional equations and their solutions consult Podlubny (1999), Ch. V.

The elastic boundary condition (1.2) implies that

$$\begin{aligned} & u(0,t) + \gamma\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0^+} \\ = \; & \int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2t^\nu)\left\{A(\beta)\left(i\beta\gamma+1\right) + B(\beta)\left(-i\beta\gamma+1\right)\right\}d\beta = 0 \end{aligned}$$

and thus

$$B(\beta) = \frac{i\beta\gamma+1}{i\beta\gamma-1}A(\beta). \qquad (2.9)$$

The initial condition (1.3) is satisfied if

$$A(\beta) = \frac{1}{2\pi} e^{-i\beta x_0} \tag{2.10}$$

and thus we obtain the following solution to problem (2.3) as

$$u_\nu^{el}(x,t;x_0,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2 t^\nu)\left\{e^{i\beta(x-x_0)} + \frac{i\beta\gamma+1}{i\beta\gamma-1}e^{-i\beta(x+x_0)}\right\}d\beta. \tag{2.11}$$

The second integral in (2.11) can be worked out as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2 t^\nu)\left\{\frac{i\beta\gamma+1}{i(\beta\gamma+i)}e^{-i\beta(x+x_0)}\right\}d\beta \qquad (2.12)\\
=\;& [\beta\gamma + i = w]\\
=\;& \frac{2}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{2\pi i w}E_{\nu,1}\left(-\frac{\lambda^2 t^\nu}{\gamma^2}\left[w^2-1-2iw\right]\right)dw\\
&+\frac{1}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{2\pi}E_{\nu,1}\left(-\frac{\lambda^2 t^\nu}{\gamma^2}\left[w^2-1-2iw\right]\right)dw.
\end{aligned}$$

By taking into account the representation of the Mittag-Leffler function as contour integral on the Hankel path Ha, the first term in the r.h.s. of (2.12) becomes

$$\begin{aligned}
&\frac{2e^{-\frac{(x+x_0)}{\gamma}}}{\gamma}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{2\pi i w}\frac{dw}{2\pi i}\int_{Ha}\frac{e^z z^{\nu-1}}{z^\nu + \frac{\lambda^2 t^\nu}{\gamma^2}\left[w^2-1-2iw\right]}dz \qquad (2.13)\\
=\;& \frac{2e^{-\frac{(x+x_0)}{\gamma}}}{\gamma}\int_0^{+\infty}dy\left\{\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{2\pi i w}e^{-y\frac{\lambda^2 t^\nu}{\gamma^2}\left(w^2-1-2iw\right)}dw\right\}\frac{1}{2\pi i}\int_{Ha}e^z z^{\nu-1}e^{-yz^\nu}dz\\
=\;& \frac{2e^{-\frac{(x+x_0)}{\gamma}}}{\gamma}\int_0^{+\infty}e^{y\frac{\lambda^2 t^\nu}{\gamma^2}}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{iw}e^{-y\frac{\lambda^2 t^\nu}{\gamma^2}\left(w^2-2iw\right)}dw\right\}W_{-\nu,1-\nu}(-y)dy.
\end{aligned}$$

In the last step we have applied the following representation of the Wright function as contour integral on Ha (see Podlubny (1999), p.37)

$$W_{-\nu,1-\nu}(x) = \frac{1}{2\pi i}\int_{Ha} e^{z+xz^\nu} z^{\nu-1}dz.$$

The integral representation

$$H_\eta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iwx-iw\eta}}{iw}dw \tag{2.14}$$

of the Heaviside function

$$H_\eta(x) = \begin{cases} 1 & x > \eta \\ 0 & x < \eta \end{cases}$$

permits us to develop the integral with respect to w in (2.13) as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{iw}e^{-y\frac{\lambda^2t^\nu}{\gamma^2}(w^2-2iw)}dw \qquad (2.15)\\
=\ &\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{iw}dw\int_{-\infty}^{+\infty}e^{iwv}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma^2}\right)^2\frac{\gamma^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi\frac{2\lambda^2t^\nu y}{\gamma^2}}}dv\\
=\ &\int_{-\infty}^{+\infty}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma^2}\right)^2\frac{\gamma^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi\frac{2\lambda^2t^\nu y}{\gamma^2}}}\frac{dv}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iwv-i\frac{w}{\gamma}(x+x_0)}}{iw}dw
\end{aligned}$$

$$\begin{aligned}
=\ &\int_{-\infty}^{+\infty}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma^2}\right)^2\frac{\gamma^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi\frac{2\lambda^2t^\nu y}{\gamma^2}}}H_{\frac{x+x_0}{\gamma}}(v)dv\\
=\ &\int_{\frac{x+x_0}{\gamma}}^{+\infty}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma^2}\right)^2\frac{\gamma^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi\frac{2\lambda^2t^\nu y}{\gamma^2}}}dv=\int_{x+x_0}^{+\infty}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma}\right)^2\frac{1}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}dv.
\end{aligned}$$

By inserting (2.15) into (2.13) we can rewrite it as

$$\begin{aligned}
&\frac{2}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_0^{+\infty}e^{y\frac{\lambda^2t^\nu}{\gamma^2}}\left\{\int_{x+x_0}^{+\infty}\frac{e^{-\left(v-\frac{2\lambda^2t^\nu y}{\gamma}\right)^2\frac{1}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}dv\right\}W_{-\nu,1-\nu}(-y)dy\\
=\ &\frac{2}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{x+x_0}^{+\infty}\frac{e^{-\frac{v^2}{2(2\lambda^2t^\nu y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}dv \qquad (2.16)\\
=\ &\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\left\{-\frac{2e^{-\frac{(x+x_0)^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}+2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty}\frac{ve^{-\frac{v^2}{2(2\lambda^2t^\nu y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda^2t^\nu y)^3}}dv\right\},
\end{aligned}$$

where, in the last step we have integrated by parts with respect to v.

We now focus our attention on the second integral in (2.12) and by performing similar steps we have that

$$\begin{aligned}
&\frac{1}{\gamma}e^{-\frac{(x+x_0)}{\gamma}}\int_{-\infty}^{+\infty}\frac{e^{-i\frac{w}{\gamma}(x+x_0)}}{2\pi}E_{\nu,1}\left(-\frac{\lambda^2t^\nu}{\gamma^2}\left[w^2-1-2iw\right]\right)dw \qquad (2.17)\\
=\ &\frac{e^{-\frac{(x+x_0)}{\gamma}}}{2\pi\gamma}\int_{-\infty}^{+\infty}e^{-i\frac{w}{\gamma}(x+x_0)}\frac{dw}{2\pi i}\int_{Ha}\frac{e^zz^{\nu-1}}{z^\nu+\frac{\lambda^2t^\nu}{\gamma^2}\left[w^2-1-2iw\right]}dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-\frac{(x+x_0)}{\gamma}}}{2\pi\gamma}\int_0^{+\infty} e^{y\frac{\lambda^2 t^\nu}{\gamma^2}}\frac{dy}{2\pi i}\int_{Ha} e^z z^{\nu-1}e^{-yz^\nu}dz\int_{-\infty}^{+\infty} e^{-\frac{iw}{\gamma}(x+x_0)-\frac{y\lambda^2 t^\nu}{\gamma^2}(w^2-2iw)}dw \\
&= \frac{e^{-\frac{(x+x_0)}{\gamma}}}{\gamma}\int_0^{+\infty} e^{y\frac{\lambda^2 t^\nu}{\gamma^2}} W_{-\nu,1-\nu}(-y)\frac{e^{-\left(\frac{x+x_0}{\gamma}-\frac{2\lambda^2 t^\nu y}{\gamma^2}\right)^2\frac{\gamma^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi\frac{2\lambda^2 t^\nu y}{\gamma^2}}}dy \\
&= \int_0^{+\infty}\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}}W_{-\nu,1-\nu}(-y)dy.
\end{aligned}
$$

In order to complete the calculations we need to evaluate also the first integral in (2.11) as follows

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2 t^\nu)e^{i\beta(x-x_0)}d\beta \qquad (2.18)\\
&= \int_0^{+\infty}\frac{dy}{2\pi i}\int_{Ha} e^z z^{\nu-1}e^{-yz^\nu}\frac{dz}{2\pi}\int_{-\infty}^{+\infty} e^{i\beta(x-x_0)-\lambda^2\beta^2 t^\nu y}d\beta \\
&= \int_0^{+\infty} W_{-\nu,1-\nu}(-y)\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}}dy.
\end{aligned}
$$

By collecting together (2.16), (2.17) and (2.18) we finally have that the solution reads

$$
\begin{aligned}
&u_\nu^{el}(x,t;x_0,0) \qquad (2.19)\\
&= \int_0^{+\infty} W_{-\nu,1-\nu}(-y)\{\frac{e^{-\frac{(x-x_0)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}} + \\
&\quad -\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}} + 2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty}\frac{ve^{-\frac{v^2}{2(2\lambda^2 t^\nu y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)^3}}dv\}dy \\
&= \frac{2}{2\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}(-\frac{y}{\lambda t^\nu})\{\frac{e^{-\frac{(x-x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + \\
&\quad -\frac{e^{-\frac{(x+x_0)^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} + 2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty}\frac{ve^{-\frac{v^2}{2(2\lambda y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda y)^3}}dv\}dy \\
&= \frac{1}{\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}(-\frac{y}{\lambda t^\nu})p^{el}(x,2\lambda y;x_0,0)dy.
\end{aligned}
$$

The last step can be explained by considering formula (2.1) and (2.2). □

The previous result shows that the solution to (2.3) can be interpreted as the distribution of the process defined as

$$
\Psi_\nu^{el}(t) = B^{el}(|\Psi_{2\nu}(t)|), \qquad t>0,
$$

where the time argument is represented by the absolute value of a stochastic process $\Psi_{2\nu}$, which is independent from B^{el} and whose density coincides with the folded solution of (2.6).

Remark 2.1
The kernel $p^{el}(x,t;x_0,0)$ represents the transition density of a Brownian motion starting from $x = x_0 > 0$ and running on the half-line $[0,\infty)$ when an elastic barrier at $x = 0$ is assumed. This means that each time the particle visits the barrier it can either be absorbed or reflected and in the last case it continues its motion. The reflection behavior of B^{el} can be expressed by means of the survival random variable T with distribution

$$\Pr\{T > t|\,\mathcal{B}_t\} = e^{\frac{1}{\gamma}L(0,t)}, \qquad \gamma < 0 \tag{2.20}$$

where

$$L(0,t) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon} meas\,\{s < t : |B(s)| < \epsilon\}$$

is the local time in zero up to time t and $\mathcal{B}_t$ the σ-field of events generated by the Brownian motion B at time t.

The third term of the kernel $p^{el}(x,t;x_0,0)$ in (2.2) has a fine representation in terms of the first passage time T_a of a standard Brownian motion through a level a. This can be shown by taking the Laplace transform, as follows:

$$\begin{aligned}
& 2e^{-\frac{(x+x_0)}{\gamma}}\int_0^\infty e^{-ut}dt\int_{x+x_0}^{+\infty}\frac{v\,e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}}dv \qquad (2.21)\\
=\;& 2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty} e^{\frac{v}{\gamma}}dv\int_0^\infty e^{-ut}\frac{v\,e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t^3}}dt\\
=\;& 2e^{-\frac{(x+x_0)}{\gamma}}\int_{x+x_0}^{+\infty} e^{-v\sqrt{2u}+\frac{v}{\gamma}}dv = \frac{2e^{-\sqrt{2u}(x+x_0)}}{\sqrt{2u}-\frac{1}{\gamma}}
\end{aligned}$$

$$= 2\int_0^{+\infty} e^{-\sqrt{2u}(x+x_0+y)+\frac{y}{\gamma}}dy = 2\int_0^{+\infty} e^{\frac{y}{\gamma}}Ee^{-uT_{x+x_0+y}}dy.$$

From the above expressions we can also write that

$$\begin{aligned}
& 2\int_0^{+\infty} e^{\frac{y}{\gamma}}e^{-\sqrt{2u}(x+x_0+y)}dy \qquad (2.22)\\
=\;& 2\int_0^{+\infty} e^{\frac{y}{\gamma}}dy\int_0^{+\infty} e^{-us}\frac{x_0e^{-\frac{x_0^2}{2s}}}{\sqrt{2\pi s^3}}ds\int_0^{+\infty} e^{-ut}\frac{(x+y)e^{-\frac{(x+y)^2}{2t}}}{\sqrt{2\pi t^3}}dt\\
=\;& 2\int_0^{+\infty} e^{\frac{y}{\gamma}}dy\int_0^{+\infty} e^{-us}\frac{x_0e^{-\frac{x_0^2}{2s}}}{\sqrt{2\pi s^3}}ds\int_s^{+\infty} e^{-u(z-s)}\frac{(x+y)e^{-\frac{(x+y)^2}{2(z-s)}}}{\sqrt{2\pi(z-s)^3}}dz\\
=\;& 2\int_0^{+\infty} e^{\frac{y}{\gamma}}dy\int_0^{+\infty} e^{-uz}dz\int_0^z \frac{x_0e^{-\frac{x_0^2}{2s}}}{\sqrt{2\pi s^3}}\,\frac{(x+y)e^{-\frac{(x+y)^2}{2(z-s)}}}{\sqrt{2\pi(z-s)^3}}ds
\end{aligned}$$

$$= \int_0^{+\infty} e^{-uz} \left\{ \int_0^z \frac{x_0 e^{-\frac{x_0^2}{2s}}}{\sqrt{2\pi s^3}} ds \int_0^{+\infty} e^{\frac{y}{\gamma}} \frac{2(x+y) e^{-\frac{(x+y)^2}{2(z-s)}}}{\sqrt{2\pi (z-s)^3}} dy \right\} dz$$

$$= \int_0^{+\infty} e^{-uz} \left\{ \frac{1}{dx} \int_0^z \frac{x_0 e^{-\frac{x_0^2}{2s}}}{\sqrt{2\pi s^3}} ds E \left\{ e^{\frac{1}{\gamma} L(0,z-s)} 1_{\left\{ \overset{+}{B}(z-s) \in dx \right\}} \middle| \overset{+}{B}(0) = x_0 \right\} \right\} dz$$

$$= \frac{1}{dx} \int_0^{+\infty} e^{-uz} \Pr \left\{ \overset{+}{B}(z) \in dx,\, T_0 < z < T \middle| \overset{+}{B}(0) = x_0 \right\} dz.$$

In the above calculations we have applied the well-known fact that the joint density of the local time $L(0,t)$ and the reflecting Brownian motion $\overset{+}{B}(t)$ has the form

$$\Pr \left\{ L(0,t) \in dv, \overset{+}{B}(t) \in du \right\} = \begin{cases} \frac{2(u+v)}{\sqrt{2\pi t^3}} e^{-\frac{(u+v)^2}{2t}} du\, dv & u, v > 0 \\ 0 \quad \text{otherwise} \end{cases}.$$

By comparing (2.21) and (2.22) we can conclude that the integral in $p^{el}(x,t;x_0,0)$ can be interpreted as the probability that a reflecting Brownian motion is in x at time t after its first visit of the barrier and before its extinction.

A derivation of the transition function by means of probabilistic arguments is presented in Ito and McKean (1965), p.46, and also in Ito and McKean (1963). An interpretation of the elastic Brownian motion as a process with a time change is hinted at in Gallavotti and McKean (1972).

The density in (2.2) can also be rewritten in the form

$$\begin{aligned} & p^{el}(x,t;x_0,0) \\ = \; & \frac{e^{-\frac{(x-x_0)^2}{2t}}}{\sqrt{2\pi t}} + \frac{e^{-\frac{(x+x_0)^2}{2t}}}{\sqrt{2\pi t}} + \frac{2}{\gamma} e^{-\frac{(x+x_0)}{\gamma}} \int_{x+x_0}^{+\infty} \frac{e^{-\frac{v^2}{2t} + \frac{v}{\gamma}}}{\sqrt{2\pi t}} dv. \end{aligned} \tag{2.23}$$

In (2.2) the elastic transition is decomposed into the sum of the absorbing component plus the part depending on the effect of the elastic barrier. In (2.23) we decompose the transition function of the elastic Brownian motion into two components, the first being that pertaining to the reflecting part. It should be pointed out that γ must be a negative constant and thus the last term in (2.23) contributes negatively to the sum.

Formula (2.23) confirms that γ must be a negative constant and is the parameter of the exponential distribution (2.20) of the survival time T.

We can evaluate the survival probability by integrating the kernel of (2.2) in $[0,\infty)$:

$$\Pr\left\{B^{el}(t)>0\right\} = \int_0^{+\infty} p^{el}(x,t;x_0,0)dx \tag{2.24}$$

$$\begin{aligned}
&= \int_0^{+\infty} \frac{e^{-\frac{(x-x_0)^2}{2t}}}{\sqrt{2\pi t}}dx - \int_0^{+\infty} \frac{e^{-\frac{(x+x_0)^2}{2t}}}{\sqrt{2\pi t}}dx \\
&\quad +2\int_0^{+\infty} e^{-\frac{(x+x_0)}{\gamma}}dx \int_{x+x_0}^{+\infty} \frac{ve^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}}dv \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2\int_{x_0}^{+\infty} \frac{ve^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}}dv \int_0^{v-x_0} e^{-\frac{(x+x_0)}{\gamma}}dx \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2\gamma\int_{x_0}^{+\infty} \frac{ve^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}}\left[e^{-\frac{x_0}{\gamma}}-e^{-\frac{v}{\gamma}}\right]dv \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2\gamma\left[\frac{e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t}}\right]_{x_0}^{+\infty} \\
&\quad +2\gamma e^{-\frac{x_0}{\gamma}}\int_{x_0}^{+\infty}\left(\frac{v}{\sqrt{t}}-\frac{1}{\gamma}+\frac{1}{\gamma}\right)\frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}}dv \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw - 2\gamma\frac{e^{-\frac{x_0^2}{2t}}}{\sqrt{2\pi t}} + \\
&\quad +2\gamma e^{-\frac{x_0}{\gamma}}\left[-\frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}}\right]_{x_0}^{+\infty} + 2e^{-\frac{x_0}{\gamma}}\int_{x_0}^{+\infty}\frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}}dv \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2e^{\frac{t}{2\gamma^2}-\frac{x_0}{\gamma}}\int_{x_0}^{+\infty}\frac{e^{-\left[\sqrt{\gamma}v-\frac{t}{\sqrt{\gamma}}\right]^2\frac{1}{2\gamma t}}}{\sqrt{2\pi t}}dv \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2e^{\frac{t}{2\gamma^2}-\frac{x_0}{\gamma}}\int_{\left(x_0-\frac{t}{\gamma}\right)/\sqrt{t}}^{+\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw.
\end{aligned}$$

We also remark that, for $\gamma\to\infty$, the survival probability (2.24) becomes

$$\begin{aligned}
&\lim_{\gamma\to\infty}\Pr\left\{B^{el}(t)>0\right\} \\
&= \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2\int_{\frac{x_0}{\sqrt{t}}}^{\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw = 1,
\end{aligned}$$

since, as it is evident from (2.2), $p^{el}(x,t;x_0,0)$ coincides in the limit with the transition function of the reflecting Brownian motion. Furthermore, for $\gamma\to 0$

the probability (2.24) tends to the survival probability of an absorbing Brownian motion, as a straightforward application of the inequalities

$$\left(\frac{1}{x}-\frac{1}{x^3}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \leq \int_x^{\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}$$

shows.

Remark 2.2

Let us the consider the simplest case where the starting point is in the origin ($x_0 = 0$): from (2.2) it is evident that, in this case, the distribution of $B^{el}(t)$ reads

$$p^{el}(x,t;0,0) = 2e^{-\frac{x}{\gamma}} \int_x^{+\infty} \frac{ve^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}}dv, \quad \gamma < 0. \tag{2.25}$$

If we denote by $T_a = \inf(t : B(t) = a)$ the first passage time of the level a, we can derive the distribution of the elastic Brownian motion at T_a as follows

$$\begin{aligned} & \Pr\left\{B^{el}(T_a) \in dx\right\} \\ = \; & E\left\{\Pr\left\{B^{el}(T_a) \in dx \middle| T_a\right\}\right\} \\ = \; & dx 2e^{-\frac{x}{\gamma}} \int_x^{+\infty} ve^{\frac{v}{\gamma}}dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t^3}} \frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}}dt \\ = \; & dx \frac{2e^{-\frac{x}{\gamma}}}{2\pi} \int_x^{+\infty} ve^{\frac{v}{\gamma}}dv \int_0^{+\infty} ye^{-\frac{y}{2}(v^2+a^2)}dy \\ = \; & dx 2^2 e^{-\frac{x}{\gamma}} \int_x^{+\infty} \frac{ve^{\frac{v}{\gamma}}}{\pi(v^2+a^2)^2}dv. \end{aligned}$$

Remark 2.3

We show now that, in view of 2.24),

$$\begin{aligned} EB^{el}(t) &= x_0 - 2\gamma \int_{\frac{x_0}{\sqrt{t}}}^{\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw + 2\gamma e^{-\frac{x_0}{\gamma}+\frac{t}{2\gamma^2}} \int_{(x_0-\frac{t}{\gamma})\frac{1}{\sqrt{t}}}^{+\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw \\ &= x_0 - 2\gamma \int_{\frac{x_0}{\sqrt{t}}}^{\infty} \varphi(w)dw + \gamma \Pr\left(B^{el}(t) > 0\right) - \gamma \int_{-\frac{x_0}{\sqrt{t}}}^{\frac{x_0}{\sqrt{t}}} \varphi(w)dw \\ &= x_0 - \gamma \Pr\left(B^{el}(t) = 0\right). \end{aligned} \tag{2.26}$$

and thus the mean position of an elastic Brownian motion lies somewhere to the right of the starting position x_0.

Formula (2.26) can be evaluated by observing that

$$\begin{aligned}
& 2\int_0^{+\infty} x e^{-\frac{x+x_0}{\gamma}} dx \int_{x+x_0}^{+\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} dv \\
= \; & 2\int_{x_0}^{+\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} dv \int_0^{v-x_0} x e^{-\frac{x_0}{\gamma}-\frac{x}{\gamma}} dx \\
= \; & 2\int_{x_0}^{+\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} \left[-\gamma(v-x_0)e^{-\frac{v}{\gamma}} + \gamma^2 (e^{-\frac{x_0}{\gamma}} - e^{-\frac{v}{\gamma}}) \right] dv
\end{aligned}$$

$$\begin{aligned}
= \; & -2\gamma \int_{x_0}^{+\infty} v(v-x_0) \frac{e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t^3}} dv - 2\gamma^2 \int_{x_0}^{\infty} \frac{v e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t^3}} dv + \\
& + 2\gamma^2 e^{-\frac{x_0}{\gamma}} \int_{x_0}^{\infty} \frac{v e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t^3}} dv \\
= \; & -2\gamma \int_{x_0}^{+\infty} \frac{e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t}} dv - 2\gamma^2 \frac{e^{-\frac{x_0^2}{2t}}}{\sqrt{2\pi t}} + 2\gamma^2 e^{-\frac{x_0}{\gamma}} \int_{x_0}^{\infty} \left(\frac{v}{t} - \frac{1}{\gamma} + \frac{1}{\gamma} \right) \frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}} dv \\
= \; & -2\gamma \int_{x_0}^{+\infty} \frac{e^{-\frac{v^2}{2t}}}{\sqrt{2\pi t}} dv - 2\gamma^2 \frac{e^{-\frac{x_0^2}{2t}}}{\sqrt{2\pi t}} + 2\gamma^2 \frac{e^{-\frac{x_0^2}{2t}}}{\sqrt{2\pi t}} + 2\gamma e^{-\frac{x_0}{\gamma}} \int_{x_0}^{\infty} \frac{e^{-\frac{v^2}{2t}+\frac{v}{\gamma}}}{\sqrt{2\pi t}} dv.
\end{aligned}$$

A change of variable then yields (2.26) once the remaining two integrals of $EB^{el}(t)$ are evaluated. the second line of (2.26) is obtained by taking into account (2.24).

For large values of t we can apply the approximation

$$\int_{(x_0-\frac{t}{\gamma})\frac{1}{\sqrt{t}}}^{+\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw \sim e^{-\frac{\left(x_0-\frac{t}{\gamma}\right)^2 \frac{1}{2t}}{2}} \frac{1}{\sqrt{2\pi}(x_0-\frac{t}{\gamma})\frac{1}{\sqrt{t}}}$$

and thus

$$\lim_{t\to\infty} EB^{el}(t) = x_0 - \gamma.$$

By means of result (2.26) we can also evaluate the mean value of the process $\Psi_\nu^{el}(t) = B^{el}(|\Psi_{2\nu}(t)|)$ related to our Cauchy problem (2.3) as follows

$$\begin{aligned}
& E\Psi_\nu^{el}(t) \\
= \; & \frac{1}{\lambda t^\nu} \int_0^{+\infty} W_{-\nu,1-\nu}(-\frac{y}{\lambda t^\nu}) \{ x_0 - 2\gamma \int_{x_0}^{\infty} \frac{e^{-\frac{v^2}{2(2\lambda y)}}}{\sqrt{2\pi(2\lambda y)}} dv \\
& + 2\gamma e^{-\frac{x_0}{\gamma}} \int_{x_0}^{\infty} \frac{e^{-\frac{v^2}{2(2\lambda y)}+\frac{v}{\gamma}}}{\sqrt{2\pi(2\lambda y)}} dv \} dy \\
= \; & x_0 - \frac{2\gamma}{\lambda} \int_{\lambda x_0}^{\infty} u_\nu(v,t) dv + \frac{2\gamma}{\lambda} e^{-\frac{x_0}{\gamma}} \int_{\lambda x_0}^{\infty} e^{\frac{v}{\gamma}} u_\nu(v,t) dv,
\end{aligned}$$

where, in the last step we have applied Theorem 2.1 of Orsingher and Beghin (2007).

Remark 2.4
We can easily give an analytic solution to problem (2.3), which is valid for any $\nu \in (0,2)$, but looses the transparent probabilistic interpretation of (2.4). We write (2.11) in the more convenient form

$$\begin{aligned}
& u_\nu^{el}(x,t;x_0,0) \qquad (2.27)\\
= \; & \frac{1}{2\pi}\int_{-\infty}^{+\infty} E_{\nu,1}(-\lambda^2\beta^2 t^\nu)\left\{ e^{i\beta(x-x_0)} + e^{-i\beta(x+x_0)} - \frac{2}{1-i\beta\gamma}e^{-i\beta(x+x_0)}\right\} d\beta \\
= \; & \frac{1}{2\lambda t^{\nu/2}} W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x-x_0|}{\lambda t^{\nu/2}}\right) + \frac{1}{2\lambda t^{\nu/2}} W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x+x_0|}{\lambda t^{\nu/2}}\right) + \\
& -2\int_0^{+\infty} dy \int_{-\infty}^{+\infty} e^{-y(1-i\beta\gamma)-i\beta(x+x_0)} E_{\nu,1}(-\lambda^2\beta^2 t^\nu) d\beta \\
= \; & \frac{1}{2\lambda t^{\nu/2}}\{ W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x-x_0|}{\lambda t^{\nu/2}}\right) + W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x+x_0|}{\lambda t^{\nu/2}}\right) + \\
& -2\int_0^{+\infty} e^{-y} W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x+x_0-y\gamma|}{\lambda t^{\nu/2}}\right) dy\}.
\end{aligned}$$

The last formula shows that the solution u_ν^{el} can be expressed by suitably combining the Wright function representing the solution u_ν of the fractional equation in $(-\infty,+\infty)$. This is analogous to what happens for the heat equation (i.e. for $\nu = 1$) where the solution (2.23) to the elastic Cauchy problem is obtained by combining in a similar way the Gaussian transition function of the free Brownian motion.

It is easy to see that in the absorbing case ($\gamma = 0$) we get from (2.27) the solution under the boundary condition $u|_{x=0^+} = 0$, that is

$$\overline{u}_\nu(x,t;x_0,0) = \frac{1}{2\lambda t^{\nu/2}}\{ W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x-x_0|}{\lambda t^{\nu/2}}\right) - W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x+x_0|}{\lambda t^{\nu/2}}\right)\}.$$

On the other hand, for $\gamma \to \infty$, the third term in the last member of (2.27) converges to zero and we get the solution under the action of the reflecting barrier:

$$\overset{+}{u}_\nu(x,t;x_0,0) = \frac{1}{2\lambda t^{\nu/2}}\{ W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x-x_0|}{\lambda t^{\nu/2}}\right) + W_{-\frac{\nu}{2},1-\frac{\nu}{2}}\left(-\frac{|x+x_0|}{\lambda t^{\nu/2}}\right)\}.$$

3 Iterated elastic Brownian motions

We examine now in detail the particular case where $\nu = \frac{1}{2^n}$ and $\lambda^2 = 2^{\frac{1}{2^n}-2}$, which leads to a first form of iterated elastic Brownian motion, where the driving process is an elastic Brownian motion composed with the classical iterated Brownian motion.

Theorem 3.1

The solution to the Cauchy problem

$$\begin{cases} \frac{\partial^{\frac{1}{2^n}} u}{\partial t^{\frac{1}{2^n}}} = 2^{\frac{1}{2^n}-2} \frac{\partial^2 u}{\partial x^2}, & x, t > 0 \\ u(0,t) + \gamma \frac{\partial u}{\partial x}\big|_{x=0^+} = 0 & \gamma < 0 \\ u(x,0) = \delta(x - x_0) \end{cases} \tag{3.1}$$

is given by

$$u^{el}_{\frac{1}{2^n}}(x,t;x_0,0) = \int_0^{+\infty} \widehat{u}_{\frac{1}{2^{n-1}}}(y,t) p^{el}(x, 2^{\frac{1}{2^n}} y; x_0, 0) dy, \tag{3.2}$$

where

$$\widehat{u}_{\frac{1}{2^{n-1}}}(y,t) = 2^n \int_0^{+\infty} \dots \int_0^{+\infty} \frac{e^{-\frac{y^2}{2z_1}}}{\sqrt{2\pi z_1}} \dots \frac{e^{-\frac{z_{n-1}^2}{2t}}}{\sqrt{2\pi t}} dz_1 \dots dz_{n-1}. \tag{3.3}$$

Proof

We specify the result of Theorem 2.1 to the case where $\nu = \frac{1}{2^n}$ and $\lambda^2 = 2^{\frac{1}{2^n}-2}$ as follows:

$$\begin{aligned} & u^{el}_{\frac{1}{2^n}}(x,t;x_0,0) \\ = \; & \frac{1}{2^{\frac{1}{2^n}-1} t^{\frac{1}{2^n}}} \int_0^{+\infty} W_{-\frac{1}{2^n}, 1-\frac{1}{2^n}}\left(-\frac{y}{2^{\frac{1}{2^n}-1} t^{\frac{1}{2^n}}}\right) p^{el}(x, 2^{\frac{1}{2^n}} y; x_0, 0) dy, \end{aligned}$$

which coincides with (3.2). The expression of the solution (3.3) can be obtained by means of formula (1.9) of Orsingher and Beghin (2007). □

We point out that (3.3) coincides with the distribution of the absolute value of $I_{n-1}(t)$, which is defined as the $(n-1)$-iterated Brownian motion

$$I_{n-1}(t) = B_1(|B_2(\dots|B_n(t)|\dots)|) \quad t > 0.$$

Therefore we can interpret the solution as the transition density of the following process

$$\Psi^{el}_n(t) = B^{el}_1(|I_{n-1}(t)|), \qquad t > 0.$$

Remark 3.1

If we now take the limit, for $n \to \infty$, of (3.2) and apply the following result

$$\begin{aligned} \lim_{n\to\infty} \widehat{u}_{\frac{1}{2^{n-1}}}(y,t) &= \lim_{n\to\infty} \frac{1}{2^{\frac{1}{2^n}-1} t^{\frac{1}{2^n}}} W_{-\frac{1}{2^n}, 1-\frac{1}{2^n}}\left(-\frac{y}{2^{\frac{1}{2^n}-1} t^{\frac{1}{2^n}}}\right) \\ &= 2W_{0,1}(-2y) = 2e^{-2y}, \end{aligned} \tag{3.4}$$

so that we get

$$\begin{aligned}
& \lim_{n\to\infty} u^{el}_{\frac{1}{2^n}}(x,t;x_0,0) \qquad (3.5)\\
= & \lim_{n\to\infty} \int_0^{+\infty} \widehat{u}_{\frac{1}{2^{n-1}}}(y,t) p^{el}(x, 2^{\frac{1}{2^n}} y; x_0, 0) dy
\end{aligned}$$

$$\begin{aligned}
= & \int_0^{+\infty} \lim_{n\to\infty} \widehat{u}_{\frac{1}{2^{n-1}}}(y,t) \{ \frac{e^{-\frac{(x-x_0)^2}{2y}}}{\sqrt{2\pi y}} - \frac{e^{-\frac{(x+x_0)^2}{2y}}}{\sqrt{2\pi y}} + 2e^{-\frac{(x+x_0)}{\gamma}} \int_{x+x_0}^{+\infty} \frac{ve^{-\frac{v^2}{2y}+\frac{v}{\gamma}}}{\sqrt{2\pi y^3}} dv \} dy \\
= & \; 2\int_0^{+\infty} e^{-2y} \{ \frac{e^{-\frac{(x-x_0)^2}{2y}}}{\sqrt{2\pi y}} - \frac{e^{-\frac{(x+x_0)^2}{2y}}}{\sqrt{2\pi y}} + 2e^{-\frac{(x+x_0)}{\gamma}} \int_{x+x_0}^{+\infty} \frac{ve^{-\frac{v^2}{2y}+\frac{v}{\gamma}}}{\sqrt{2\pi y^3}} dv \} dy \\
= & \; \frac{2e^{-2|x-x_0|}}{2} - \frac{2e^{-2|x+x_0|}}{2} + \\
& +4e^{-\frac{|x+x_0|}{\gamma}} \int_{x+x_0}^{+\infty} e^{\frac{v}{\gamma}} dv \int_0^{\infty} e^{-2y} \frac{ve^{-\frac{v^2}{2y}}}{\sqrt{2\pi y^3}} dy
\end{aligned}$$

$$\begin{aligned}
= & \; e^{-2|x-x_0|} - e^{-2|x+x_0|} + 4e^{-\frac{x+x_0}{\gamma}} \int_{x+x_0}^{+\infty} e^{-v\left(2-\frac{1}{\gamma}\right)} dv \\
= & \; e^{-2|x-x_0|} - e^{-2|x+x_0|} + \frac{4\gamma}{2\gamma-1} e^{-2(x+x_0)} \\
= & \; e^{-2|x-x_0|} + \frac{2\gamma+1}{2\gamma-1} e^{-2|x+x_0|},
\end{aligned}$$

for $x, x_0 > .$

Alternatively we can take the limit of (2.27), getting the same result:

$$\begin{aligned}
& \lim_{n\to\infty} u^{el}_{\frac{1}{2^n}}(x,t;x_0,0) \\
= & \; W_{0,1}(-2|x-x_0|) + W_{0,1}(-2|x+x_0|) + \\
& -2\int_0^{+\infty} e^{-y} W_{0,1}(-2|x+x_0-y\gamma|) dy \\
= & \; e^{-2|x-x_0|} + e^{-2(x+x_0)} - 2e^{-(x+x_0)} \int_0^{+\infty} e^{-y+\gamma y} dy \\
= & \; e^{-2|x-x_0|} + e^{-2(x+x_0)} - 2\frac{e^{-(x+x_0)}}{1-2\gamma} \\
= & \; e^{-2|x-x_0|} + \frac{2\gamma+1}{2\gamma-1} e^{-2(x+x_0)}.
\end{aligned}$$

The most striking fact about (3.5) is that the limiting distribution does not depend on t.

The results obtained so far suggest us to introduce and examine new compositions of processes such as $B_1^{el}(|B_2(t)|)$, $B_1^{el}(|B_2^{el}(t)|)$ and $B_1(|B_2^{el}(t)|)$, which are similar to the iterated Brownian motion. These processes represent generalizations in different directions of the iterated Brownian motion. Many properties of $I_1(t) = B_1(|B_2(t)|)$ have been analyzed by Burdzy (1994) (the fourth-order variation), by Burdzy and San Martìn (1995) (the law of the iterated logarithm), by Khoshnevisian and Lewis (1996) (the modulus of continuity) and by Allouba and Zheng (2001). Applications of the iterated Brownian motion can be found in De Blassie (2004).

The iterated processes of different forms emerging here can be imagined as limits of compositions of independent random walks. In the simplest case a random walk with a reflecting barrier on the y axis represents the time with upward and downward steps of length Δs ("upward" means that time moves from the past to the future and viceversa for "downward"). At each instant we can consider a random walk on a line parallel to the x axes, that, every Δs units of time, moves rightward or backwards. Thus the particle occupies the position x if a sufficient time elapse has passed and we must sum up (integrate) on all the random walks passing through x. In the limit this construction leads to the iterated Brownian motion. Moreover, if at $x = 0$ some form of barrier is considered, we have reflecting, absorbing or elastic random walks composed with the random walk representing the time evolution.

The section below is devoted to the analysis of these processes (and their extensions) and to the related boundary-value problems for the fractional diffusion equation. For the sake of simplicity we consider elastic Brownian motions starting at $x = 0$.

Theorem 3.2

The distribution of the process

$$B_1^{el}(|B_2(t)|), \qquad t > 0,$$

given by

$$\begin{aligned} & \Pr\left\{B_1^{el}(|B_2(t)|) \in dx\right\} \\ = \quad & dx 2^2 e^{-\frac{x}{\gamma}} \int_0^{+\infty} ds \int_x^{+\infty} \frac{v e^{-\frac{v^2}{2s}+\frac{v}{\gamma}}}{\sqrt{2\pi s^3}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} dv, \end{aligned} \tag{3.6}$$

coincides with the solution $u_{\frac{1}{2}}^{el}(x, t; 0, 0)$ of the Cauchy problem

$$\begin{cases} \frac{\partial^{\frac{1}{2}} u}{\partial t^{\frac{1}{2}}} = \frac{1}{2^{3/2}} \frac{\partial^2 u}{\partial x^2}, & x, t > 0 \\ u(0, t) + \gamma \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0^+} = 0 & \gamma < 0. \\ u(x, 0) = \delta(x) \end{cases} \tag{3.7}$$

Proof Instead of the reasoning used in Theorem 2.1, we resort here to the Laplace transform

$$L(x, \eta) = \int_0^{+\infty} e^{-\eta t} u(x, t) dt, \tag{3.8}$$

which converts problem (3.7) into

$$\begin{cases} -\frac{1}{\sqrt{\eta}}\delta(x)+\sqrt{\eta}L(x,\eta)=\frac{1}{2^{3/2}}\frac{\partial^2 L}{\partial x^2} \\ L(0,\eta)+\gamma\left.\frac{\partial L(x,\eta)}{\partial x}\right|_{x=0^+}=0 \end{cases}. \tag{3.9}$$

The solution to equation (3.9), for $x>0$, is clearly equal to

$$L(x,\eta)=Ae^{x\sqrt{2\sqrt{2\eta}}}+Be^{-x\sqrt{2\sqrt{2\eta}}} \tag{3.10}$$

and, in view of the boundedness of the solution we must assume that $A=0$.

From (3.9), by integrating in the interval $[0,\varepsilon)$, we have that

$$\begin{aligned} & -\frac{1}{\sqrt{\eta}}\int_0^\varepsilon \delta(x)dx+\sqrt{\eta}\int_0^\varepsilon L(x,\eta)dx \qquad (3.11) \\ = {} & \frac{1}{2^{3/2}}\int_0^\varepsilon \frac{\partial^2 L(x,\eta)}{\partial x^2}dx \\ = {} & \frac{1}{2^{3/2}}\left[-\sqrt{2\sqrt{2\eta}}Be^{-\varepsilon\sqrt{2\sqrt{2\eta}}}-\left.\frac{\partial L(x,\eta)}{\partial x}\right|_{x=0^+}\right]. \end{aligned}$$

By taking into account that L is a bounded function, from (3.11) we get that

$$B+\gamma\left(\frac{2^{3/2}}{\sqrt{\eta}}-B\sqrt{2\sqrt{2\eta}}\right)=0$$

and thus

$$B=-\frac{2^{3/2}\gamma}{\sqrt{\eta}}\frac{1}{1-\gamma\sqrt{2\sqrt{2\eta}}}=\frac{2^2}{\sqrt{2\eta}\left(\sqrt{2\sqrt{2\eta}}-\frac{1}{\gamma}\right)}.$$

Therefore the solution to problem (3.9) is given by

$$L(x,\eta)=\frac{2^2e^{-x\sqrt{2\sqrt{2\eta}}}}{\sqrt{2\eta}\left(\sqrt{2\sqrt{2\eta}}-\frac{1}{\gamma}\right)} \tag{3.12}$$

and this coincides with the Laplace transform of (3.6). □

Remark 3.2

We show now that the distribution of $B_1^{el}(|B_2(t)|)$ can be expressed by means of the law of the iterated Brownian motion $I(t)$. For the particular case where

$x_0 = 0$ we get from (2.25), for $x > 0$,

$$\begin{aligned}
& \Pr\left\{B_1^{el}(|B_2(t)|) \in dx\right\} && (3.13)\\
= \; & 2^2 e^{-\frac{x}{\gamma}} \int_0^{+\infty} ds \int_x^{+\infty} \frac{v e^{-\frac{v^2}{2s}+\frac{v}{\gamma}}}{\sqrt{2\pi s^3}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} dv \\
= \; & 2^2 e^{-\frac{x}{\gamma}} \{ \int_0^{+\infty} \left[-\frac{e^{-\frac{v^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} e^{\frac{v}{\gamma}} \right]_{v=x}^{v=+\infty} ds + \\
& + \frac{1}{\gamma} \int_x^{+\infty} dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2s}+\frac{v}{\gamma}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds \} \\
= \; & 2^2 \int_0^{+\infty} \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds + \\
& + \frac{2^2 e^{-\frac{x}{\gamma}}}{\gamma} \int_x^{+\infty} e^{\frac{v}{\gamma}} dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2s}+\frac{v}{\gamma}}}{\sqrt{2\pi s}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds
\end{aligned}$$

$$\begin{aligned}
= \; & 2 \Pr\{I(t) \in dx\} + \frac{2e^{-\frac{x}{\gamma}}}{\gamma} \int_x^{+\infty} e^{\frac{v}{\gamma}} \Pr\{I(t) \in dv\} \\
= \; & 2 \Pr\{I(t) \in dx\} + \frac{2}{\gamma} \int_0^{+\infty} e^{\frac{y}{\gamma}} \Pr\{I(t) \in d(x+y)\} \\
= \; & \Pr\{|I(t)| \in dx\} + \frac{1}{\gamma} \int_0^{+\infty} e^{\frac{y}{\gamma}} \Pr\{|I(t)| \in d(x+y)\}
\end{aligned}$$

Since γ is a negative constant the previous relationship shows that the distribution of $B^{el}(|B(t)|)$ can be obtained from that of $I(t)$ by an appropriate correction (due to the partially absorbing effect of each visit of the barrier).

The same result could be also derived by considering that the process $B^{el}(|B(t)|)$ is a particular case of $\Psi_n^{el}(t) = B_1^{el}(|I_{n-1}(t)|)$ for $n = 1$ (i.e. for $\nu = 1/2$), with the additional assumption that $x_0 = 0$. Therefore by specializing (3.2) we get

$$\begin{aligned}
& u_{\frac{1}{2}}^{el}(x, t; 0, 0) && (3.14)\\
= \; & 2 \int_0^{+\infty} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} p^{el}(x, 2^{\frac{1}{2}} y; 0, 0) dy
\end{aligned}$$

and, by inserting (2.25) we obtain again (3.13).

We consider now the process obtained by composing two independent elastic Brownian motions B_1^{el} and B_2^{el} starting from the origin, as

$$B_1^{el}(B_2^{el}(t)), \qquad t > 0.$$

In this case the role of time is played by an elastic Brownian motion and its related clock can be stopped during the visits of the origin. Once the clock is stopped the driving process is forced to be captured by the elastic trap because it behaves like $B_1^{el}(0)$ for the remaining time elapse.

In the following result we prove that its density satisfies a Cauchy problem identical to (3.7), but with an additional term in the boundary condition.

Theorem 3.3
The transition density of the process $B_1^{el}(B_2^{el}(t))$, $t > 0$, given by

$$\begin{aligned} & \Pr\left\{B_1^{el}(B_2^{el}(t)) \in dx\right\} && (3.15) \\ = \quad & 2dxe^{-\frac{x}{\gamma_1}} \int_x^{+\infty} ve^{\frac{v}{\gamma_1}} dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2s}}}{\sqrt{2\pi s^3}} \left(2e^{-\frac{s}{\gamma_2}} \int_s^{+\infty} \frac{we^{-\frac{w^2}{2t}+\frac{w}{\gamma_2}}}{\sqrt{2\pi t^3}} dw \right) ds \end{aligned}$$

solves the following Cauchy problem

$$\begin{cases} \frac{\partial^{\frac{1}{2}} u}{\partial t^{\frac{1}{2}}} = \frac{1}{2^{3/2}} \frac{\partial^2 u}{\partial x^2}, \qquad x, t > 0 \\ u(0,t) + \gamma_1 \left. \frac{\partial u(x,t)}{\partial x} \right|_{x=0^+} + f(t) = 0 \\ u(x,0) = \delta(x) \end{cases}, \tag{3.16}$$

where

$$f(t) = \frac{2\gamma_1}{\gamma_2} E_{\frac{1}{2},1}\left(\frac{\sqrt{t}}{\sqrt{2}\gamma_2} \right), \quad \gamma_1, \gamma_2 < 0. \tag{3.17}$$

Proof We start by taking the Laplace transform of (3.15) with respect to t:

$$\begin{aligned} & \int_0^{+\infty} e^{-\eta t} \Pr\left\{B_1^{el}(B_2^{el}(t)) \in dx\right\} dt && (3.18) \\ = \quad & 2dxe^{-\frac{x}{\gamma_1}} \int_x^{+\infty} ve^{\frac{v}{\gamma_1}} dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2s}}}{\sqrt{2\pi s^3}} \left(2e^{-\frac{s}{\gamma_2}} \int_0^{+\infty} e^{-\eta t} dt \int_s^{+\infty} w \frac{e^{\frac{w}{\gamma_2}-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw \right) ds \\ = \quad & 2^2 dxe^{-\frac{x}{\gamma_1}} \int_x^{+\infty} ve^{\frac{v}{\gamma_1}} dv \int_0^{+\infty} \frac{e^{-\frac{v^2}{2s}}}{\sqrt{2\pi s^3}} e^{-\frac{s}{\gamma_2}} ds \int_s^{+\infty} e^{\frac{w}{\gamma_2} - w\sqrt{2\eta}} dw \\ = \quad & 2^2 dxe^{-\frac{x}{\gamma_1}} \int_x^{+\infty} e^{\frac{v}{\gamma_1}} dv \int_0^{+\infty} v \frac{e^{-\frac{v^2}{2s} - s\sqrt{2\eta}}}{\sqrt{2\pi s^3}} \frac{1}{\sqrt{2\eta} - \frac{1}{\gamma_2}} ds \\ = \quad & 2^2 dxe^{-\frac{x}{\gamma_1}} \int_x^{+\infty} e^{\frac{v}{\gamma_1}} \frac{1}{\sqrt{2\eta} - \frac{1}{\gamma_2}} e^{-v\sqrt{2\sqrt{2\eta}}} dv \\ = \quad & \frac{2^2 dx}{\sqrt{2\eta} - \frac{1}{\gamma_2}} \frac{e^{-x\sqrt{2\sqrt{2\eta}}}}{\sqrt{2\sqrt{2\eta}} - \frac{1}{\gamma_1}}. \end{aligned}$$

We now pass to the Laplace transform of (3.16) which leads to

$$\begin{cases} -\frac{1}{\sqrt{\eta}} \delta(x) + \sqrt{\eta} L(x,\eta) = \frac{1}{2^{3/2}} \frac{\partial^2 L}{\partial x^2} \\ L(0,\eta) + \gamma_1 \left. \frac{\partial L(x,\eta)}{\partial x} \right|_{x=0^+} + \int_0^{+\infty} e^{-\eta t} f(t) dt = 0 \end{cases}. \tag{3.19}$$

The solution to the above equation, in view of its boundedness, takes the following form

$$L(x,\eta) = Be^{-x\sqrt{2\sqrt{2\eta}}}$$

and the constant B can be determined by integrating the first equation in (3.19) on $[0,\varepsilon]$ and then letting $\varepsilon \to 0$:

$$-\frac{1}{\sqrt{\eta}} = \frac{1}{2^{3/2}}\left[-B\sqrt{2\sqrt{2\eta}} - \left.\frac{\partial L}{\partial x}\right|_{x=0^+}\right]. \tag{3.20}$$

By inserting (3.20) into (3.19) we get that

$$B = \frac{\frac{2^2}{\sqrt{2\eta}} + \frac{1}{\gamma_1}\int_0^{+\infty} e^{-\eta t} f(t)dt}{\sqrt{2\sqrt{2\eta}} - \frac{1}{\gamma_1}}.$$

Thus the solution to problem (3.16) becomes

$$\begin{aligned} & L(x,\eta) \\ = \quad & \left(\frac{2^2}{\sqrt{2\eta}} + \frac{1}{\gamma_1}\int_0^{+\infty} e^{-\eta t} f(t)dt\right) \frac{e^{-x\sqrt{2\sqrt{2\eta}}}}{\sqrt{2\sqrt{2\eta}} - \frac{1}{\gamma_1}}. \end{aligned} \tag{3.21}$$

We need now to determine the explicit expression for $f(t)$, such that (3.21) and (3.18) coincide; after some manipulation we obtain that

$$\begin{aligned} \int_0^{+\infty} e^{-\eta t} f(t)dt &= \frac{2^2\gamma_1}{\gamma_2}\frac{1}{\sqrt{2\eta}(\sqrt{2\eta} - \frac{1}{\gamma_2})} \\ &= \frac{2\gamma_1}{\gamma_2}\frac{1}{\eta - \sqrt{\eta}\frac{1}{\sqrt{2}\gamma_2}}. \end{aligned}$$

It can be checked that

$$\int_0^{+\infty} e^{-\eta t} E_{\frac{1}{2},1}\left(\frac{\sqrt{t}}{\sqrt{2}\gamma_2}\right) dt = \frac{1}{\eta - \sqrt{\eta}\frac{1}{\sqrt{2}\gamma_2}}, \quad for\ \eta > \frac{1}{2\gamma_2^2}$$

and then (3.17) follows (for Laplace transforms of Mittag-Leffler functions see Podlubny (1999), p.21, formula (1.80)). □

In the following theorem we obtain a connection between the two processes $B_1^{el}(B_2^{el}(t))$ and $B_1^{el}(B_2(t))$.

Theorem 3.4
For the iterated elastic Brownian motion the following relationship holds

$$\begin{aligned} & \Pr\left\{B_1^{el}(B_2^{el}(t)) \in dx\right\} \\ = \quad & \frac{1}{2\gamma_2^2}\int_0^t \Pr\left\{B_2^{el}(s) > 0\right\} \Pr\left\{B_1^{el}(|B_2(t-s)|) \in dx\right\} ds, \end{aligned} \tag{3.22}$$

for $x, t > 0$ and $\gamma_2 < 0$.
Proof The Laplace transform (3.18) can be rearranged as

$$\int_0^{+\infty} e^{-\eta t} \Pr\left\{B_1^{el}(B_2^{el}(t)) \in dx\right\} dt \tag{3.23}$$
$$= \frac{\sqrt{2\eta}}{\sqrt{2\eta} - \frac{1}{\gamma_2}} \frac{2^2 dx e^{-x\sqrt{2\sqrt{2\eta}}}}{\sqrt{2\eta}\left(\sqrt{2\sqrt{2\eta}} - \frac{1}{\gamma_1}\right)}.$$

The second term in the right hand side of (3.23) corresponds to the Laplace transform (3.12) of $B_1^{el}(|B_2(t)|)$. In order to determine the inverse Laplace transform of the additional term we write

$$\begin{aligned} \frac{1}{1 - \frac{1}{\gamma_2\sqrt{2\eta}}} &= \sum_{k=0}^{\infty} \left(\frac{1}{\gamma_2\sqrt{2\eta}}\right)^k \\ &= \int_0^{+\infty} e^{-\eta t} \sum_{k=0}^{\infty} \frac{t^{\frac{k}{2}-1}}{\Gamma\left(\frac{k}{2}\right)} \frac{1}{\left(\gamma_2\sqrt{2\eta}\right)^k} dt, \end{aligned} \tag{3.24}$$

and this permits us to conclude that it coincides with

$$g(t) = \frac{1}{t} E_{\frac{1}{2},0}\left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}}\right). \tag{3.25}$$

The previous function can be rewritten as follows

$$\begin{aligned} g(t) &= \frac{1}{t} \sum_{k=1}^{\infty} \left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}}\right)^k \frac{1}{\Gamma\left(\frac{k}{2}\right)} \\ &= \frac{1}{t} \sum_{r=0}^{\infty} \left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}}\right)^{r+1} \frac{1}{\Gamma\left(\frac{r}{2} + \frac{1}{2}\right)} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{t} \sum_{r=0}^{\infty} \left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}}\right)^{r+1} \frac{\Gamma\left(\frac{r}{2}\right) 2^r}{2\sqrt{\pi}\Gamma(r)} \\ &= \frac{1}{2\sqrt{\pi}t} \int_0^{+\infty} e^{-w} \sum_{r=1}^{\infty} \left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}}\right)^{r+1} \frac{w^{\frac{r}{2}-1} 2^r}{(r-1)!} dw \\ &= \frac{1}{2\gamma_2^2\sqrt{\pi}} \int_0^{+\infty} e^{-w} w^{-\frac{1}{2}} e^{\frac{1}{\gamma_2}\sqrt{2tw}} dw. \end{aligned}$$

By the change of variable $y = \sqrt{2w}$ and by comparing (2.24) for $x_0 = 0$, formula (3.22) is then obtained. $\square$

Formula (3.22) shows that $B_1^{el}(B_2^{el}(t))$ can reach a point x if the "time" process B_2^{el} has not been absorbed up to time s and then it behaves as $B_1^{el}(|B_2(t)|)$, in the interval (s,t).

The mean value of the iterated elastic Brownian motion starting at $x_0 = 0$ can be obtained, by recalling (2.26), as follows

$$\begin{aligned} & E\left(B_1^{el}(B_2^{el}(t))\right) \qquad (3.26) \\ = & \int_0^{+\infty} EB_1^{el}(s)\Pr\left\{B_2^{el}(t)\in ds\right\} \\ = & -\gamma_1\int_0^{+\infty}\Pr\left\{B_1^{el}(s)=0\right\}\Pr\left\{B_2^{el}(t)\in ds\right\}. \end{aligned}$$

From (3.26) and (2.26) we can easily check that $EB_1^{el}(s)$ and thus $E\left(B_1^{el}(B_2^{el}(t))\right)$ are non-negative: indeed we rewrite the first one follows

$$\begin{aligned} EB_1^{el}(s) &= -\gamma_1 + 2\gamma_1 e^{\frac{s}{2\gamma_1^2}}\int_{-\frac{\sqrt{s}}{\gamma_1}}^{+\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw \\ &= \left[y = w+\sqrt{s}/\gamma_1\right] \\ &= -2\gamma_1\int_0^{+\infty}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\left(1-e^{\frac{y\sqrt{s}}{\gamma_1}}\right)dy > 0, \end{aligned}$$

for $\gamma_1 < 0$ and for any $s > 0$.

We generalize the previous results to the n-times iterated elastic Brownian motion (with starting point at $x=0$) and we show that its density (consisting of $2n+1$ integrals), given by

$$\begin{aligned} & \Pr\left\{B_1^{el}(B_2^{el}(\dots(B_{n+1}^{el}(t))\dots))\in dx\right\} \qquad (3.27) \\ = \; & dx 2^{n+1}e^{-\frac{x}{\gamma_1}}\int_x^{+\infty}v_1 e^{\frac{v_1}{\gamma_1}}dv_1\int_0^{+\infty}\frac{e^{-\frac{v_1^2}{2s_1}}}{\sqrt{2\pi s_1^3}}e^{-\frac{s_1}{\gamma_2}}ds_1\int_{s_1}^{+\infty}v_2 e^{\frac{v_2}{\gamma_2}}dv_2\,\cdot \\ & \cdot\int_0^{+\infty}\frac{e^{-\frac{v_2^2}{2s_2}}}{\sqrt{2\pi s_2^3}}ds_2\dots\int_0^{+\infty}e^{-\frac{s_n}{\gamma_{n+1}}}\frac{v_n e^{-\frac{v_n^2}{2s_n}}}{\sqrt{2\pi s_n^3}}ds_n\int_{s_n}^{+\infty}v_{n+1}e^{\frac{v_{n+1}}{\gamma_{n+1}}}dv_{n+1}\frac{e^{-\frac{v_{n+1}^2}{2t}}}{\sqrt{2\pi t^3}}, \end{aligned}$$

satisfies similar relationships.

For $n=1$ the previous expression clearly gives the one-time iterated elastic Brownian motion studied above.

We give now the explicit Laplace transform of the distribution (3.27):

$$\int_0^{+\infty} e^{-\eta t} \Pr\left\{B_1^{el}(B_2^{el}(...(B_{n+1}^{el}(t))...)) \in dx\right\} dt \tag{3.28}$$

$$= dx 2^{n+1} e^{-\frac{x}{\gamma_1}} \int_x^{+\infty} v_1 e^{\frac{v_1}{\gamma_1}} dv_1 \int_0^{+\infty} \frac{e^{-\frac{v_1^2}{2s_1}}}{\sqrt{2\pi s_1^3}} ds_1 e^{-\frac{s_1}{\gamma_2}} \int_{s_1}^{+\infty} v_2 e^{\frac{v_2}{\gamma_2}} dv_2 \cdot$$

$$\cdot \int_0^{+\infty} \frac{e^{-\frac{v_2^2}{2s_2}}}{\sqrt{2\pi s_2^3}} ds_2 ... \int_0^{+\infty} e^{-\frac{s_n}{\gamma_{n+1}}} \frac{v_n e^{-\frac{v_n^2}{2s_n}}}{\sqrt{2\pi s_n^3}} ds_n \int_{s_n}^{+\infty} e^{\frac{v_{n+1}}{\gamma_{n+1}}} e^{-v_{n+1} 2^{\frac{1}{2}} \eta^{\frac{1}{2}}} dv_{n+1}$$

$$= dx 2^{n+1} e^{-\frac{x}{\gamma_1}} \int_x^{+\infty} v_1 e^{\frac{v_1}{\gamma_1}} dv_1 \int_0^{+\infty} \frac{e^{-\frac{v_1^2}{2s_1}}}{\sqrt{2\pi s_1^3}} e^{-\frac{s_1}{\gamma_2}} ds_1 \int_{s_1}^{+\infty} v_2 e^{\frac{v_2}{\gamma_2}} dv_2 \cdot$$

$$\cdot \int_0^{+\infty} \frac{e^{-\frac{v_2^2}{2s_2}}}{\sqrt{2\pi s_2^3}} ds_2 ... \int_0^{+\infty} \frac{e^{-s_n 2^{\frac{1}{2}} \eta^{\frac{1}{2}}}}{2^{\frac{1}{2}} \eta^{\frac{1}{2}} - \frac{1}{\gamma_{n+1}}} \frac{v_n e^{-\frac{v_n^2}{2s_n}}}{\sqrt{2\pi s_n^3}} ds_n$$

$$= \frac{dx 2^{n+1} e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}}{\prod_{j=1}^{n+1} \left(2^{1-\frac{1}{2^{n-j+2}}} \eta^{\frac{1}{2^{n-j+2}}} - \frac{1}{\gamma_j}\right)}.$$

If we let $\gamma_1, \gamma_2, ...\gamma_{n+1} \to \infty$ we get from (3.28) that

$$\lim_{\gamma_1,\gamma_2,...\gamma_{n+1}\to\infty} \int_0^{+\infty} e^{-\eta t} \Pr\left\{B_1^{el}(B_2^{el}(...(B_{n+1}^{el}(t))...)) \in dx\right\} dt$$

$$= \int_0^{+\infty} e^{-\eta t} \Pr\left\{|B_1(|B_2(...(|B_{n+1}(t)|)...)|)| \in dx\right\} dt \tag{3.29}$$

$$= \frac{dx 2^{n+1} e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}}{2^{n+1} \left(\frac{\eta}{2}\right)^{\sum_{j=1}^{n+1} \frac{1}{2^{n-j+2}}}}$$

$$= \frac{dx e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}}{\left(\frac{\eta}{2}\right)^{1-\frac{1}{2^{n+1}}}}.$$

If we let $n \to \infty$ in (3.29) we obtain that

$$\lim_{n\to\infty} \int_0^{+\infty} e^{-\eta t} \Pr\left\{|B_1(|B_2(...(|B_{n+1}(t)|)...)|)| \in dx\right\} dt$$

$$= dx \frac{2e^{-2x}}{\eta} = 2e^{-2x} dx \int_0^{+\infty} e^{-\eta t} dt.$$

We thus conclude that

$$\lim_{n\to\infty} \Pr\left\{|B_1(|B_2(...(|B_{n+1}(t)|)...)|)| \in dx\right\}$$

$$= 2e^{-2x} dx, \qquad x > 0$$

and this is in accordance with the results of Remark 3.4 of Orsingher and Beghin (2007).

In the same spirit of Theorem 3.4 we can prove the following result.

Theorem 3.5
For the n-times iterated elastic Brownian motion the following relationship holds, for $x>0$,

$$\begin{aligned} & \Pr\left\{B_1^{el}(B_2^{el}(...(B_{n+1}^{el}(t))...))\in dx\right\} \qquad (3.30)\\ = & \frac{1}{2\gamma_{n+1}^2}\int_0^t \Pr\left\{B_{n+1}^{el}(s)>0\right\}\Pr\left\{B_1^{el}(B_2^{el}(...(|B_{n+1}(t-s)|)...))\in dx\right\}. \end{aligned}$$

Proof We start with the Laplace transform

$$\begin{aligned} & \int_0^{+\infty} e^{-\eta t}\Pr\left\{B_1^{el}(B_2^{el}(...(|B_{n+1}(t)|)...))\in dx\right\}dt \qquad (3.31)\\ = & \; dx 2^{n+1}e^{-\frac{x}{\gamma_1}}\int_x^{+\infty} v_1 e^{\frac{v_1}{\gamma_1}}dv_1 \int_0^{+\infty}\frac{e^{-\frac{v_1^2}{2s_1}}}{\sqrt{2\pi s_1^3}}ds_1\cdot\\ & \cdot e^{-\frac{s_1}{\gamma_2}}\int_{s_1}^{+\infty} v_2 e^{\frac{v_2}{\gamma_2}}dv_2\int_0^{+\infty}\frac{e^{-\frac{v_2^2}{2s_2}}}{\sqrt{2\pi s_2^3}}ds_2 ... e^{-\frac{s_{n-1}}{\gamma_n}}\int_{s_{n-1}}^{+\infty} v_n e^{\frac{v_n}{\gamma_n}}dv_n\int_0^{+\infty}\frac{e^{-\frac{v_n^2}{2t}}e^{-\eta t}}{\sqrt{2\pi t}}dt\\ = & \frac{2^{n+1}}{\sqrt{2\eta}}\frac{dx e^{-2^{1-\frac{1}{2^{n+1}}}\eta^{\frac{1}{2^{n+1}}}x}}{\prod_{j=1}^n\left(2^{1-\frac{1}{2^{n-j+2}}}\eta^{\frac{1}{2^{n-j+2}}}-\frac{1}{\gamma_j}\right)}\\ = & \frac{2^{\frac12}\eta^{\frac12}-\frac{1}{\gamma_{n+1}}}{\sqrt{2\eta}}\int_0^{+\infty} e^{-\eta t}\Pr\left\{B_1^{el}(B_2^{el}(...(B_{n+1}^{el}(t))...))\in dx\right\}dt, \end{aligned}$$

where the last step is obtained by comparing (3.31) with (3.28). The conclusion follows from (3.31) if we note that

$$\frac{1}{1-\frac{1}{\gamma_{n+1}}\sqrt{2\eta}}$$

can be expanded as

$$g(t)=\frac{1}{t}E_{\frac12,0}\left(\frac{1}{\gamma_{n+1}}\sqrt{\frac{t}{2}}\right).$$

□

We are now interested in finding the boundary-value problem which is satisfied by the density (3.27) and we follow the same steps performed in Theorem 3.3.

Theorem 3.6
The transition density of the n-times iterated elastic Brownian motion, given in (3.27) solves the following Cauchy problem

$$\begin{cases} \frac{\partial^{\frac{1}{2^n}} u}{\partial t^{\frac{1}{2^n}}} = 2^{\frac{1}{2^n}-2} \frac{\partial^2 u}{\partial x^2}, & x, t > 0 \\ u(0,t) + \gamma_1 \left. \frac{\partial u(x,t)}{\partial x} \right|_{x=0^+} + f_n(t) = 0 & , \\ u(x,0) = \delta(x) \end{cases} \tag{3.32}$$

where $f_n(t)$ is a function with Laplace transform

$$\begin{aligned} & \int_0^{+\infty} e^{-\eta t} f_n(t) dt \qquad (3.33) \\ = \quad & \gamma_1 \left[2^{n+1} \prod_{j=2}^{n+1} \frac{1}{2^{1-\frac{1}{2^{n-j+2}}} \eta^{\frac{1}{2^{n-j+2}}} - \frac{1}{\gamma_j}} - 2^{2-\frac{1}{2^n}} \eta^{\frac{1}{2^n}-1} \right]. \end{aligned}$$

Proof The Laplace transform of (3.32) leads to

$$\begin{cases} \eta^{\frac{1}{2^n}} L(x,\eta) - \eta^{\frac{1}{2^n}-1} \delta(x) = 2^{\frac{1}{2^n}-2} \frac{\partial^2 L}{\partial x^2} \\ L(0,\eta) + \gamma_1 \left. \frac{\partial L(x,\eta)}{\partial x} \right|_{x=0^+} + \int_0^{+\infty} e^{-\eta t} f_n(t) dt = 0 \end{cases} \tag{3.34}$$

and, by considering the boundedness of the solution, we get

$$L(x,\eta) = B e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x}, \qquad x > 0.$$

By integrating the first equation in (3.34) on $[0, \varepsilon]$ and then letting $\varepsilon \to 0$

$$-\eta^{\frac{1}{2^n}-1} = 2^{\frac{1}{2^n}-2} \left[-B \eta^{\frac{1}{2^{n+1}}} 2^{1-\frac{1}{2^{n+1}}} - \left. \frac{\partial L}{\partial x} \right|_{x=0^+} \right] \tag{3.35}$$

we get

$$B + \gamma_1 \left[\eta^{\frac{1}{2^n}-1} 2^{2-\frac{1}{2^n}} - B \eta^{\frac{1}{2^{n+1}}} 2^{1-\frac{1}{2^{n+1}}} \right] + \int_0^{+\infty} e^{-\eta t} f_n(t) dt = 0,$$

which gives

$$L(x,\eta) = \frac{e^{-2^{1-\frac{1}{2^{n+1}}} \eta^{\frac{1}{2^{n+1}}} x} \left[\eta^{\frac{1}{2^n}-1} 2^{2-\frac{1}{2^n}} + \frac{1}{\gamma_1} \int_0^{+\infty} e^{-\eta t} f_n(t) dt \right]}{\eta^{\frac{1}{2^{n+1}}} 2^{1-\frac{1}{2^{n+1}}} - \frac{1}{\gamma_1}}. \tag{3.36}$$

By equating (3.36) to (3.28) we get

$$\begin{aligned} & 2^{2-\frac{1}{2^n}} \eta^{\frac{1}{2^n}-1} + \frac{1}{\gamma_1} \int_0^{+\infty} e^{-\eta t} f_n(t) dt \qquad (3.37) \\ = \quad & \frac{2^{n+1}}{\prod_{j=2}^{n+1} \left(2^{1-\frac{1}{2^{n-j+2}}} \eta^{\frac{1}{2^{n-j+2}}} - \frac{1}{\gamma_j} \right)} \end{aligned}$$

and (3.33). $\square$

Remark 3.3
The results of theorems 3.4 and 3.6 prove that the distribution of processes like $B_1^{el}(B_2^{el}(...))$, obtained by composing independent elastic Brownian motions are related to fractional differential equations with non-homogeneous elastic boundary conditions. On the other hand processes of the form $B_1^{el}(|I_{n-1}|)$ emerge when an homogeneous boundary condition is considered.

For $n = 1$ the previous result reduces to Theorem 3.3, since the function f_n appearing in (3.33) coincides with (3.17): indeed we can rewrite it as follows

$$\begin{aligned} \int_0^{+\infty} e^{-\eta t} f_1(t)dt &= \gamma_1 \left[\frac{2^2}{\sqrt{2}\left(\sqrt{\eta} - \frac{1}{\gamma_2\sqrt{2}}\right)} - \frac{2^{3/2}}{\sqrt{\eta}} \right] \\ &= \gamma_1 2^2 \left\{ \int_0^{+\infty} e^{-\eta t} \left[\frac{1}{\sqrt{2t}} E_{\frac{1}{2},\frac{1}{2}}\left(\frac{1}{\gamma_2}\sqrt{\frac{t}{2}} \right) dt - \frac{t^{-\frac{1}{2}}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)} \right] dt \right\}. \end{aligned}$$

Therefore we get

$$\begin{aligned} f_1(t) &= 2^2\gamma_1 \left[\frac{\gamma_2}{t} \sum_{k=0}^{\infty} \left(\frac{t}{2\gamma_2^2} \right)^{\frac{k+1}{2}} \frac{1}{\Gamma\left(\frac{k+1}{2}\right)} - \frac{\gamma_2 t^{-\frac{1}{2}}}{\gamma_2\sqrt{2}\Gamma\left(\frac{1}{2}\right)} \right] \\ &= \frac{2^2\gamma_1\gamma_2}{t} \sum_{k=1}^{\infty} \left(\frac{t}{2\gamma_2^2} \right)^{\frac{k+1}{2}} \frac{1}{\Gamma\left(\frac{k-1}{2}+1\right)} \\ &= \frac{2^2\gamma_1\gamma_2}{t} \sum_{r=0}^{\infty} \left(\frac{t}{2\gamma_2^2} \right)^{\frac{r}{2}+1} \frac{1}{\Gamma\left(\frac{r}{2}+1\right)} \\ &= \frac{2\gamma_1}{\gamma_2} E_{\frac{1}{2},1}\left(\frac{\sqrt{t}}{\sqrt{2}\gamma_2} \right), \end{aligned}$$

which coincides with $f(t)$ given in (3.17), obtained by a different approach.

For $n = 2$, the Laplace transform (3.33) can be still inverted as follows

$$\begin{aligned} &\int_0^{+\infty} e^{-\eta t} f_2(t)dt = \\ &= 2^3\gamma_1 \left[\frac{1}{2^{1-\frac{1}{2^2}}\left(\eta^{\frac{1}{4}} - \frac{1}{\gamma_2 2^{3/4}}\right) 2^{\frac{1}{2}}\left(\eta^{\frac{1}{2}} - \frac{1}{\gamma_3 2^{1/2}}\right)} - 2^{-1\frac{1}{2^2}}\eta^{\frac{1}{2^2}-1} \right] \\ &= 2^3\gamma_1 \{ \frac{1}{2^{\frac{5}{2^2}}} \int_0^{+\infty} e^{-\eta t} [\int_0^t s^{-\frac{3}{4}} E_{\frac{1}{4},\frac{1}{4}}\left(\frac{s^{1/4}}{\gamma_2 2^{3/4}} \right) (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\left(\frac{(t-s)^{1/2}}{\gamma_3 2^{1/2}} \right) ds + \\ &\quad -dt \int_0^t \frac{s^{-\frac{3}{4}}(t-s)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)} ds] dt \} \end{aligned}$$

so that we get an explicit form for $f_2(t)$ in terms of convolutions of two Mittag-Leffler functions

$$f_2(t) = 2^{3/4}\gamma_1 \int_0^t s^{-\frac{3}{4}}(t-s)^{-\frac{1}{2}} \left[E_{\frac{1}{4},\frac{1}{4}} \left(\frac{s^{1/4}}{\gamma_2 2^{3/4}} \right) E_{\frac{1}{2},\frac{1}{2}} \left(\frac{(t-s)^{1/2}}{\gamma_3 2^{1/2}} \right) - \frac{1}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)} \right] ds.$$

For an arbitrary value of n the form of $f_n(t)$ is much more complicated and can be expressed as convolutions of Mittag-Leffler functions of different order.

4 Fractional diffusion equations with drift and boundary conditions

In this section we add to the fractional diffusion equation (1.1) a term representing a drift of intensity μ. The presence of drift in time-fractional diffusion equations is the source of interesting extensions of the previous results. We thus examine fractional equations of the form

$$\frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial u}{\partial x}$$

either on the whole line, or on the half-lines $[0, +\infty)$, $(-\infty, a]$ and with different forms of boundary conditions.

The drift makes the consideration of elastic boundary conditions extremely hard and we restrict ourselves to the reflecting and absorbing barriers only.

The special case $\nu = \frac{1}{2^n}$ leads to processes of the following forms: $B^{\mu/\lambda}(|I_{n-1}(t)|)$, $\overline{B}^{\mu/\lambda}(|I_{n-1}(t)|)$ and $\overset{+}{B}{}^{\mu/\lambda}(|I_{n-1}(t)|)$, where $B^{\mu/\lambda}$, $\overline{B}^{\mu/\lambda}$ and $\overset{+}{B}{}^{\mu/\lambda}$ are, respectively, free, absorbing and reflecting Brownian motions, independent from the iterated Brownian motion I_{n-1} and endowed with drift μ/λ.

In all these cases, in the limit for $n \to \infty$, we obtain asymmetric distributions composed by combinations of exponentials where the dependence from t is cancelled.

We start with the case where only the initial condition is assumed and, for the sake of simplicity, we write $p(x,t)$ and $u_\nu(x,t)$ instead of $p(x,t;0,0)$ and $u_\nu(x,t;0,0)$, when the starting point is $x_0 = 0$.

Theorem 4.1

The following initial value problem, for $0 < \nu < 1$,

$$\begin{cases} \frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial u}{\partial x} \\ u(x,0) = \delta(x) \end{cases}, \qquad x, \mu \in \mathbb{R}, \quad t > 0 \tag{4.1}$$

is solved by

$$\begin{aligned} u_\nu^\mu(x,t) &= \frac{1}{\lambda t^\nu} \int_0^{+\infty} \frac{e^{-\frac{(x-\frac{\mu z}{\lambda})^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} W_{-\nu,1-\nu}(-\frac{z}{\lambda t^\nu}) dz \\ &= \int_0^{+\infty} p^\mu(x,z) \widehat{u}_{2\nu}(z,t) dz \end{aligned} \tag{4.2}$$

where $\widehat{u}_{2\nu}(z,t)$ is given in (2.5) and $u_{2\nu}(z,t)$ is the solution to (2.6). The kernel $p^{\mu/\lambda}(x,z)$ represents the transition density of a Brownian motion $B^{\mu/\lambda}$ with a drift of intensity $\frac{\mu}{\lambda}$ and an infinitesimal variance equal to 2λ, i.e. coincides with the solution to

$$\begin{cases} \frac{\partial p}{\partial t} = \lambda \frac{\partial^2 p}{\partial x^2} - \frac{\mu}{\lambda}\frac{\partial p}{\partial x} \\ p(x,0) = \delta(x) \end{cases}, \qquad x, \mu \in \mathbb{R}, \quad \lambda, t > 0. \tag{4.3}$$

Proof

By applying to (4.1) the Laplace transform and taking into account that for the Dzherbashyan-Caputo fractional derivative the following formula holds

$$\begin{aligned} &\int_0^{+\infty} e^{-\eta t} \frac{\partial^\nu u(x,t)}{\partial t^\nu} dt \\ = \; & \eta^\nu \int_0^{+\infty} e^{-\eta t} u(x,t) dt - \sum_{k=0}^{m-1} \eta^{\nu-1-k} \left. \frac{\partial^k u(x,t)}{\partial t^k} \right|_{t=0}, \end{aligned}$$

(where $m = \lfloor \nu \rfloor + 1$), we obtain that

$$\begin{aligned} &\int_0^{+\infty} e^{-\eta t} dt \int_{-\infty}^{+\infty} e^{i\beta x} u_\nu^\mu(x,t) dx \\ = \; & \frac{\eta^{\nu-1}}{\eta^\nu + \lambda^2\beta^2 - i\beta\mu}. \end{aligned} \tag{4.4}$$

By inverting the Laplace transform we get

$$\int_{-\infty}^{+\infty} e^{i\beta x} u_\nu^\mu(x,t) dx = E_{\nu,1}\left(-(\lambda^2\beta^2 - i\beta\mu)t^\nu\right) \tag{4.5}$$

so that the solution can be extracted by inverting the Fourier transform as follows:

$$\begin{aligned} & u_\nu^\mu(x,t) \\ = \; & \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\beta x} E_{\nu,1}\left(-(\lambda^2\beta^2 - i\beta\mu)t^\nu\right) d\beta \\ = \; & \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\beta x} \frac{d\beta}{2\pi i} \int_{Ha} \frac{e^z z^{\nu-1}}{z^\nu + t^\nu \left[\lambda^2\beta^2 - i\beta\mu\right]} dz \\ = \; & \frac{1}{2\pi i} \int_{Ha} e^z z^{\nu-1} \frac{dz}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\beta x}}{z^\nu + t^\nu \left[\lambda^2\beta^2 - i\beta\mu\right]} d\beta \\ = \; & \frac{1}{2\pi i} \int_{Ha} e^z z^{\nu-1} \frac{dz}{2\pi} \int_{-\infty}^{+\infty} d\beta \int_0^{+\infty} e^{-i\beta x - y\left[z^\nu + t^\nu\left(\lambda^2\beta^2 - i\beta\mu\right)\right]} dy \\ = \; & \int_0^{+\infty} dy \frac{1}{2\pi i} \int_{Ha} e^{-yz^\nu} e^z z^{\nu-1} \frac{e^{-\frac{(x-\mu t^\nu y)^2}{2(2t^\nu\lambda^2 y)}}}{\sqrt{2\pi(2t^\nu\lambda^2 y)}} dz \\ = \; & \int_0^{+\infty} \frac{e^{-\frac{(x-\mu t^\nu y)^2}{2(2t^\nu\lambda^2 y)}}}{\sqrt{2\pi(2t^\nu\lambda^2 y)}} W_{-\nu,1-\nu}(-y) dy, \end{aligned} \tag{4.6}$$

which gives (4.2), after the change of variable $z = t^\nu \lambda y$. $\square$

Remark 4.1

The previous result shows that $u_\nu^\mu(x,t)$ can be interpreted as the distribution of the process

$$\mathcal{T}_\nu^\mu(t) = B^{\mu/\lambda}\left(|\mathcal{T}_{2\nu}(t)|\right), \quad t>0 \tag{4.7}$$

where $\mathcal{T}_{2\nu}(t)$ is the process whose law is obtained by folding the solution to the fractional equation (2.6). The drift appears only in the distribution of the driving Brownian motion $B^{\mu/\lambda}$ and we note also that if the drift is absent we get $\mathcal{T}_\nu^0(t) = \mathcal{T}_\nu(t) = B\left(|\mathcal{T}_{2\nu}(t)|\right)$ (where $B = B^0$), as in Theorem 2.3 of Orsingher and Beghin (2007).

Remark 4.2

We can derive an alternative expression for the solution $u_\nu^\mu(x,t)$ where, with respect to (4.2), the roles of time and space are interchanged. We prove that the following relationship holds

$$u_\nu^\mu(x,t) = \int_0^{+\infty} u_{2\nu}^\mu(x,z)\frac{e^{-\frac{z2}{4t}}}{\sqrt{\pi t}}dz, \tag{4.8}$$

by evaluating the Fourier transform of the r.h.s. of (4.8)

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{i\beta x}\int_0^{+\infty} u_{2\nu}^\mu(x,z)\frac{e^{-\frac{z2}{4t}}}{\sqrt{\pi t}}dzdx \qquad (4.9)\\
=& \int_0^{+\infty} E_{2\nu,1}\left(-\left(\lambda^2\beta^2 - i\beta\mu\right)z^{2\nu}\right)\frac{e^{-\frac{z2}{4t}}}{\sqrt{\pi t}}dz\\
=& \sum_{k=0}^\infty \frac{\left(-\left(\lambda^2\beta^2-i\beta\mu\right)\right)^k}{\Gamma(2k\nu+1)\sqrt{\pi t}}\int_0^{+\infty} z^{2\nu k}e^{-\frac{z2}{4t}}dz
\end{aligned}$$

$$\begin{aligned}
=& \sum_{k=0}^\infty \frac{\left(-\left(\lambda^2\beta^2-i\beta\mu\right)\right)^k\sqrt{t}}{\Gamma(2k\nu+1)\sqrt{\pi t}}2^{2\nu k}t^{\nu k}\Gamma\left(k\nu+\frac{1}{2}\right)\\
=& \sum_{k=0}^\infty \frac{\left(-\left(\lambda^2\beta^2-i\beta\mu\right)t^\nu\right)^k}{2k\nu\Gamma(2k\nu)\sqrt{\pi}}2^{2\nu k}\sqrt{\pi}2^{1-2k\nu}\frac{\Gamma(2k\nu)}{\Gamma(k\nu)}\\
=& \sum_{k=0}^\infty \frac{\left(-\left(\lambda^2\beta^2-i\beta\mu\right)t^\nu\right)^k}{\Gamma(k\nu+1)} = \int_{-\infty}^{+\infty} e^{i\beta x}u_\nu^\mu(x,t)dx.
\end{aligned}$$

Formula (4.8) suggests that the solution to (4.1) can be interpreted as the distribution of the process

$$\mathcal{T}_\nu^\mu(t) = \mathcal{T}_{2\nu}^\mu\left(|B(t)|\right), \quad t>0, \tag{4.10}$$

which is an alternative form of (4.7)

Remark 4.3

We consider now the particular case where $\nu = \frac{1}{2^n}$ and $\lambda^2 = 2^{\frac{1}{2^n}-2}$ (we maintain λ in the distribution of the driving process, only for typographical reasons): by applying Theorem 4.1 together with Theorem 2.2 of Orsingher and Beghin (2007) we can show that $u^{\mu}_{\frac{1}{2^n}}(x,t)$ can be written as

$$\begin{aligned} u^{\mu}_{\frac{1}{2^n}}(x,t) &= \int_0^{+\infty} \frac{e^{-\frac{(x-\frac{\mu z}{\lambda})^2}{2(2\lambda z)}}}{\sqrt{\pi \lambda z}} \widehat{u}_{\frac{1}{2^{n-1}}}(z,t)dz \qquad (4.11) \\ &= 2^n \int_0^{+\infty} \frac{e^{-\frac{(x-\frac{\mu z}{\lambda})^2}{2(2\lambda z)}}}{\sqrt{\pi \lambda z}} \left\{ \int_0^{+\infty} \frac{e^{-\frac{z^2}{2w_1}}}{\sqrt{2\pi w_1}} dw_1 ... \int_0^{+\infty} \frac{e^{-\frac{w_{n-1}^2}{2t}}}{\sqrt{2\pi t}} dw_{n-1} \right\} dz \end{aligned}$$

and thus can be interpreted as the distribution of the process

$$\mathcal{T}^{\mu}_{\frac{1}{2^n}}(t) = B^{\mu/\lambda}\left(|I_{n-1}(t)|\right), \quad t > 0. \qquad (4.12)$$

In particular, for $n = 1$, the process (4.12) coincides with the iterated Brownian motion $B_1^{\mu/\lambda}\left(|B_2(t)|\right)$, where the driving process possesses drift equal to μ/λ.

Remark 4.4

In order to study the asymptotic behavior of the solution we specialize the Fourier transform (4.5) of $u^{\mu}_{\nu}(x,t)$ for $\nu = \frac{1}{2^n}$ and let $n \to \infty$, so that we get

$$\begin{aligned} &\lim_{n\to\infty} \int_{-\infty}^{+\infty} e^{i\beta x} u^{\mu}_{\frac{1}{2^n}}(x,t)dx \qquad (4.13) \\ &= \lim_{n\to\infty} E_{\frac{1}{2^n},1}\left(-(\lambda^2\beta^2 - i\beta\mu)t^{\frac{1}{2^n}}\right) \\ &= E_{0,1}\left(-(\lambda^2\beta^2 - i\beta\mu)\right) = \frac{1}{1+\lambda^2\beta^2 - i\beta\mu}. \end{aligned}$$

By taking the inverse Fourier transform of (4.13) we obtain that

$$\begin{aligned} \lim_{n\to\infty} u^{\mu}_{\frac{1}{2^n}}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-i\beta x}}{1+\frac{\mu^2}{4\lambda^2}+\left(\lambda\beta - \frac{i\mu}{2\lambda}\right)^2} d\beta \qquad (4.14) \\ &= \left[w = \lambda\beta - \frac{i\mu}{2\lambda} \right] \\ &= \frac{1}{2\pi\lambda}\int_{-\infty}^{+\infty} \frac{e^{-\frac{ix}{\lambda}\left[w+\frac{i\mu}{2\lambda}\right]}}{1+\frac{\mu^2}{4\lambda^2}+w^2} dw \\ &= \left[y = \frac{w}{\sqrt{1+\frac{\mu^2}{4\lambda^2}}} \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{\frac{\mu x}{2\lambda^2}}}{2\pi\lambda}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{1+\frac{\mu^2}{4\lambda^2}}}\frac{e^{-\frac{ixy}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}}}{1+y^2}dy \\
&= \frac{e^{\frac{\mu x}{2\lambda^2}}}{2\lambda\sqrt{1+\frac{\mu^2}{4\lambda^2}}}e^{-\frac{|x|}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}} = \frac{e^{\frac{1}{2\lambda^2}\left[\mu x-|x|\sqrt{4\lambda^2+\mu^2}\right]}}{\sqrt{4\lambda^2+\mu^2}} \\
&= \begin{cases} \frac{e^{-\frac{x}{2\lambda^2}\left[\sqrt{4\lambda^2+\mu^2}-\mu\right]}}{\sqrt{4\lambda^2+\mu^2}} & x>0 \\ \frac{e^{\frac{x}{2\lambda^2}\left[\sqrt{4\lambda^2+\mu^2}+\mu\right]}}{\sqrt{4\lambda^2+\mu^2}} & x<0 \end{cases}.
\end{aligned}
$$

Formula (4.14) coincides with the solution of the limiting equation

$$
u = \lambda^2\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial u}{\partial x}
$$

obtained from (4.1), for $\nu = \frac{1}{2^n}$, with $\lim_{n\to\infty}\frac{\partial^{1/2^n}u}{\partial t^{1/2^n}} = u$, as should be.

We recognize in (4.14) an asymmetric exponential random variable X (independent from t) and we evaluate below its moments. By denoting $A = \frac{\sqrt{4\lambda^2+\mu^2}-\mu}{2\lambda^2}$ and $B = \frac{\sqrt{4\lambda^2+\mu^2}+\mu}{2\lambda^2}$ we obtain the mean value as follows

$$
\begin{aligned}
EX &= \frac{1}{\sqrt{4\lambda^2+\mu^2}}\left[\int_0^{+\infty} xe^{-xA}dx + \int_{-\infty}^0 xe^{xB}dx\right] \\
&= \frac{1}{\sqrt{4\lambda^2+\mu^2}}\left[\frac{1}{A^2}-\frac{1}{B^2}\right] \\
&= \frac{1}{\sqrt{4\lambda^2+\mu^2}}\left[\left(\frac{2\lambda^2}{\sqrt{4\lambda^2+\mu^2}-\mu}\right)^2 - \left(\frac{2\lambda^2}{\sqrt{4\lambda^2+\mu^2}+\mu}\right)^2\right] \\
&= \frac{2^2\lambda^4}{\sqrt{4\lambda^2+\mu^2}}\left[\frac{\left(\sqrt{4\lambda^2+\mu^2}+\mu\right)^2}{(4\lambda^2)^2} - \frac{\left(\sqrt{4\lambda^2+\mu^2}-\mu\right)^2}{(4\lambda^2)^2}\right] \\
&= \frac{2^2\mu\sqrt{4\lambda^2+\mu^2}}{2^2\sqrt{4\lambda^2+\mu^2}} = \mu.
\end{aligned}
$$

Analogously, for the second moment we get

$$
\begin{aligned}
EX^2 &= \frac{1}{\sqrt{4\lambda^2+\mu^2}}\left[\int_0^{+\infty} x^2e^{-xA}dx + \int_{-\infty}^0 x^2e^{xB}dx\right] \\
&= \frac{1}{\sqrt{4\lambda^2+\mu^2}}\left[\int_0^{+\infty} x^2e^{-xA}dx + \int_0^{+\infty} x^2e^{-xB}dx\right] \\
&= \frac{2}{\sqrt{4\lambda^2+\mu^2}}\left[\frac{1}{A^3}+\frac{1}{B^3}\right]
\end{aligned}
$$

$$= \frac{2}{\sqrt{4\lambda^2+\mu^2}}\left[\left(\frac{2\lambda^2}{\sqrt{4\lambda^2+\mu^2}-\mu}\right)^3+\left(\frac{2\lambda^2}{\sqrt{4\lambda^2+\mu^2}+\mu}\right)^3\right]$$

$$= \frac{2^4\lambda^6}{\sqrt{4\lambda^2+\mu^2}}\left[\frac{\left(\sqrt{4\lambda^2+\mu^2}+\mu\right)^3}{(4\lambda^2)^3}+\frac{\left(\sqrt{4\lambda^2+\mu^2}-\mu\right)^3}{(4\lambda^2)^3}\right]$$

$$= \frac{1}{2^2\sqrt{4\lambda^2+\mu^2}}\left[2(\sqrt{4\lambda^2+\mu^2})^3+3\cdot 2\mu^2\sqrt{4\lambda^2+\mu^2}\right]$$

$$= \frac{1}{2}\left[4\lambda^2+\mu^2+3\mu^2\right]=2(\lambda^2+\mu^2).$$

The variance of the asymmetric exponential random variable is therefore equal to $VarX = 2\lambda^2+\mu^2$. We can arrive at the same conclusion by taking the derivatives of the function (4.13) for $\beta = 0$.

We consider now the time-fractional diffusion equation with drift μ, subject to reflecting boundary conditions. We will study two different formulations of this problem: in the first one we impose the following condition

$$\mu u(0,t) = \lambda^2 \frac{\partial u}{\partial x}\bigg|_{x=0^+}, \tag{4.15}$$

which reveals the presence of a reflecting barrier in the origin.

Moreover we assume that our motion starts now from a point $x_0 > 0$, so that we modify the initial condition as follows

$$u(x,0) = \delta(x-x_0), \qquad x_0 > 0. \tag{4.16}$$

Theorem 4.2

The solution to the following Cauchy problem, for $0<\nu<1$,

$$\begin{cases} \frac{\partial^\nu u}{\partial t^\nu} = \lambda^2\frac{\partial^2 u}{\partial x^2}-\mu\frac{\partial u}{\partial x} \\ \mu u(0,t) = \lambda^2\frac{\partial u}{\partial x}\big|_{x=0^+} \\ u(x,0)=\delta(x-x_0) \end{cases}, \qquad x,\mu\in\mathbb{R}, \quad t,x_0>0 \tag{4.17}$$

coincides with

$$\overset{+\mu}{u}_\nu(x,t;x_0,0) \tag{4.18}$$

$$= \frac{1}{\lambda t^\nu}\int_0^{+\infty} W_{-\nu,1-\nu}\left(-\frac{z}{\lambda t^\nu}\right)\left\{\frac{e^{-\frac{(x-x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}+e^{-\frac{x_0\mu}{\lambda^2}}\frac{e^{-\frac{(x+x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}+\right.$$

$$\left.-\frac{\mu}{\lambda^2}\int_{-\infty}^{-x_0} e^{\frac{\mu w}{\lambda^2}}\frac{e^{-\frac{\left(x-w-\frac{\mu}{\lambda}z\right)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}dw\right\}dz$$

$$= \int_0^{+\infty} \overset{+\mu/\lambda}{p}(x,z;x_0,0)\widehat{u}_{2\nu}(z,t)dz$$

where $\widehat{u}_{2\nu}$ is defined in (2.5).
Proof We first give the general solution to (4.17), obtained by means of the method of separation of variables. For $u_\nu(x,t) = X(t)T(t)$, we have the ordinary differential equations

$$\frac{\partial^\nu T}{\partial t^\nu} = -\beta^2 T \tag{4.19}$$

and

$$\lambda^2 X'' - \mu X' + \beta^2 X = 0, \tag{4.20}$$

whose solutions are

$$\begin{cases} X(x) = e^{\frac{x\mu}{2\lambda^2}} \left(A e^{\frac{x}{2\lambda^2}\sqrt{\mu^2-4\lambda^2\beta^2}} + B e^{-\frac{x}{2\lambda^2}\sqrt{\mu^2-4\lambda^2\beta^2}} \right) \\ T(t) = C E_{\nu,1}(-\beta^2 t^\nu) \end{cases}.$$

We can therefore write the general solution as

$$\begin{aligned} & u_\nu(x,t) && (4.21) \\ = \; & e^{\frac{x\mu}{2\lambda^2}} \left\{ \int_{-\infty}^{+\infty} E_{\nu,1}(-\beta^2 t^\nu) \left[A e^{\frac{x}{2\lambda^2}\sqrt{\mu^2-4\lambda^2\beta^2}} + B e^{-\frac{x}{2\lambda^2}\sqrt{\mu^2-4\lambda^2\beta^2}} \right] d\beta \right\} \\ = \; & \left[\gamma^2 = 4\lambda^2\beta^2 - \mu^2 \right] \\ = \; & e^{\frac{x\mu}{2\lambda^2}} \int_{-\infty}^{+\infty} E_{\nu,1}(\left(-\frac{t^\nu}{4\lambda^2} \left(\mu^2 + \gamma^2 \right) \right) \left[A(\gamma) e^{\frac{i\gamma x}{2\lambda^2}} + B(\gamma) e^{-\frac{i\gamma x}{2\lambda^2}} \right] d\gamma. \end{aligned}$$

By imposing the conditions in (4.17) we get

$$\begin{aligned} & \overset{+\mu}{u}_\nu(x,t;x_0,0) && (4.22) \\ = \; & \frac{e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)} \int_{-\infty}^{+\infty} E_{\nu,1}(-\frac{t^\nu}{4\lambda^2} \left(\mu^2 + \beta^2 \right)) \left\{ e^{\frac{i\beta}{2\lambda^2}(x-x_0)} + e^{-\frac{i\beta}{2\lambda^2}(x+x_0)} - \frac{2\mu e^{-\frac{i\beta}{2\lambda^2}(x+x_0)}}{i\beta+\mu} \right\} d\beta, \end{aligned}$$

since the unknown constants must be in this case

$$B = \frac{i\beta - \mu}{i\beta + \mu} A$$

and

$$A = \frac{e^{-\frac{i\beta x_0}{2\lambda^2}} e^{-\frac{\mu x_0}{2\lambda^2}}}{2\pi(2\lambda^2)}.$$

The third term in (4.22) can be developed as follows

$$
\begin{aligned}
& -\frac{2\mu e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)}\int_{-\infty}^{+\infty}E_{\nu,1}(-\frac{t^\nu}{4\lambda^2}\left(\mu^2+\beta^2\right))\frac{e^{-\frac{i\beta}{2\lambda^2}(x+x_0)}}{i\beta+\mu}d\beta \\
= & -\frac{2\mu e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)}\int_{-\infty}^{+\infty}\frac{e^{-\frac{i\beta}{2\lambda^2}(x+x_0)}d\beta}{i(\beta-i\mu)}\frac{1}{2\pi i}\int_{Ha}e^z z^{\nu-1}dz\int_0^{+\infty}e^{-y\left[z^\nu+\frac{t^\nu}{4\lambda^2}\left(\mu^2+\beta^2\right)\right]}dy \\
= & [\beta-i\mu=v] \\
= & -\frac{2\mu e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)}\int_0^{+\infty}e^{-\frac{yt^\nu\mu^2}{4\lambda^2}}dy\frac{1}{2\pi i}\int_{Ha}e^z z^{\nu-1}e^{-yz^\nu}dz\int_{-\infty}^{+\infty}\frac{e^{-\frac{yt^\nu}{4\lambda^2}(i\mu+v)^2}}{iv}e^{-\frac{i(x+x_0)}{2\lambda^2}(i\mu+v)}dv \\
= & -\frac{2\mu e^{\frac{\mu(x-x_0)+\mu(x+x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)}\int_0^{+\infty}\frac{dy}{2\pi i}\int_{Ha}e^z z^{\nu-1}e^{-yz^\nu}dz\int_{-\infty}^{+\infty}\frac{e^{-\frac{yt^\nu}{4\lambda^2}\left(v^2+2\mu vi\right)}}{iv}e^{-\frac{iv(x+x_0)}{2\lambda^2}}dv
\end{aligned}
$$

$$
\begin{aligned}
= & -\frac{2\mu e^{\frac{\mu x}{\lambda^2}}}{2\pi(2\lambda^2)}\int_0^{+\infty}\frac{dy}{2\pi i}\int_{Ha}e^z z^{\nu-1}e^{-yz^\nu}dz\int_{-\infty}^{+\infty}\frac{e^{-\frac{iv(x+x_0)}{2\lambda^2}}}{iv}dv\,\cdot \\
& \cdot\int_{-\infty}^{+\infty}e^{ivw}\frac{e^{-\left(w+\frac{\mu yt^\nu}{2\lambda^2}\right)^2\frac{2\lambda^2}{2yt^\nu}}}{\sqrt{2\pi\frac{t^\nu y}{2\lambda^2}}}dw \\
= & -\frac{\mu e^{\frac{\mu x}{\lambda^2}}}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{-\infty}^{+\infty}\frac{e^{-\left(w+\frac{\mu yt^\nu}{2\lambda^2}\right)^2\frac{2\lambda^2}{2yt^\nu}}}{\sqrt{2\pi\frac{t^\nu y}{2\lambda^2}}}\frac{dw}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ivw-iv\frac{x+x_0}{2\lambda^2}}}{iv}dv \\
= & -\frac{\mu e^{\frac{\mu x}{\lambda^2}}}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{-\infty}^{+\infty}\frac{e^{-\left(w+\frac{\mu yt^\nu}{2\lambda^2}\right)^2\frac{2\lambda^2}{2yt^\nu}}}{\sqrt{2\pi\frac{t^\nu y}{2\lambda^2}}}H_{\frac{x+x_0}{2\lambda^2}}(w)dw
\end{aligned}
$$

$$
\begin{aligned}
= & -\frac{\mu e^{\frac{\mu x}{\lambda^2}}}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{\frac{x+x_0}{2\lambda^2}}^{+\infty}\frac{e^{-\left(w+\frac{\mu yt^\nu}{2\lambda^2}\right)^2\frac{2\lambda^2}{2yt^\nu}}}{\sqrt{2\pi\frac{t^\nu y}{2\lambda^2}}}dw \\
= & \left[2\lambda^2 w=x-w'\right] \\
= & -\frac{\mu e^{\frac{\mu x}{\lambda^2}}}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{-\infty}^{-x_0}\frac{e^{-\frac{\left(x-w+\mu yt^\nu\right)^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}dw \\
= & -\frac{\mu}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{-\infty}^{-x_0}e^{\frac{\mu w}{\lambda^2}}\frac{e^{-\frac{\left(x-w-\mu yt^\nu\right)^2}{2(2\lambda^2t^\nu y)}}}{\sqrt{2\pi(2\lambda^2t^\nu y)}}dw.
\end{aligned}
$$

As far as the first term in (4.22) is concerned, we can write it as

$$
\begin{aligned}
& \frac{e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\pi(2\lambda^2)} \int_{-\infty}^{+\infty} e^{\frac{i\beta}{2\lambda^2}(x-x_0)} d\beta \frac{1}{2\pi i} \int_{Ha} e^z z^{\nu-1} dz \int_0^{+\infty} e^{-y\left[z^\nu + \frac{t^\nu}{4\lambda^2}\left(\mu^2+\beta^2\right)\right]} dy \\
= & \frac{e^{\frac{\mu(x-x_0)}{2\lambda^2}}}{2\lambda^2} \int_0^{+\infty} e^{-\frac{yt^\nu \mu^2}{4\lambda^2}} \frac{dy}{2\pi i} \int_{Ha} e^z z^{\nu-1} e^{-yz^\nu} dz \int_{-\infty}^{+\infty} \frac{e^{\frac{i\beta}{2\lambda^2}(x-x_0)}}{2\pi} e^{-\frac{yt^\nu}{4\lambda^2}\beta^2} d\beta \\
= & e^{\frac{\mu(x-x_0)}{2\lambda^2}} \int_0^{+\infty} e^{-\frac{yt^\nu \mu^2}{4\lambda^2}} W_{-\nu,1-\nu}(-y) dy \int_{-\infty}^{+\infty} \frac{e^{i\beta(x-x_0)}}{2\pi} e^{-\frac{y2\lambda^2 t^\nu}{2}\beta^2} d\beta \\
= & e^{\frac{\mu(x-x_0)}{2\lambda^2}} \int_0^{+\infty} e^{-\frac{yt^\nu \mu^2}{4\lambda^2}} \frac{e^{-\frac{(x-x_0)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}} W_{-\nu,1-\nu}(-y) dy \\
= & \int_0^{+\infty} \frac{e^{-\frac{(x-x_0-\mu t^\nu y)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}} W_{-\nu,1-\nu}(-y) dy.
\end{aligned}
$$

Analogously the second term becomes

$$
e^{-\frac{x_0\mu}{\lambda^2}} \int_0^{+\infty} \frac{e^{-\frac{(x+x_0-\mu t^\nu y)^2}{2(2\lambda^2 t^\nu y)}}}{\sqrt{2\pi(2\lambda^2 t^\nu y)}} W_{-\nu,1-\nu}(-y) dy,
$$

so that the solution coincides with (4.18), after the change of variable $y = z/\lambda t^\nu$.
$\square$

Remark 4.5
Let us check that (4.18) integrates to one, with respect to x: it is sufficient to calculate the integral below

$$
\begin{aligned}
& \int_0^{+\infty} \overset{+}{p}{}^{\mu/\lambda}(x, z; x_0, 0) dx \qquad (4.23)\\
= & \int_0^{+\infty} \frac{e^{-\frac{(x-x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} dx + e^{-\frac{x_0\mu}{\lambda^2}} \int_0^{+\infty} \frac{e^{-\frac{(x+x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} dx + \\
& -\frac{\mu}{\lambda^2} \int_0^{+\infty} dx \int_{-\infty}^{-x_0} e^{\frac{\mu w}{\lambda^2}} \frac{e^{-\frac{(x-w-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi 2(2\lambda z)}} dw \\
= & \int_{-\frac{x_0+\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy + e^{-\frac{x_0\mu}{\lambda^2}} \int_{\frac{x_0-\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy + \\
& -\frac{\mu}{\lambda^2} \int_{-\infty}^{-x_0} e^{\frac{\mu w}{\lambda^2}} dw \int_{-\frac{w+\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\end{aligned}
$$

The last integral in (4.23) can be rewritten, by inverting the order of integration, as follows

$$
\begin{aligned}
& -\frac{\mu}{\lambda^2} \int_{\frac{x_0-\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \int_{-\sqrt{2\lambda z}y-\frac{\mu}{\lambda}z}^{-x_0} e^{\frac{\mu w}{\lambda^2}} dw \qquad (4.24) \\
= \quad & -e^{\frac{\mu x_0}{\lambda^2}} \int_{\frac{x_0-\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy + \int_{\frac{x_0-\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}-\frac{\mu}{\lambda^2}\left(\sqrt{2\lambda z}y+\frac{\mu}{\lambda}z\right)}}{\sqrt{2\pi}} dy \\
= \quad & -e^{\frac{\mu x_0}{\lambda^2}} \int_{\frac{x_0-\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy + \int_{\frac{x_0+\frac{\mu}{\lambda}z}{\sqrt{2\lambda z}}}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy,
\end{aligned}
$$

where, in the last step, the following change of variable

$$y' = y + \frac{2\mu z}{\lambda} \frac{1}{\sqrt{2\lambda z}}$$

has been introduced. By putting pieces together we have that (4.23) and thus (4.18) both integrate to one.

The solution (4.18) is thus a proper probability density and coincides with the distribution of the process $\overset{+}{B}{}^{\mu/\lambda}\left(|\mathcal{T}_{2\nu}(t)|\right)$, which for $\nu = \frac{1}{2^n}$ becomes $\overset{+}{B}{}^{\mu/\lambda}\left(|I_{n-1}(t)|\right)$. It can also be expressed in terms of the solution of equation (4.1) (without boundary conditions) as follows

$$
\begin{aligned}
& \overset{+}{u}{}^{\mu}_{\nu}(x,t;x_0,0) \qquad (4.25) \\
= \quad & u^{\mu}_{\nu}(x,t;x_0,0) + e^{-\frac{x_0\mu}{\lambda^2}} u^{\mu}_{\nu}(x,t;-x_0,0) + \\
& -\frac{\mu}{\lambda^2} \int_{-\infty}^{-x_0} e^{\frac{\mu w}{\lambda^2}} u^{\mu}_{\nu}(x,t;w,0).
\end{aligned}
$$

Formula (4.25) shows that the solution of the boundary-value problem envisaged in Theorem 4.2 can be constructed by superposing solutions of equation (4.1) emanating from the sources at $x = x_0$ (with unit intensity), at $x = -x_0$ (with intensity $e^{-\frac{x_0\mu}{\lambda^2}}$) and a continuum of negative sources of exponentially decaying intensity placed on the half-line $(-\infty, -x_0)$.

For $\nu = 1$, formula (4.25) represents the distribution of a reflecting Brownian motion with drift expressed in terms of the transition function of a free Brownian motion.

Remark 4.6

In the case where the equation (4.1) is subject to the absorbing condition $u(0,t) = 0$, it can be proved with little effort that the solution to the corresponding boundary-value problem reads

$$
\begin{aligned}
&\overline{u}^{\mu}_{\nu}(x,t;x_0,0) && (4.26)\\
= \;& \frac{1}{\lambda t^{\nu}}\int_0^{+\infty} W_{-\nu,1-\nu}\left(-\frac{z}{\lambda t^{\nu}}\right)\left\{\frac{e^{-\frac{(x-x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} - e^{-\frac{x_0\mu}{\lambda^2}}\frac{e^{-\frac{(x+x_0-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}\right\}dz\\
= \;& \int_0^{+\infty}\overline{p}^{\mu/\lambda}(x,z;x_0,0)\widehat{u}_{2\nu}(z,t)dz.
\end{aligned}
$$

The second boundary-value problem with a reflecting barrier that we will consider is expressed by the following condition

$$
\left.\frac{\partial u}{\partial x}\right|_{x=a} - \mu u(a,t) = 0,
$$

which means that a reflecting barrier is placed in a. We also assume that the motion starts from zero.

Theorem 4.3

The solution to the following problem, for $0 < \nu < 1$,

$$
\begin{cases}
\frac{\partial^{\nu} u}{\partial t^{\nu}} = \lambda^2\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial u}{\partial x} \\
u(x,0) = \delta(x) \\
\left.\frac{\partial u}{\partial x}\right|_{x=a} - \mu u(a,t) = 0
\end{cases}, \qquad x < a, \quad t > 0, \tag{4.27}
$$

is given by

$$
\begin{aligned}
&\overset{+}{u}{}^{\mu,a}_{\nu}(x,t) && (4.28)\\
= \;& \int_0^{+\infty}\left[\frac{e^{-\frac{(x-\frac{\mu z}{\lambda})^2}{4\lambda z}}}{\sqrt{4\pi\lambda z}} + e^{\frac{a\mu}{\lambda^2}}\frac{e^{-\frac{(x-2a-\frac{\mu z}{\lambda})^2}{4\lambda z}}}{\sqrt{4\pi\lambda z}}\right]\widehat{u}_{2\nu}(z,t)dz + \\
& + \frac{\mu}{\lambda^2}\int_0^{+\infty}\left[\int_{2a}^{+\infty} e^{\frac{\mu}{\lambda^2}(v-a)}\frac{e^{-(x-v-y\frac{\mu}{\lambda})^2\frac{1}{4\lambda y}}}{\sqrt{4\pi y\lambda}}dv\right]\widehat{u}_{2\nu}(y,t)dy,
\end{aligned}
$$

where $\widehat{u}_{2\nu}(y,t)$ coincides with (2.5).

Proof The general solution to equation (4.27) can be written down as in (4.21) and then

$$
\begin{aligned}
&\frac{\partial u}{\partial x} && (4.29)\\
= \;& \frac{\mu e^{\frac{\mu x}{2\lambda^2}}}{2\lambda^2}\int_{-\infty}^{+\infty} E_{\nu,1}\left(-\frac{t^{\nu}}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\left[A(\gamma)e^{\frac{i\gamma x}{2\lambda^2}} + B(\gamma)e^{-\frac{i\gamma x}{2\lambda^2}}\right]d\gamma\\
& + e^{\frac{\mu x}{2\lambda^2}}\int_{-\infty}^{+\infty} E_{\nu,1}\left(-\frac{t^{\nu}}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\left[\frac{i\gamma}{2\lambda^2}A(\gamma)e^{\frac{i\gamma x}{2\lambda^2}} - \frac{i\gamma}{2\lambda^2}B(\gamma)e^{-\frac{i\gamma x}{2\lambda^2}}\right]d\gamma\\
= \;& \frac{e^{\frac{\mu x}{2\lambda^2}}}{2\lambda^2}\int_{-\infty}^{+\infty} E_{\nu,1}\left(-\frac{t^{\nu}}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\left[\frac{A(\gamma)}{2}(\mu+i\gamma)e^{\frac{i\gamma x}{2\lambda^2}} + \frac{B(\gamma)}{2}(\mu-i\gamma)e^{-\frac{i\gamma x}{2\lambda^2}}\right]d\gamma.
\end{aligned}
$$

Therefore, by applying the conditions in (4.27), we get the following relationship between the unknown constants

$$\left[\frac{1}{2}\left(\mu+i\gamma\right)-\mu\right]Ae^{\frac{i\gamma}{2\lambda^2}a}=-\left[\frac{1}{2}\left(\mu-i\gamma\right)-\mu\right]Be^{-\frac{i\gamma}{2\lambda^2}a}$$

$$B=\frac{-\mu+i\gamma}{\mu+i\gamma}e^{\frac{i\gamma}{\lambda^2}a}A$$

$$A=\frac{1}{2\pi}\frac{1}{2\lambda^2}.$$

We insert the previous expression into (4.29) and then put $\gamma'=\frac{\gamma+i\mu}{2\lambda^2}$ so that we get

$$\begin{aligned}
&\overset{+}{u}{}_{\nu}^{\mu,a}(x,t) && (4.30)\\
=\;& \frac{e^{\frac{\mu x}{2\lambda^2}}}{4\pi\lambda^2}\int_{-\infty}^{+\infty}E_{\nu,1}\left(-\frac{t^\nu}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\left[e^{i\gamma x}+e^{-\frac{i\gamma(x-2a)}{2\lambda^2}}-\frac{2\mu}{\mu+i\gamma}e^{-\frac{i\gamma(x-2a)}{2\lambda^2}}\right]d\gamma\\
=\;& \int_0^{+\infty}\left[\frac{e^{-\frac{(x-\frac{\mu z}{\lambda})^2}{4\lambda z}}}{\sqrt{4\pi\lambda z}}+e^{\frac{a\mu}{\lambda^2}}\frac{e^{-\frac{(x-2a-\frac{\mu z}{\lambda})^2}{4\lambda z}}}{\sqrt{4\pi\lambda z}}\right]\widehat{u}_{2\nu}(z,t)dz+\\
&-\frac{2\mu e^{\frac{\mu x}{2\lambda^2}}}{4\pi\lambda^2}\int_{-\infty}^{+\infty}E_{\nu,1}\left(-\frac{t^\nu}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\frac{d\gamma}{\mu+i\gamma}e^{-\frac{i\gamma(x-2a)}{2\lambda^2}},
\end{aligned}$$

where, for the first term we have applied (4.6).

The last line in (4.30) can be rewritten as

$$\begin{aligned}
&-\frac{2\mu e^{\frac{\mu x}{2\lambda^2}}}{4\pi\lambda^2}\int_{-\infty}^{+\infty}\frac{d\gamma}{\mu+i\gamma}e^{\frac{i\gamma(2a-x)}{2\lambda^2}}\left[\frac{1}{2\pi i}\int_{Ha}\frac{e^z z^{\nu-1}}{z^\nu+\frac{t^\nu}{4\lambda^2}\left(\mu^2+\gamma^2\right)}dz\right] && (4.31)\\
=\;& -\frac{2\mu e^{\frac{\mu x}{2\lambda^2}}}{4\pi\lambda^2}\int_0^{+\infty}dy\int_{-\infty}^{+\infty}\frac{d\gamma}{\mu+i\gamma}e^{\frac{i\gamma(2a-x)}{2\lambda^2}}\frac{1}{2\pi i}\int_{Ha}e^z z^{\nu-1}e^{-y\left[z^\nu+\frac{t^\nu}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right]}dz\\
=\;& \int_0^{+\infty}\frac{e^{-\frac{yt^\nu\mu^2}{4\lambda^2}}}{2\pi i}dy\int_{Ha}e^z z^{\nu-1}e^{-yz^\nu}dz\int_{-\infty}^{+\infty}\left(-\frac{\mu e^{\frac{\mu x}{2\lambda^2}}}{\lambda^2}\right)\frac{1}{2\pi}\frac{d\gamma}{\mu+i\gamma}e^{\frac{i\gamma(2a-x)}{2\lambda^2}}e^{-\frac{yt^\nu\gamma^2}{4\lambda^2}}.
\end{aligned}$$

The last integral in (4.31) can be evaluated as follows

$$\begin{aligned}
&-\frac{\mu e^{\frac{\mu x}{2\lambda^2}}}{\lambda^2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{\frac{i\gamma(2a-x)}{2\lambda^2}}}{i(\gamma-i\mu)}e^{-\frac{yt^\nu\gamma^2}{4\lambda^2}}d\gamma && (4.32)\\
=\;& [\gamma-i\mu=w]\\
=\;& -\frac{\mu e^{\frac{\mu x}{2\lambda^2}}}{\lambda^2}\frac{e^{\frac{\mu(x-2a)}{2\lambda^2}}}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iw\frac{(2a-x)}{2\lambda^2}}}{iw}e^{-\frac{yt^\nu(i\mu+w)^2}{4\lambda^2}}dw\\
=\;& -\frac{\mu}{\lambda^2}\frac{e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iw\frac{(2a-x)}{2\lambda^2}-iw\frac{yt^\nu\mu}{2\lambda^2}}}{iw}e^{-\frac{yt^\nu w^2}{4\lambda^2}}dw
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu}{\lambda^2}\frac{e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iw\frac{(2a-x)}{2\lambda^2}}}{iw}dw\int_{-\infty}^{+\infty}e^{iwz}\frac{e^{-(z+\frac{yt^\nu\mu}{2\lambda^2})^2\frac{\lambda^2}{yt^\nu}}}{\sqrt{2\pi\frac{yt^\nu}{2\lambda^2}}}dz\\
&= -\frac{\mu}{\lambda^2}\frac{e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-(z+\frac{yt^\nu\mu}{2\lambda^2})^2\frac{\lambda^2}{yt^\nu}}}{\sqrt{2\pi\frac{yt^\nu}{2\lambda^2}}}dz\int_{-\infty}^{+\infty}e^{iwz}\frac{e^{iw\frac{(2a-x)}{2\lambda^2}}}{iw}dw\\
&= \frac{\mu}{\lambda^2}e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{-(z+\frac{yt^\nu\mu}{2\lambda^2})^2\frac{\lambda^2}{yt^\nu}}}{\sqrt{2\pi\frac{yt^\nu}{2\lambda^2}}}H_{\frac{2a}{2\lambda^2}}\left(\frac{x}{2\lambda^2}-z\right)dz\\
&= \left[v=\frac{x}{2\lambda^2}-z\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mu}{\lambda^2}e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{-(\frac{x}{2\lambda^2}-v+\frac{yt^\nu\mu}{2\lambda^2})^2\frac{\lambda^2}{yt^\nu}}}{\sqrt{2\pi\frac{yt^\nu}{2\lambda^2}}}H_{\frac{2a}{2\lambda^2}}(v)\,dv\\
&= \frac{\mu}{\lambda^2}e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}\int_{\frac{2a}{2\lambda^2}}^{+\infty}\frac{e^{-(\frac{x}{2\lambda^2}-v+\frac{yt^\nu\mu}{2\lambda^2})^2\frac{\lambda^2}{yt^\nu}}}{\sqrt{2\pi\frac{yt^\nu}{2\lambda^2}}}dv\\
&= \frac{\mu}{\lambda^2}e^{\frac{2\mu(x-a)}{2\lambda^2}+\frac{yt^\nu\mu^2}{4\lambda^2}}\int_{2a}^{+\infty}\frac{e^{-(x-v+yt^\nu\mu)^2\frac{1}{4\lambda^2yt^\nu}}}{\sqrt{4\pi yt^\nu\lambda^2}}dv\\
&= \frac{\mu}{\lambda^2}e^{\frac{yt^\nu\mu^2}{4\lambda^2}}\int_{2a}^{+\infty}\frac{e^{-(x-v-yt^\nu\mu)^2\frac{1}{4\lambda^2yt^\nu}}}{\sqrt{4\pi yt^\nu\lambda^2}}e^{\frac{\mu}{\lambda^2}(v-a)}dv.
\end{aligned}
$$

In the last step we have carried out the following manipulations:

$$
\begin{aligned}
& (x-v+yt^\nu\mu)^2-4\mu(x-a)t^\nu y-y^2t^{2\nu}\mu^2\\
=\;& (x-v-yt^\nu\mu+2yt^\nu\mu)^2-4\mu(x-a)t^\nu y-y^2t^{2\nu}\mu^2\\
=\;& (x-v-yt^\nu\mu)^2+4yt^\nu\mu(x-v-yt^\nu\mu)-4\mu(x-a)t^\nu y+3y^2t^{2\nu}\mu^2\\
=\;& (x-v-yt^\nu\mu)^2+4yt^\nu\mu(-v+a)-y^2t^{2\nu}\mu^2.
\end{aligned}
$$

By inserting (4.32) into (4.31) we get

$$
\begin{aligned}
& \frac{\mu}{\lambda^2}\int_0^{+\infty}\frac{dy}{2\pi i}\int_{Ha} e^z z^{\nu-1}e^{-yz^\nu}dz\int_{2a}^{+\infty}\frac{e^{-(x-v-yt^\nu\mu)^2\frac{1}{4\lambda^2 yt^\nu}}}{\sqrt{4\pi yt^\nu\lambda^2}}e^{\frac{\mu}{\lambda^2}(v-a)}dv \\
= & \frac{\mu}{\lambda^2}\int_0^{+\infty}W_{-\nu,1-\nu}(-y)dy\int_{2a}^{+\infty}\frac{e^{-(x-v-yt^\nu\mu)^2\frac{1}{4\lambda^2 yt^\nu}}}{\sqrt{4\pi yt^\nu\lambda^2}}e^{\frac{\mu}{\lambda^2}(v-a)}dv \qquad (4.33) \\
= & \frac{\mu}{\lambda^2}\int_{2a}^{+\infty}e^{\frac{\mu}{\lambda^2}(v-a)}dv\int_0^{+\infty}W_{-\nu,1-\nu}(-y)\frac{e^{-(x-v-yt^\nu\mu)^2\frac{1}{4\lambda^2 yt^\nu}}}{\sqrt{4\pi yt^\nu\lambda^2}}dy \\
= & \frac{\mu}{\lambda^2}\int_{2a}^{+\infty}e^{\frac{\mu}{\lambda^2}(v-a)}dv\int_0^{+\infty}\frac{1}{\lambda t^\nu}W_{-\nu,1-\nu}(-\frac{y'}{\lambda t^\nu})\frac{e^{-(x-v-y'\frac{\mu}{\lambda})^2\frac{1}{4\lambda y'}}}{\sqrt{4\pi y'\lambda}}dy' \\
= & \frac{\mu}{\lambda^2}\int_{2a}^{+\infty}e^{\frac{\mu}{\lambda^2}(v-a)}dv\int_0^{+\infty}\widehat{u}_{2\nu}(y,t)\frac{e^{-(x-v-y\frac{\mu}{\lambda})^2\frac{1}{4\lambda y}}}{\sqrt{4\pi y\lambda}}dy.
\end{aligned}
$$

Finally we substitute the last line of (4.30) with (4.33) and we obtain (4.28). □

Remark 4.7

The comments following Theorem 4.2 can easily be adapted to the solutions of diffusion equation (4.27) with reflecting boundary conditions at $x = a$ and starting point $x = 0$.

Finally let us consider the case where, together with the presence of a drift, an absorbing barrier is assumed in $a > 0$.

Theorem 4.4

The solution to the following boundary-initial value problem, for $a > 0$ and $0 < \nu < 1$,

$$
\begin{cases} \frac{\partial^\nu u}{\partial t^\nu} = \lambda^2\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial u}{\partial x} \\ u(x,0) = \delta(x) \\ u(a,t) = 0 \end{cases}, \qquad x < a, \quad t > 0, \qquad (4.34)
$$

is given by

$$
\begin{aligned}
& \overline{u}_\nu^{\mu,a}(x,t) \qquad (4.35) \\
= & \int_0^{+\infty}\left[\frac{e^{-\frac{(x-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} - e^{\frac{a\mu}{\lambda^2}}\frac{e^{-\frac{(x-2a-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}\right]\widehat{u}_{2\nu}(z,t)dz.
\end{aligned}
$$

Proof

The application of the boundary and initial conditions to (4.21) leads to

$$
B(\gamma) = -A(\gamma)e^{\frac{x\gamma a}{\lambda^2}} \qquad A(\gamma) = \frac{1}{2\pi(2\lambda^2)}.
$$

Thus the solution to (4.34) reads

$$
\begin{aligned}
&\overline{u}_\nu^{\mu,a}(x,t) \qquad (4.36)\\
= \;& \frac{e^{\frac{x\mu}{2\lambda^2}}}{4\pi\lambda^2}\left\{\int_{-\infty}^{+\infty} E_{\nu,1}\left(-\frac{t^\nu}{4\lambda^2}\left(\mu^2+\gamma^2\right)\right)\left[e^{\frac{i\gamma x}{2\lambda^2}} - e^{-\frac{i\gamma(x-2a)}{2\lambda^2}}\right]d\gamma\right\}\\
= \;& \frac{e^{\frac{x\mu}{2\lambda^2}}}{4\pi\lambda^2}\int_{-\infty}^{+\infty}\left[e^{\frac{i\gamma x}{2\lambda^2}} - e^{-\frac{i\gamma(x-2a)}{2\lambda^2}}\right]\frac{d\gamma}{2\pi i}\int_{Ha} e^z z^{\nu-1}dz\int_0^{+\infty} e^{-y(z^\nu+\frac{t^\nu}{4\lambda^2}(\mu^2+\gamma^2))}dy\\
= \;& \frac{e^{\frac{x\mu}{2\lambda^2}}}{2\lambda^2}\int_0^{+\infty} e^{-y\frac{t^\nu}{4\lambda^2}\mu^2}\frac{dy}{2\pi i}\int_{Ha} e^{z-yz^\nu}z^{\nu-1}2\lambda^2\left[\frac{e^{-\frac{x^2}{2(2yt^\nu\lambda^2)}}}{\sqrt{2\pi(2yt^\nu\lambda^2)}} - \frac{e^{-\frac{(x-2a)^2}{2(2yt^\nu\lambda^2)}}}{\sqrt{2\pi(2yt^\nu\lambda^2)}}\right]dz\\
= \;& \int_0^{+\infty} e^{-y\frac{t^\nu\mu^2}{4\lambda^2}}\left[\frac{e^{-\frac{(x-\mu yt^\nu)^2}{2(2yt^\nu\lambda^2)}}}{\sqrt{2\pi(2yt^\nu\lambda^2)}} - e^{\frac{a\mu}{\lambda^2}}\frac{e^{-\frac{(x-2a-\mu yt^\nu)^2}{2(2yt^\nu\lambda^2)}}}{\sqrt{2\pi(2yt^\nu\lambda^2)}}\right]e^{\frac{\mu^2t^\nu y}{4\lambda^2}}\frac{dy}{2\pi i}\int_{Ha} e^{z-yz^\nu}z^{\nu-1}dz\\
= \;& \left[z = y\lambda^2t^\nu\right]\\
= \;& \frac{1}{\lambda t^\nu}\int_0^{+\infty}\left[\frac{e^{-\frac{(x-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}} - e^{\frac{a\mu}{\lambda^2}}\frac{e^{-\frac{(x-2a-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{2\pi(2\lambda z)}}\right]W_{-\nu,1-\nu}(-\frac{z}{\lambda t^\nu})dz,
\end{aligned}
$$

for $x < a$, which coincides with (4.35). □

Remark 4.8

Formula (4.35) suggests the following interpretation for the process governed by (4.34):

$$\mathcal{T}_\nu^{a,\mu}(t) = \overline{B}^{a,\mu/\lambda}\left(|\mathcal{T}_{2\nu}(t)|\right), \quad t > 0, \tag{4.37}$$

where $\overline{B}^{a,\mu/\lambda}$ denotes a Brownian motion with drift of intensity μ/λ and an absorbing barrier. We note that result (4.37) is analogue to (4.7).

The survival probability of the process (4.37) is equal to $1-e^{a\mu/\lambda^2}$ for $\mu < 0$, as can be ascertained from (4.35), by means of the transformation $w = \frac{x-\mu yt^\nu}{\sqrt{2yt^\nu\lambda^2}}$ and letting $t \to +\infty$.

Remark 4.9

If we assume that $\nu = \frac{1}{2^n}$ and $\lambda^2 = 2^{\frac{1}{2^n}-2}$, from (4.35) we get

$$\begin{aligned} & \overline{u}^{a,\mu}_{\frac{1}{2^n}}(x,t) \qquad (4.38) \\ = & \int_0^{+\infty} \left[\frac{e^{-\frac{(x-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{4\pi\lambda z}} - e^{\frac{a\mu}{\lambda^2}} \frac{e^{-\frac{(x-2a-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{4\pi\lambda z}} \right] \widehat{u}_{\frac{1}{2^{n-1}}}(z,t)dz \\ = & 2^n \int_0^{+\infty} \frac{\left[e^{-\frac{(x-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}} - e^{\frac{a\mu}{\lambda^2}-\frac{(x-2a-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}} \right]}{\sqrt{4\pi\lambda z}} \cdot \\ & \cdot \left\{ \int_0^{+\infty} \frac{e^{-\frac{z^2}{2w_1}}}{\sqrt{2\pi w_1}} dw_1 ... \int_0^{+\infty} \frac{e^{-\frac{w_{n-1}^2}{2t}}}{\sqrt{2\pi t}} dw_{n-1} \right\} dz, \end{aligned}$$

which represents the counterpart of (4.11), when an absorbing barrier at $x = a$ is considered (we have left λ in the density of the driving process, only for typographical reasons). Therefore in this case the process can be expressed as

$$\mathcal{T}^{a,\mu}_{\frac{1}{2^n}}(t) = \overline{B}^{a,\mu/\lambda}\left(|I_{n-1}(t)|\right), \quad t > 0. \qquad (4.39)$$

Remark 4.10 We study the limiting behavior of (4.35) for $\nu = \frac{1}{2^n}$ and $n \to \infty$, for arbitrary values of λ. By considering the last line of (4.36), we obtain the following asymptotic distribution

$$\begin{aligned} & \lim_{n\to\infty} \overline{u}^{a,\mu}_{\frac{1}{2^n}}(x,t) \qquad (4.40) \\ = & \frac{1}{\lambda} \int_0^{+\infty} \left[\frac{e^{-\frac{(x-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{4\pi\lambda z}} - e^{\frac{a\mu}{\lambda^2}} \frac{e^{-\frac{(x-2a-\frac{\mu}{\lambda}z)^2}{2(2\lambda z)}}}{\sqrt{4\pi\lambda z}} \right] e^{-\frac{z}{\lambda}} dz \\ = & \frac{e^{\frac{\mu x}{2\lambda^2}}}{\lambda} \int_0^{+\infty} \frac{e^{-\frac{x^2}{4\lambda z}-\frac{\mu^2 z}{4\lambda^3}-\frac{z}{\lambda}}}{\sqrt{4\pi\lambda z}} dz - e^{\frac{a\mu}{\lambda^2}} \frac{e^{\frac{\mu(x-2a)}{2\lambda^2}}}{\lambda} \int_0^{+\infty} \frac{e^{-\frac{(x-2a)^2}{4\lambda z}-\frac{\mu^2 z}{4\lambda^3}-\frac{z}{\lambda}}}{\sqrt{4\pi\lambda z}} dz \\ = & \frac{e^{\frac{\mu x}{2\lambda^2}}}{\lambda} \frac{e^{-\frac{|x|}{\sqrt{2\lambda}}\sqrt{\frac{2}{\lambda}\left(1+\frac{\mu^2}{4\lambda^2}\right)}}}{2\sqrt{1+\frac{\mu^2}{4\lambda^2}}} - e^{\frac{a\mu}{\lambda^2}} \frac{e^{\frac{\mu(x-2a)}{2\lambda^2}}}{\lambda} \frac{e^{-\frac{|x-2a|}{\sqrt{2\lambda}}\sqrt{\frac{2}{\lambda}\left(1+\frac{\mu^2}{4\lambda^2}\right)}}}{2\sqrt{1+\frac{\mu^2}{4\lambda^2}}} \\ = & \frac{e^{\frac{\mu x}{2\lambda^2}}}{\sqrt{4\lambda^2+\mu^2}} \left[e^{-\frac{|x|}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}} - e^{-\frac{|x-2a|}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}} \right], \end{aligned}$$

for $x < a$.

We integrate the previous expression in $(-\infty, a)$, for $a > 0$, and we get

$$\int_{-\infty}^{0} \frac{e^{x\left(\frac{\mu}{2\lambda^2}+\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)}}{\sqrt{4\lambda^2+\mu^2}} dx + \int_{0}^{a} \frac{e^{x\left(\frac{\mu}{2\lambda^2}-\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)}}{\sqrt{4\lambda^2+\mu^2}} dx + \tag{4.41}$$

$$-\frac{e^{-\frac{2a}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}}}{\sqrt{4\lambda^2+\mu^2}} \int_{-\infty}^{a} e^{x\left(\frac{\mu}{2\lambda^2}+\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)} dx$$

$$\begin{aligned} = \quad & \frac{1}{\sqrt{4\lambda^2+\mu^2}} \frac{1}{\frac{\mu}{2\lambda^2}-\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}} \left[e^{a\left(\frac{\mu}{2\lambda^2}-\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)} - 1 \right] + \\ & + \frac{1}{\sqrt{4\lambda^2+\mu^2}} \frac{1}{\frac{\mu}{2\lambda^2}+\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}} \left[1 - e^{a\left(\frac{\mu}{2\lambda^2}-\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)} \right] \\ = \quad & 1 - e^{a\left(\frac{\mu}{2\lambda^2}-\frac{1}{\lambda}\sqrt{1+\frac{\mu^2}{4\lambda^2}}\right)}. \end{aligned}$$

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