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A simple check for VAR representations of DSGE models

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ABSTRACT. The present paper shows that there is a simple way to check whether a DSGE model can be represented by a finite order VAR. This consists in verifying that the eigenvalues of a certain matrix defined in Fernández-Villaverde et al. (2007) are all equal to zero. Further we show that this condition is equivalent to the one in Ravenna (2007), which is, however, not easily applicable.

1. Introduction

The analysis of how an economy reacts to shocks plays a central role in macroeconomics. Since the seminal work of Sims (1980), the mechanisms of propagation of economic shocks have been analyzed empirically using vector autoregressive (VAR) models. During the last thirty years, a vast literature has provided evidence on the effects of monetary policy, fiscal policy and other economically relevant shocks via the analysis of impulse response functions (IRFs) derived from VARs, see e.g. Blanchard and Quah (1989), Blanchard and Perotti (2002), Uhlig (2005), Mountford and Uhlig (2009), Perotti (2008). On the theoretical side, dynamic stochastic general equilibrium (DSGE) models have recently gained a central role in formalizing these mechanisms of propagation in a coherent theoretical framework; in many cases, one uses the 'stylized facts' derived from a VAR as a guidance for isolating properties that a theoretical model would need to possess.

A growing number of papers (Chari et al., 2005, Christiano et al., 2006, Kapetanios et al., 2007, Fernández-Villaverde et al., 2007, Ravenna, 2007) remarks that this requires that the data-generating process consistent with the DSGE admits a finite order VAR representation, and the following very basic question is posed: is it always possible to capture the economic shocks of a DSGE via the residuals of a VAR? That is, does a reduced form VAR always contain the economic shocks of the DSGE among its structural interpretations? This difficulty is related to the problem of non-invertibility (or non-fundamentalness) of economic models, see Hansen and Sargent (1980, 1991), Lippi and Reichlin (1993, 1994) for early treatments of the issue, and precedes the identification step.

The present paper shows that there is a simple way to check whether a DSGE model can be represented by a finite order VAR. This consists in verifying that all the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) are equal to zero. The same problem is addressed and answered in Ravenna (2007) by giving a unimodularity condition which is, however, not easily applicable in practice. In Section 3 we further show that our condition is equivalent to the latter and thus it provides an easy way to implement it.

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Moreover, our results are very much related to those in Fernández-Villaverde et al. (2007), who discuss when a DSGE model admits an infinite order VAR representation, and can be viewed as clarifying the missing link between their condition and the one proposed in Ravenna (2007).

2. The 'ABCD' setup

Let an equilibrium of an economic model have the state space representation (see e.g. Uhlig (1999) for an exposition of how to obtain it):

$$\begin{aligned} x_t &= Ax_{t-1} + Bw_t \\ y_t &= Cx_{t-1} + Dw_t \\ w_t &= Hw_{t-1} + \varepsilon_t \end{aligned} \tag{1}$$

where x_t is an $n_x \times 1$ vector of possibly unobserved variables, y_t is an $n_y \times 1$ vector of observed variables, w_t is an $n_w \times 1$ autoregressive process, and ε_t is an $n_\varepsilon \times 1$ vector white noise of economic shocks, i.e. $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t')$ is a diagonal matrix.

Assumption 2.1. *Assume that* (1) *satisfies the following requirements:*

i) $n_\varepsilon = n_y$, *i.e. the number of economic shocks is equal to the number of observables;*
ii) D *is an invertible matrix;*
iii) *the system is minimal.*[1]

We are interested in characterizing situations in which the structural shocks of the DSGE match up with those of a finite order VAR on the observable y_t. That is, we wish to give a necessary and sufficient condition for (1) to admit a finite order VAR representation

$$y_t = \sum_{j=1}^{k} A_j y_{t-j} + u_t, \qquad u_t = Q\varepsilon_t, \tag{2}$$

where the reduced form errors u_t are a linear combination of the economic shocks of the DSGE ε_t and Q is an invertible matrix.

Two recent articles discuss this issue: Fernández-Villaverde et al. (2007) study the case in which $H = 0$ and show that if all eigenvalues of the matrix $A - BD^{-1}C$ are less than one in modulus then (2) holds for $k = \infty$. This is the so-called 'poor man's invertibility condition'. Ravenna (2007) states that (2) holds for a finite k if and only the matrix polynomial[2]

$$|I - Az|I + C(I - Az)^{\text{adj}} BD^{-1} z, \qquad z \in \mathbb{C}, \tag{3}$$

is unimodular, i.e. its determinant is a constant different from zero.

We observe that the unimodularity condition in Ravenna (2007) is not of immediate application; in fact, one would need to compute determinant and adjoint of matrix functions of z and then verify that for all $z \in \mathbb{C}$ the determinant of the resulting matrix polynomial in (3) is a constant different from zero. This cannot be done numerically. On the contrary, the 'poor man's invertibility condition' in Fernández-Villaverde et al.

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[1] That is, n_x is as small as possible. One can easily check minimality of (1) via the condition

$$\text{rank}\begin{pmatrix} B & AB & \cdots & A^{n_x-1}B \end{pmatrix} = \text{rank}\begin{pmatrix} C' & A'C' & \cdots & (A')^{n_x-1}C' \end{pmatrix} = n_x,$$

see e.g. Ch.6 in Kailath (1980). See Franchi and Paruolo (2012a,b) for non-minimal systems.

[2] $|M|$ and M^{adj} indicate determinant and adjoint of M.

(2007) is of straightforward application as it only requires to compute the eigenvalues of a given matrix. Moreover, we note that the relation between the 'poor man's invertibility condition' and the unimodularity condition is not clear: since the former ensures an infinite order VAR and the latter a finite order one, the intuition is that the condition in Ravenna (2007) is stronger than the one in Fernández-Villaverde et al. (2007).

3. Our result

This section presents two results: in Proposition 3.1 we show that a necessary and sufficient condition for a DSGE to admit a finite order VAR representation is that the eigenvalues of the matrix $A - BD^{-1}C$ defined in Fernández-Villaverde et al. (2007) are all equal to zero. This condition is of immediate application. In Proposition 3.2 we further show that this requirement is equivalent to the unimodularity condition in Ravenna (2007) and thus it may be viewed as an easy way to implement it. Combining the two results, one understands the relation between the 'poor man's invertibility condition' in Fernández-Villaverde et al. (2007) and the unimodularity condition in Ravenna (2007); in particular, one sees why the former is weaker than the latter.

Proposition 3.1. *A finite order VAR representation for y_t exists if and only if the eigenvalues of $F = A - BD^{-1}C$ are all equal to zero, that is F is nilpotent.*

It is interesting to observe how this result is related to the one in Fernández-Villaverde et al. (2007), who show that if the eigenvalues of F are all less than one in modulus, then y_t admits an infinite order VAR representation. Our result shows that if all the eigenvalues of F are not only stable but also equal to zero, then the VAR is of finite order. In minimal systems the converse also holds; that is, nilpotency of F characterizes the existence of finite order VAR representations of DSGE models. Hence what is needed in order to eliminate the infinitely many lags in Fernández-Villaverde et al. (2007) is the stronger condition that F is not only stable but also nilpotent.

Next we discuss the relation between our condition and the one in Ravenna (2007).

Proposition 3.2. *The following statements are equivalent:*

i) $|I - Az|I + C(I - Az)^{\mathrm{adj}} BD^{-1} z$ *is unimodular;*
ii) F *is nilpotent.*

Ravenna (2007) shows that a finite order VAR representation for y_t exists if and only if the unimodularity condition in *i*) holds. Here it is shown that *i*) holds if and only if F is nilpotent. Apart from mathematical simplicity, this result is of practical importance; in fact, it implies that if one wants to check the condition in Ravenna (2007) he can simply verify whether the eigenvalues of F are all equal to zero. Finally, we observe that Propositions 3.1 and 3.2 allow to link the results in Fernández-Villaverde et al. (2007) to those in Ravenna (2007). In fact, one immediately sees why the unimodularity condition is stronger than the 'poor man's invertibility condition' and thus able to eliminate the infinitely many lags from the autoregressive representation.

4. Conclusions

In the present paper we have shown that there is a simple check for finite order VAR representations of DSGE models. This consists in verifying that the matrix defined in Fernández-Villaverde et al. (2007) is nilpotent. This condition is shown to be equivalent to the one in Ravenna (2007) and in minimal systems it characterizes the existence of finite order VAR representations of DSGE models. The results of the paper can also be viewed as providing the missing link between those in Fernández-Villaverde et al. (2007) and those in Ravenna (2007).

Appendix A. Proofs

The proofs of Propositions 3.1 and 3.2 are based on the following lemmas.

Lemma A.1. *$F(z) = I - Fz$ is unimodular if and only if F is nilpotent.*

Proof. Observe that $\lambda_0 \neq 0$ is an eigenvalue of F if and only if $z_0 = \lambda_0^{-1}$ is a root of $|I - Fz| = 0$. We next show that if the eigenvalues of F are all equal to zero, then $F(z)$ is unimodular. Suppose that this is not the case, namely that there exists $z_0 \neq 0$ such that $|F(z_0)| = 0$. Because $I - Fz_0 = (-z_0)(F - z_0^{-1}I)$, one has $|F - z_0^{-1}I| = 0$, i.e. $\lambda_0 = z_0^{-1} \neq 0$ is an eigenvalue of F. This contradicts the hypothesis and hence $F(z)$ must be unimodular. Similarly one proves necessity. $\square$

Lemma A.2. *Let $\lambda_0 \neq 0$ be an eigenvalue of F and define $z_0 = \lambda_0^{-1}$; then*

$$(I - Fz)^{-1} = \frac{G(z)}{(z - z_0)^m g(z)}, \qquad m \geq 1, \qquad g(z_0) \neq 0, \qquad G(z_0) = u\varphi v' \neq 0, \tag{4}$$

where u, v are bases of the right and left eigenspaces[3] *of F corresponding to λ_0. Moreover, if* (1) *is minimal, then*

$$\operatorname{rank}(v'B) = \operatorname{rank}(Cu) = n_x - r, \tag{5}$$

where $r = \operatorname{rank}(F - \lambda_0 I)$.

Proof. Let $\lambda_0 \neq 0$ be an eigenvalue of F and define $z_0 = \lambda_0^{-1}$. One can write

$$|I - Fz| = (z - z_0)^a g(z), \qquad a \geq 1, \qquad g(z_0) \neq 0,$$

$$(I - Fz)^{\text{adj}} = (z - z_0)^b G(z), \qquad 0 \leq b < a, \qquad G(z_0) \neq 0;$$

cancelling common factors from determinant and adjoint, one then has

$$(I - Fz)^{-1} = \frac{G(z)}{(z - z_0)^m g(z)}, \qquad m = a - b.$$

The identity $(I - Fz)(I - Fz)^{\text{adj}} = (I - Fz)^{\text{adj}}(I - Fz) = |I - Fz|I$ delivers

$$(I - Fz)G(z) = G(z)(I - Fz) = (z - z_0)^m g(z) I,$$

which evaluated for $z = z_0$ gives

$$(F - \lambda_0 I)G(z_0) = G(z_0)(F - \lambda_0 I) = 0,$$

because $I - Fz_0 = (-z_0)(F - \lambda_0 I)$. The last equation implies that the non-zero columns (rows) of $G(z_0)$ are right (left) eigenvectors of F corresponding to λ_0; hence $G(z_0) = u\varphi v'$, where $\varphi \neq 0$ and u, v are bases of the right and left eigenspaces of F corresponding to λ_0. With this notation, one can write the rank factorization

[3] We say that $x \neq 0$ is a right (left) eigenvector corresponding to the eigenvalue λ_0 if $(F - \lambda_0 I)x = 0$ ($x'(F - \lambda_0 I) = 0$). By right (left) eigenspace of F corresponding to λ_0 we mean the space generated by its right (left) eigenvectors.

$(F-\lambda_0 I) = v_\perp u'_\perp$, where $x_\perp$ indicates a basis of the orthogonal complement of the space generated by x. This completes the proof of the first statement.

Next assume (1) is minimal; Lancaster and Rodman (1995, Theorems 4.3.3 and 6.1.5) show that this means

$$\mathrm{rank}\begin{pmatrix} A-\lambda I & B \end{pmatrix} = \mathrm{rank}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = n_x, \qquad \forall \lambda \in \mathbb{C}.$$

We observe that this is equivalent to

$$\mathrm{rank}\begin{pmatrix} F-\lambda I & B \end{pmatrix} = \mathrm{rank}\begin{pmatrix} F-\lambda I \\ C \end{pmatrix} = n_x, \qquad \forall \lambda \in \mathbb{C};$$

in fact $(A-\lambda I : B)$ and $(F-\lambda I : B)$ are connected by the invertible transformation

$$\begin{pmatrix} A-\lambda I & B \end{pmatrix}\begin{pmatrix} I_{n_x} & 0 \\ -D^{-1}C & I_{n_y} \end{pmatrix} = \begin{pmatrix} F-\lambda I & B \end{pmatrix}.$$

Similarly one shows that $\mathrm{rank}(A'-\lambda I : C') = \mathrm{rank}(F'-\lambda I : C') = n_x$. Next use the projection identity $I = P_{v_\perp} + P_v = v_\perp \bar{v}'_\perp + \bar{v}v'$, where $P_x = x(x'x)^{-1}x'$ is the projection on the space generated by x and $\bar{x} = x(x'x)^{-1}$, to write

$$\begin{pmatrix} F-\lambda_0 I & B \end{pmatrix} = \begin{pmatrix} v_\perp u'_\perp & v_\perp \bar{v}'_\perp B + \bar{v}v'B \end{pmatrix} = \begin{pmatrix} v_\perp & \bar{v} \end{pmatrix}\begin{pmatrix} u'_\perp & \bar{v}'_\perp B \\ 0 & v'B \end{pmatrix}.$$

Because $\mathrm{rank}(F-\lambda_0 I : B) = n_x$ and $(v_\perp : \bar{v})$ is invertible, this shows that $\mathrm{rank}(v'B) = n_x - r$, where $r = \mathrm{rank}(F-\lambda_0 I)$. Similarly, because

$$\begin{pmatrix} F-\lambda_0 I \\ C \end{pmatrix} = \begin{pmatrix} v_\perp & 0 \\ C\bar{u}_\perp & Cu \end{pmatrix}\begin{pmatrix} u'_\perp \\ \bar{u}' \end{pmatrix},$$

one has that $\mathrm{rank}(Cu) = n_x - r$. □

Proof of Proposition 3.1. Let $z_t = Dw_t$ and rewrite (1) as

$$\begin{aligned} x_t &= Fx_{t-1} + BD^{-1}y_t, \qquad F = A - BD^{-1}C, \\ y_t &= Cx_{t-1} + z_t \\ z_t &= Mz_{t-1} + D\varepsilon_t, \qquad M = DHD^{-1}; \end{aligned}$$

combining the first two equations one finds $(I - C(I-FL)^{-1}BD^{-1}L)y_t = z_t$ and thus using the third

$$(I-ML)T(L)y_t = D\varepsilon_t, \qquad T(z) = I - C(I-Fz)^{-1}BD^{-1}z, \qquad z \in \mathbb{C}. \tag{6}$$

Note that (6) is a finite order VAR if and only if $T(z)$ is a polynomial matrix.

Suff. We first prove that if F is nilpotent then a finite order VAR representation for y_t exists. Since $F(z) = I - Fz$ being unimodular is equivalent to F being nilpotent, see Lemma A.1, if F is nilpotent then

$$(I-Fz)^{-1} = \frac{(I-Fz)^{\mathrm{adj}}}{|I-Fz|} = (I-Fz)^{\mathrm{adj}}$$

is a matrix polynomial and hence the same holds for $T(z)$.

Nec. We now show that if a finite order VAR representation for y_t exists then F must be nilpotent. First we observe that $T(z)$ is a polynomial matrix if and only if $C(I-Fz)^{-1}B$ is a polynomial matrix. Next we show that if $C(I-Fz)^{-1}B$ is a polynomial matrix then F must be nilpotent. Suppose that this is not the case, namely assume that there exists $\lambda_0 \neq 0$ eigenvalue of F. Write $G(z) = \sum_{j=0}^{k} G_j(z-z_0)^j$, so that $G_0 = G(z_0)$; then, see (4) in Lemma A.2,

$$C(I-Fz)^{-1}B = \frac{CG(z)B}{(z-z_0)^m g(z)} = \frac{1}{g(z)}\left(\frac{CG_0B}{(z-z_0)^m} + \frac{CG_1B}{(z-z_0)^{m-1}} + \cdots + \frac{CG_kB}{(z-z_0)^{m-k}}\right), \qquad G_0 = u\varphi v' \neq 0.$$

Because $T(z)$ is a polynomial matrix it must be that at least $CG_0B = 0$, i.e. that $Cu\varphi v'B = 0$. However, in minimal systems Cu and $B'v$ have full column rank, see (5) in Lemma A.2, and this implies $\varphi = 0$, which is a contradiction because $G_0 \neq 0$. Hence it must be that F is nilpotent. □

Proof of Proposition 3.2. Corollary 2.2 in Ravenna (2007) shows that $|I-Az|I + C(I-Az)^{\text{adj}}BD^{-1}z$ is unimodular if and only if the same holds for $I - Fz$. The statement then follows from Lemma A.1. □

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