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On ABCs (and Ds) of VAR representations of DSGE models

MASSIMO FRANCHI AND PAOLO PARUOLO

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Abstract. This paper presents a necessary and sufficient condition for non-invertibility of DSGE models, i.e. for the impossibility of recovering the structural shocks of a DSGE via a VAR. We contrast this condition with the so-called poor man's invertibility condition in Fernández-Villaverde et al. (2007), which is, in general, only a sufficient condition for invertibility. Situations when the poor man's invertibility condition becomes equivalent to the present condition (and hence also necessary) are discussed. The permanent income model is used to illustrate results in the paper.

1. Introduction

Economic shocks of Dynamic Stochastic General Equilibrium (DSGE) models cannot always be recovered from Vector AutoRegressions (VAR). This situation has been discussed e.g. in Chari et al. (2005), Christiano et al. (2006), Kapetanios et al. (2007), Ravenna (2007), and it is related to the non-fundamentalness of economic models, see Hansen and Sargent (1980, 1991), Lippi and Reichlin (1993, 1994) for early treatments of the problem.

In this context Fernández-Villaverde et al. (2007) have proposed a condition for non-invertibility of DSGE models, called the 'poor man's invertibility condition'. This condition is applied e.g. in Leeper et al. (2009), Schmitt-Grohé (2010), Kurmann and Otrok (2011), Sims (2012) to specific models. Fernández-Villaverde et al. (2007) show that if the poor man's invertibility condition holds then the model is invertible, i.e. that the condition is sufficient. They also show that if the poor man's invertibility condition does not hold and some additional conditions are satisfied, then the model is non-invertible. Hence the poor

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man's invertibility condition is in general only a sufficient condition for DSGE models to be invertible.

The permanent income model provides an example for which the set of additional conditions does not hold, the poor man's invertibility condition is violated and the model is invertible; this example motivates our analysis and it is used throughout the paper to illustrate various statements.

A novel necessary and sufficient condition for invertibility is formulated and its relation with the condition in Fernández-Villaverde et al. (2007) is discussed. We also propose a new strategy for checking fundamentalness. In the last section of the paper, we show that under the same set of additional conditions the new condition coincides with the poor man's invertibility condition. All proofs are deferred to the Appendix.

2. Model generalities: the square case

Following Fernández-Villaverde et al. (2007) we consider an equilibrium of an economic model with representation

$$\begin{aligned} x_{t+1} &= Ax_t + Bw_{t+1} \\ y_{t+1} &= Cx_t + Dw_{t+1} \end{aligned} \tag{1}$$

where w_{t+1} is a Gaussian white noise with identity covariance matrix, $u_t = x_t, y_t, w_t$ have dimension $n_u \times 1$, $n_w = n_y$ and D is non-singular. This is called the square case.

It is of interest to characterize situations in which y_{t+1} admits representation

$$y_{t+1} = \sum_{j=1}^{\infty} A_j y_{t+1-j} + G w_{t+1}, \tag{2}$$

where the sequence $\{A_j\}_{j=1}^{\infty}$ is square summable and G is a non-singular matrix.[1] In this case the economic model in (1) has the property that its structural shocks w_{t+1} can be recovered from the reduced form errors of the infinite order VAR representation of y_{t+1}. When (2) holds, (1) is called invertible (or fundamental), see Hansen and Sargent (1980), Lippi and Reichlin (1993, 1994).

[1]Ravenna (2007) studies the finite order VAR case, see also Franchi and Paruolo (2012).

Writing the second equation in (1) as $w_{t+1} = D^{-1}(y_{t+1} - Cx_t)$ and substituting it in the first equation, Fernández-Villaverde et al. (2007) obtain the equivalent formulation

$$\begin{aligned} x_{t+1} &= Fx_t + BD^{-1}y_{t+1}, \qquad F := A - BD^{-1}C, \\ y_{t+1} &= Cx_t + Dw_{t+1}. \end{aligned} \tag{3}$$

They call the condition of stability[2] of F the 'poor man's invertibility condition'.

We observe that (3) implies the transfer function

$$T(L)y_{t+1} = Dw_{t+1}, \qquad T(z) := I_{n_y} - C(I_{n_x} - Fz)^{-1}BD^{-1}z, \qquad z \in \mathbb{C}; \tag{4}$$

this leads to the following statement.

Proposition 2.1. *If $T(z)$ is regular,[3] then (1) is invertible.*

3. Motivating examples

The examples in this section illustrate that the poor man's invertibility condition is in general not necessary for the recovery of structural shocks from the VAR. This motivates the rest of the analysis.

3.1. **Permanent income model.**

Consider the permanent income model

$$\begin{aligned} c_{t+1} &= c_t + \sigma_w(1 - R^{-1})w_{t+1} \\ \widetilde{y}_{t+1} &= \sigma_w w_{t+1} \end{aligned}$$

where c_{t+1} is consumption, $\widetilde{y}_{t+1}$ is labor income and $R > 1$ is the gross interest rate. If one lets $x_{t+1} = c_{t+1}$ and $y_{t+1} = \widetilde{y}_{t+1}$, then $A = 1$, $B = \sigma_w(1 - R^{-1})$, $C = 0$, and $D = \sigma_w$ imply $F = A = 1$, so that the poor man's invertibility condition does not hold. However, the model is fundamental because (2) is satisfied with $A_j = 0$, $j \geq 1$, and $G = \sigma_w$. This illustrates that a violation of poor man's invertibility condition does not necessarily imply non-invertibility of the economic model, i.e. that the poor man's invertibility condition is not a check for the non-invertibility of (1).

[2]A square matrix is called stable when all its eigenvalues are stable, i.e. of modulus strictly less than one; if it is has unstable eigenvalues then it is called unstable.

[3]A function $M(z)$ is called regular if it is finite in the unit disc, i.e. for all $z \in \mathbb{C}$ such that $|z| < 1 + \delta$, for some $\delta > 0$; observe that if $M(z)$ is regular, then the sequence $\{M_j\}_{j=0}^{\infty}$ in $M(z) = \sum_{j=0}^{\infty} M_j z^j$ is square summable.

3.2. **A second motivating example.** Let $n_x = 2, n_y = n_w = 1$ and take

$$A = \frac{1}{10}\begin{pmatrix} 9 & -3 \\ -3 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad C = \frac{1}{10}\begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad D = 1;$$

then

$$F = A - BC = \frac{2}{5}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

has eigenvalues $\{\frac{2}{5}, \frac{6}{5}\}$ and $\lambda := \frac{6}{5}$ is unstable. Observe that one can rank-decompose $F - \lambda I$ as $F - \lambda I = \alpha\beta'$, with $\alpha = B$, $\beta = -\frac{2}{5}\alpha$, and one can choose $\alpha_\perp = \beta_\perp = (1 : -1)'$ as bases of the orthogonal complements of $\operatorname{col}\alpha$ and $\operatorname{col}\beta$;[4] then Theorem 3 in Johansen (2009) implies

$$(I - Fz)^{-1} = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp\frac{1}{(z - \lambda^{-1})} + H(z),$$

where $H(z) = \sum_{j=0}^{\infty} H_j z^j$ is regular. Because $B = \alpha$ one has $\alpha'_\perp B = 0$ and thus $(I - Fz)^{-1}B = H(z)B$ is regular. Then $T(z)$ in (4) is regular and

$$y_{t+1} = C\sum_{j=0}^{\infty} H_j B y_{t-j} + w_{t+1}$$

is the infinite order VAR representation of y_{t+1} whose reduced form errors are the structural shocks. Again here, the poor man's invertibility condition does not hold and the model is invertible.

4. A check for non-invertibility

In this section we provide a necessary and sufficient condition for invertibility of (1). This is presented in Proposition 4.2. In Proposition 4.3 we further show that if F is unstable and (1) is invertible then any unstable eigenvalue of F is also an eigenvalue of A; the converse does not hold. Finally, the case in which F has simple unstable eigenvalues is discussed in Corollary 4.5.

The properties of the transfer function $T(z)$ in (4) depend on those of $(I - Fz)^{-1} = \operatorname{adj}(I - Fz)/|I - Fz|$. The roots of $|I - Fz| = 0$ are poles of $(I - Fz)^{-1}$; because $|I - Fz| = 0$ if and only if $z = \lambda_u^{-1}$, where $\{\lambda_u\}$ are the eigenvalues of F, if F is stable then $(I - Fz)^{-1}$ is regular because $\min_u |\lambda_u^{-1}| > 1$. Hence $T(z)$ is regular and, by Proposition 2.1, this implies that (1) is invertible. However, the examples in Section 3 show that the converse does not

[4] $\operatorname{col}\alpha$ indicates the column range space of the matrix α.

hold. This leads to the following proposition, which states that the poor man's invertibility condition is sufficient for invertibility.

Proposition 4.1. *If F is stable, then* (1) *is invertible; the converse does not hold.*

Observe that $T(z)$ can be regular, and thus (1) is invertible, even if F is unstable. In fact, when F is unstable, $(I - Fz)^{-1}$ has poles in the (closed) unit disc and thus it is non-regular, but those singularities may be absent from $C(I - Fz)^{-1}B$ due to the presence of B and C. In this case $C(I - Fz)^{-1}B$ is regular and thus the same holds for $T(z)$; by Proposition 2.1, this implies that (1) is invertible. The condition in Proposition 4.2 below builds on this observation.

We first introduce notation: let λ_u, $u = 1, \dots, q$, be all the distinct, unstable eigenvalues of F, $|\lambda_u| \geq 1$. Next apply the partial fraction expansion[5]

(5)

$$(I-Fz)^{-1} = P(z)+H(z), \qquad P(z) = \sum_{u=1}^{q} P_u(z), \qquad P_u(z) = \sum_{j=1}^{m_u} \frac{P_{\lambda_u, m_u - j}}{(z - \lambda_u^{-1})^j}, \qquad P_{\lambda_u,0} \neq 0.$$

Here $H(z)$ is regular, $P(z)$ is the sum of the principal parts $P_u(z)$ of $(I - Fz)^{-1}$ at $z = \lambda_u^{-1}$ and m_u is the order of the pole of $(I - Fz)^{-1}$ at $z = \lambda_u^{-1}$. Note that if F is stable, $q = 0$ implies $P(z) = 0$ and hence $(I - Fz)^{-1} = H(z)$ is regular. We are now able to state the main characterization result.

Proposition 4.2. *Let λ_u, $u = 1, \dots, q$, be all the distinct, unstable eigenvalues of F, $|\lambda_u| \geq 1$; then* (1) *is invertible if and only if*

$$CP_u(z)B = 0, \qquad u = 1, \dots, q, \tag{6}$$

where $P_u(z)$ is the principal part of $(I - Fz)^{-1}$ at $z = \lambda_u^{-1}$, see (5). *When F is stable,* (6) *is automatically satisfied.*

The next proposition shows that if F is unstable and (2) holds, then all the unstable eigenvalues of F are common to A.

Proposition 4.3. *If* (1) *is invertible, then each unstable eigenvalue of F is an eigenvalue of A.*

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[5]See e.g. Fischer and Lieb (2012, Ch. III.2).

The converse does not hold; that is, if the unstable eigenvalues of F are eigenvalues of A it does not follow that (1) is invertible.

The previous results suggest the following procedure as a check for invertibility of (1).

Remark 4.4. *In order to check whether* (1) *is invertible or not, proceed as follows: compute the eigenvalues of F; if F is stable, conclude that* (1) *is invertible* (*by Proposition* 4.1)*. If F is unstable, compute the eigenvalues of A; if there is an unstable eigenvalue of F which is not an eigenvalue of A, conclude that* (1) *is non-invertible* (*by Proposition* 4.3)*. If each unstable eigenvalue of F is an eigenvalue of A, check the condition in Proposition* 4.2*; if it is satified, conclude that* (1) *is invertible, otherwise that it is non-invertible.*

Of course one could simply check (6) directly.

For simple unstable eigenvalues, the condition in Proposition 4.2 simplifies as follows.

Corollary 4.5. *If λ_u is a simple eigenvalue of F, $|\lambda_u| \geq 1$; then $CP_u(z)B = 0$ in* (6) *is equivalent to*

$$C\beta_{u,\perp}(\alpha'_{u,\perp}\beta_{u,\perp})^{-1}\alpha'_{u,\perp}B = 0,$$

where α_u, β_u are defined by the rank factorization $F - \lambda_u I = \alpha_u\beta'_u$ and $\alpha_{u,\perp}, \beta_{u,\perp}$ are bases of the orthogonal complements of $\operatorname{col}\alpha$ *and* $\operatorname{col}\beta$*.*

We illustrate the procedure in Remark 4.4 on the permanent income model reported in Section 3 and on the version used in Fernández-Villaverde et al. (2007); in the latter they let $x_{t+1} = c_{t+1}$, $y_{t+1} = \widetilde{y}_{t+1} - c_{t+1}$ and hence they have $A = 1$, $B = \sigma_w(1 - R^{-1})$, $C = -1$, $D = \sigma_w R^{-1}$, and $F = R > 1$.

Both versions satisfy the assumptions of Corollary 4.5. If one lets $y_{t+1} = \widetilde{y}_{t+1} - c_{t+1}$, then $F = R > 1$ is a simple unstable eigenvalue of F; because $A = 1$, one concludes that the model is non-invertible. If one lets $y_{t+1} = \widetilde{y}_{t+1}$, then $F = A = 1$ and one finds $\alpha = \beta = 0$, $\alpha_\perp = \beta_\perp = 1$. Because $CB = 0$, the condition in Corollary 4.5 applies and hence one concludes that (1) is invertible.

5. When the poor man's invertibility condition provides a check for non-invertibility

In this section we show that if (1) is stabilizable and detectable[6] then the condition in Proposition 4.2 coincides with the poor man's invertibility condition. Remark that these are the conditions used in Fernández-Villaverde et al. (2007, Sec. C) to ensure the asymptotic stability and time invariance of the Kalman filter, see e.g. Anderson and Moore (1979, Sec. 4.4) and Lancaster and Rodman (1995, Ch. 17). This result is given in Proposition 5.2. In Corollaries 5.3 and 5.4 we present two direct consequences of it when (1) is stable, or controllable and observable. We conclude this section by discussing the domains of applicability of the two conditions.

The following definition[7] is based on the characterization results in Lancaster and Rodman (1995, Theorems 4.3.3 and 4.5.6).

Definition 5.1. *The economic model* (1) *is called stabilizable if* $\mathrm{rank}(A - \lambda I : B) = n_x$ *for all* $|\lambda| \geq 1$*; if this condition holds for all* $\lambda \in \mathbb{C}$*, model* (1) *is called controllable. The economic model in* (1) *is called detectable if* $\mathrm{rank}(A' - \lambda I : C') = n_x$ *for all* $|\lambda| \geq 1$*; if this condition holds for all* $\lambda \in \mathbb{C}$*, model* (1) *is called observable.*

Note that a controllable system is necessarily stabilizable, but not viceversa; hence stabilizability is a weaker concept than controllability. The same relation holds between the notions of detectability and observability. The next proposition shows that when (1) is stabilizable and detectable, the condition in Proposition 4.2 coincides with the poor man's invertibility condition.

Proposition 5.2. *Assume that* (1) *is stabilizable and detectable; then it is invertible if and only if* F *is stable.*

A first direct consequence of this proposition follows from the fact that if A is stable then (1) is stabilizable and detectable.

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[6]Recall that the pair (A, B) is called stabilizable if there exists K such that $A + BK$ is stable, and that the pair (C, A) is called detectable if the pair (A', C') is stabilizable, see e.g. Lancaster and Rodman (1995, Ch. 4).

[7]Recall that the pair (A, B) is called controllable if $\mathrm{rank}(B : AB : A^2B : \cdots : A^{n_x-1}B) = n_x$ and the pair (C, A) is called observable if the pair (A', C') is controllable, see e.g. Lancaster and Rodman (1995, Ch. 4).

Corollary 5.3. *If A is stable, then* (1) *is invertible if and only if F is stable.*

Similarly, if (1) is controllable and observable, then it is stabilizable and detectable; hence the following statement.

Corollary 5.4. *If* (1) *is controllable and observable, then it is invertible if and only if F is stable.*

Proposition 5.2 shows that if the economic model is stabilizable and detectable, then the poor man's invertibility condition provides a check for non-invertibility. Because stability of A implies stabilizability and detectability of (1), Corollary 5.3 shows that the poor man's invertibility condition provides a check for non-invertibility also in this case. Similarly, it is also a valid check when (1) is controllable and observable, as stated in Corollary 5.4.

Conversely, if (1) is either non-stabilizable and/or non-detectable, then the poor man's invertibility condition is not necessary for invertibility and thus it cannot be used to check for non-invertibility. Unlike for the poor man's invertibility condition, the condition in Proposition 4.2 applies to any square case, irrespectively of its stability and/or detectability. When (1) is stabilizable and detectable, the two conditions coincide.

We illustrate these facts via the permanent income model: if one lets $y_{t+1} = \widetilde{y}_{t+1} - c_{t+1}$, the model is controllable and observable; in fact $\lambda = A = 1$, $B = \sigma_w(1 - R^{-1})$ and $C = -1$ imply $\text{rank}(0 : \sigma_w(1 - R^{-1})) = \text{rank}(0 : -1) = 1$. In this case the two conditions agree. If one lets $y_{t+1} = \widetilde{y}_{t+1}$, then $\lambda = A = 1$, $B = \sigma_w(1 - R^{-1})$ and $C = 0$ imply $\text{rank}(0 : \sigma_w(1 - R^{-1})) = 1$ and $\text{rank}(0 : 0) = 0$; hence the model is controllable but not detectable and one cannot use the poor man's invertibility condition. The second counterexample is neither stabilizable nor detectable and hence the poor man's invertibility condition cannot be applied as a check for fundamentalness. In any of the three cases, one can proceed as described in Remark 4.4, or simply check condition (6).

6. Conclusions

In the present paper we have illustrated that the poor man's invertibility condition in Fernández-Villaverde et al. (2007) is only a sufficient but not necessary condition for invertibility. A violation of this condition does not necessarily imply non-invertibility of the DSGE, unless additional conditions hold. The condition presented in Section 4 is shown to provide a check (i.e. a necessary and sufficient condition) for non-invertibility in any square case. When the economic model is stabilizable and detectable, the two conditions coincide.

Appendix A. Proofs

Proof of Proposition 2.1. Regularity of $T(z)$ implies that the VAR coefficients are square summable, see footnote 4. ∎

Proof of Proposition 4.1. If F is stable, then $(I - Fz)^{-1}$ is regular and so is $T(z)$; then apply Proposition 2.1. ∎

Proof of Proposition 4.2. Consider (4) and (5); if $CP_u(z)B = 0$, $u = 1, \ldots, q$, then $C(I - Fz)^{-1}B = CH(z)B$ is regular and hence the same holds for $T(z)$, and Proposition 2.1 applies. Conversely, assume $T(z)$ is regular; then the same holds for $C(I - Fz)^{-1}B$ and thus $CP_u(z)B = 0$, $u = 1, \ldots, q$. The last statement follows from the equivalence of F stable and $q = 0$. ∎

Proof of Proposition 4.3. Assume (1) is invertible; then, see Proposition 4.2, one has $CP_u(z)B = 0$ for $u = 1, \ldots, q$. By (5), this is equivalent to $CP_{\lambda_u, m_u - j}B = 0$ for $j = 1, \ldots, m_u$ and $u = 1, \ldots, q$; hence in particular $CP_{\lambda_u,0}B = 0$, where $P_{\lambda_u,0} \neq 0$. Write $I - Fz = (I - F\lambda_u^{-1}) - F(z - \lambda_u^{-1})$ and (5) as $(I - Fz)^{-1} = P_u(z) + P_{-u}(z)$, where $P_{-u}(z) = \sum_{v=1, v\neq u}^{q} P_v(z) + H(z)$; then $(I - Fz)(I - Fz)^{-1} = I$ implies

$$(I - F\lambda_u^{-1})P_u(z) + (I - F\lambda_u^{-1})P_{-u}(z) - (z - \lambda_u^{-1})F(I - Fz)^{-1} = I. \tag{7}$$

Substituting $P_u(z)$ from (5) one has

$$(I - F\lambda_u^{-1})P_u(z) = \frac{(I - F\lambda_u^{-1})P_{\lambda_u,0}}{(z - \lambda_u^{-1})^{m_u}} + \sum_{j=1}^{m_u - 1} \frac{(I - F\lambda_u^{-1})P_{\lambda_u, m_u - j}}{(z - \lambda_u^{-1})^{j}};$$

because $(I - F\lambda_u^{-1})P_{\lambda_u,0}$ is the only term in (7) that loads $(z - \lambda_u^{-1})^{-m_u}$, then (7) implies $(I - F\lambda_u^{-1})P_{\lambda_u,0} = 0$. Similarly, starting from $(I - Fz)^{-1}(I - Fz) = I$ one finds that $P_{\lambda_u,0}(I - F\lambda_u^{-1}) = 0$. Hence $(I - F\lambda_u^{-1})P_{\lambda_u,0} = P_{\lambda_u,0}(I - F\lambda_u^{-1}) = 0$. Because λ_u is an eigenvalue of F, one can write $F - \lambda_u I = \alpha\beta'$, where α, β are $n_x \times r$ full column rank matrices, and $r = \operatorname{rank}(F - \lambda_u I) < n_x$; one then has $P_{\lambda_u,0} = \beta_\perp \varphi \alpha_\perp' \neq 0$, where $\alpha_\perp, \beta_\perp$ are bases of the orthogonal complements of α, β and φ is some matrix, see e.g. Franchi and Paruolo (2011).

Next let $\varphi = \xi\eta'$, where ξ, η are $(n_x - r) \times r_1$ full column rank matrices and $r_1 = \operatorname{rank}\varphi \leq n_x - r$; then one has $P_{\lambda_u,0} = \beta_1\alpha_1'$, where $\alpha_1 := \alpha_\perp\eta, \beta_1 := \beta_\perp\xi$ have full column rank r_1. Let $\alpha_2 := \bar{\alpha}_\perp\eta_\perp, \beta_2 := \bar{\beta}_\perp\xi_\perp$ and use the projection identities[8] $I_{n_x} = \alpha\bar{\alpha}' + \bar{\alpha}_1\alpha_1' + \alpha_2\bar{\alpha}_2' = \beta\bar{\beta}' + \bar{\beta}_1\beta_1' + \beta_2\bar{\beta}_2'$ to write $B = \alpha B_0 + \bar{\alpha}_1 B_1 + \alpha_2 B_2$, $C = C_0\beta' + C_1\bar{\beta}_1' + C_2\beta_2'$; with this notation one finds that $CP_{\lambda_u,0}B = 0$ is equivalent to $C_1B_1 = 0$. The dimensions of C_1 and B_1 are respectively $n_y \times r_1$ and $r_1 \times n_y$.

Because $\operatorname{rank} C_1 = r_1$ implies $B_1 = 0$ and $\operatorname{rank} B_1 = r_1$ implies $C_1 = 0$, from $C_1B_1 = 0$ it follows that B_1 and C_1 cannot have simultaneously full rank r_1. This implies that

$$\begin{pmatrix} I_r & B_0 \\ 0 & B_1 \\ 0 & B_2 \end{pmatrix}, \qquad \begin{pmatrix} I_r & 0 & 0 \\ C_0 & C_1 & C_2 \end{pmatrix}$$

cannot have simultaneously rank n_x. Hence

$$\begin{aligned} A - \lambda_u I = F - \lambda_u I + BD^{-1}C &= \begin{pmatrix} \alpha & BD^{-1} \end{pmatrix} \begin{pmatrix} \beta' \\ C \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \bar{\alpha}_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} I_r & B_0 \\ 0 & B_1 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I_r & 0 & 0 \\ C_0 & C_1 & C_2 \end{pmatrix} \begin{pmatrix} \beta' \\ \bar{\beta}_1' \\ \beta_2' \end{pmatrix} \end{aligned}$$

is singular. Thus λ_u is also an eigenvalue of A. ■

Proof of Corollary 4.5. If λ_u is a simple eigenvalue of F, there exist α_u, β_u of full column rank $r = \operatorname{rank}(F - \lambda_u I) < n_x$ such that $F - \lambda_u I = \alpha_u\beta_u'$ and $|\alpha_{u,\perp}'\beta_{u,\perp}| \neq 0$; moreover, see Theorem 3 in Johansen (2009), $(I - Fz)^{-1}$ has a pole of order one at $z = \lambda_u^{-1}$ and $P_{\lambda_u,0} = \beta_{u,\perp}(\alpha_{u,\perp}'\beta_{u,\perp})^{-1}\alpha_{u,\perp}'$. The statement then follows from Proposition 4.2. ■

Proof of Proposition 5.2. If F is stable, see Proposition 4.1. Next we show that if (1) is stabilizable and detectable, then the condition in Proposition 4.2 cannot hold; this implies that (1) is invertible only if F stable. Observe that $\operatorname{rank}(A - \lambda I : B) = \operatorname{rank}(F - \lambda I : B)$; in fact

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix} \begin{pmatrix} I_{n_x} & 0 \\ -D^{-1}C & I_{n_y} \end{pmatrix} = \begin{pmatrix} F - \lambda I & B \end{pmatrix}.$$

Because (1) is stabilizable, $\operatorname{rank}(F - \lambda I : B) = n_x$ for all $|\lambda| \geq 1$. Similarly, because (1) is detectable, one finds that $\operatorname{rank}(F' - \lambda I : C') = \operatorname{rank}(A' - \lambda I : C') = n_x$ for all $|\lambda| \geq 1$.

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[8]In the following we use the notation $\bar{\gamma} = \gamma(\gamma'\gamma)^{-1}$ for any full column rank matrix γ.

Let now λ be an unstable eigenvalue of F, $|\lambda| \geq 1$, and write $F - \lambda I = \alpha\beta'$, where α, β are $n_x \times r$ matrices and $r = \operatorname{rank}(F - \lambda I) < n_x$, and let $\alpha_\perp, \beta_\perp$ be bases of the orthogonal complements of $\operatorname{col}\alpha, \operatorname{col}\beta$. Use the projection identities $I = \alpha\bar{\alpha}' + \bar{\alpha}_\perp\alpha'_\perp = \bar{\beta}\beta' + \beta_\perp\bar{\beta}'_\perp$ to write $B = \alpha\widetilde{B}_1 + \bar{\alpha}_\perp\widetilde{B}_2$ and $C = \widetilde{C}_1\beta' + \widetilde{C}_2\bar{\beta}'_\perp$. Next we show that $\operatorname{rank}\widetilde{B}_2 = \operatorname{rank}\widetilde{C}_2 = n_x - r$; in fact

$$\begin{pmatrix} F - \lambda I & B \end{pmatrix} = \begin{pmatrix} \alpha\beta' & \alpha\widetilde{B}_1 + \bar{\alpha}_\perp\widetilde{B}_2 \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha}_\perp \end{pmatrix}\begin{pmatrix} \beta' & \widetilde{B}_1 \\ 0 & \widetilde{B}_2 \end{pmatrix},$$

$$\begin{pmatrix} F - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} \alpha\beta' \\ \widetilde{C}_1\beta' + \widetilde{C}_2\bar{\beta}'_\perp \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \widetilde{C}_1 & \widetilde{C}_2 \end{pmatrix}\begin{pmatrix} \beta' \\ \bar{\beta}'_\perp \end{pmatrix}.$$

In the proof of Proposition 4.3 it is shown that (1) is invertible if and only if $C_1B_1 = 0$, where $B_1 := \eta'\alpha'_\perp B, C_1 := C\beta_\perp\xi$ and $\varphi = \xi\eta' \neq 0$. Because $C_1B_1 = C\beta_\perp\xi\eta'\alpha'_\perp B = \widetilde{C}_2\varphi\widetilde{B}_2 = 0$ and $\varphi \neq 0$, this contradicts $\operatorname{rank}\widetilde{B}_2 = \operatorname{rank}\widetilde{C}_2 = n_x - r$; hence if the economic model is stabilizable and detectable, then the condition in Proposition 4.2 cannot hold. This implies that if (1) is invertible and it is also stabilizable and detectable, F cannot have unstable eigenvalues. ■

Proof of Corollary 5.3. If A is stable, then $\operatorname{rank}(A - \lambda I) = n_x$ for all $|\lambda| \geq 1$; hence (1) is stabilizable and detectable and Proposition 5.2 applies. ■

Proof of Corollary 5.4. If (1) is controllable and observable then it is stabilizable and detectable; hence Proposition 5.2 applies. ■

References

Anderson, B. and J. Moore (1979). *Optimal Filtering.* Prentice Hall.

Chari, V., P. Kehoe, and E. McGrattan (2005). A critique of structural VARs using real business cycle theory. Working Papers 631, Federal Reserve Bank of Minneapolis.

Christiano, L. J., M. Eichenbaum, and R. Vigfusson (2006). Assessing Structural VARs. In *D. Acemoglu, K. Rogoff, M. Woodford (eds.) NBER Macroeconomics Annual 2006, Volume 21*, pp. 1–106. The MIT Press.

Fernández-Villaverde, J., J. Rubio-Ramírez, T. Sargent, and M. Watson (2007). ABCs (and Ds) of Understanding VARs. *American Economic Review 97*, 1021–26.

Fischer, W. and I. Lieb (2012). *A Course in Complex Analysis: from Basic Results to Advanced Topics.* Vieweg+Teubner Verlag.

Franchi, M. and P. Paruolo (2011). Inversion of regular analytic matrix functions: local Smith form and subspace duality. *Linear Algebra and its Applications 435*, 2896–2912.

Franchi, M. and P. Paruolo (2012). A characterization of DSGE models with a finite order VAR reduced form. Mimeo.

Hansen, L. and T. Sargent (1980). Formulating and estimating dynamic linear rational expectations models. *Journal of Economic Dynamics and Control 2*, 7 – 46.

Hansen, L. and T. Sargent (1991). Two difficulties in interpreting vector autoregressions. In L. Hansen and T. Sargent (Eds.), *Rational Expectations Econometrics*, pp. 77–119. Westview Press.

Johansen, S. (2009). Representation of cointegrated autoregressive process with application to fractional process. *Econometric Reviews 28*, 121–145.

Kapetanios, G., A. Pagan, and A. Scott (2007). Making a match: Combining theory and evidence in policy-oriented macroeconomic modeling. *Journal of Econometrics 136*, 565 – 594.

Kurmann, A. and C. Otrok (2011). News Shocks and the Term Structure of Interest Rates: A Challenge for DSGE Models. Working Paper.

Lancaster, P. and L. Rodman (1995). *Algebraic Riccati Equations*. Claredon Press.

Leeper, E. M., T. B. Walker, and S.-C. S. Yang (2009). Fiscal foresight and information flows. NBER Working Papers 14630.

Lippi, M. and L. Reichlin (1993). The dynamic effects of aggregate demand and supply: comment. *American Economic Review 83*, 644–652.

Lippi, M. and L. Reichlin (1994). Var analysis, non-fundamental representations, Blaschke matrices. *Journal of Econometrics 63*, 307–325.

Ravenna, F. (2007). Vector autoregressions and reduced form representations of DSGE models. *Journal of Monetary Economics 54*, 2048–2064.

Schmitt-Grohé, S. (2010). Comment. *NBER Macroeconomics Annual 24*, pp. 475–490.

Sims, E. (2012). News, Non-Invertibility, and Structural VARs. *Advances in Econometrics*, forthcoming.