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Dipartimento di Scienze Statistiche
Sezione di Statistica Economica ed Econometria

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May 30, 2013

Abstract. Solutions of DSGE models are usually represented by state space forms. This note shows that if one wishes to determine whether the observables of the model admit a finite order VAR representation, minimality of the state space representation of the solution matters. More specifically, we first provide a counterexample to Proposition 2.1 and Corollary 2.2 in Ravenna (2007), which state that in the square case a finite order VAR exists if and only if a 'unimodularity condition' holds. Our counterexample shows that the proposed condition is not necessary for the existence of a finite order VAR representation. That is, if the state space representation of the solution is non-minimal, the observables of the DSGE may admit a finite order VAR representation even though the unimodularity condition fails. It is further shown that if the state space representation of the solution is minimal, then the unimodularity condition is necessary. Given that a minimal state space representation always exists, before applying the unimodularity condition one simply needs to check whether the state space representation of the solution is minimal and if not transform it into an equivalent minimal form. A discussion of how to perform such reduction is presented and further it is shown that the economic interpretation of the system is not affected by this transformation. An interpretation of the results in terms of the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) for the poor man's invertibility condition is also provided. The analysis is then applied to the Smets and Wouters (2007) model.

1. Introduction

Solutions of linear or (log-)linearized dynamic stochastic general equilibrium (DSGE) models are usually represented by state space forms. In the square case, i.e. when the number of economic shocks is equal to the number of observed variables and the matrix that loads the former into the latter is non-singular, it is often of interest to determine whether the shocks of the DSGE can be recovered via a vector autoregressive (VAR) model on the observables. Instances of this analysis are found in Chari et al. (2005), Christiano et al. (2006), Fernández-Villaverde et al. (2007), Kapetanios et al. (2007), Ravenna (2007), Schmitt-Grohé (2010), Kurmann and Otrok (2011), Sims (2012), Leeper et al. (2013) among others. This issue is related to the

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fact that economic models may be non-invertible (or economic shocks be non-fundamental), as early stressed in Hansen and Sargent (1980), Hansen and Sargent (1991), Lippi and Reichlin (1993), Lippi and Reichlin (1994).

The present paper shows that if one wishes to determine whether a VAR representation of a DSGE model exists, minimality of the state space representation of the solution matters. More specifically, we first provide a counterexample to Proposition 2.1 and Corollary 2.2 in Ravenna (2007), which state that in the square case a finite order VAR exists if and only if a 'unimodularity condition' holds. Our counterexample shows that the proposed condition is not necessary for the existence of a finite order VAR representation. That is, if the state space representation of the solution is non-minimal, the observables of the DSGE may admit a finite order VAR representation even though the unimodularity condition fails.

Non-minimal state space representations are natural outcomes of standard solution methods, such as Sims (2002). This is illustrated in Komunjer and Ng (2011), where it is shown that the state space representation of the solution of the models in Christiano et al. (2005), An and Schorfheide (2007), Smets and Wouters (2007), García-Cicco et al. (2010) are all non-minimal. Indeed, minimality of the state space representation is important in other contexts as well. For example, it is key to establish and to employ the necessary and sufficient conditions for identification in Komunjer and Ng (2011) and thus it is also relevant when choosing the variables to estimate singular DSGE models as proposed in Canova et al. (2013): dropping observable variables in order to make a singular system become square can have an impact on minimality and hence on the conditions for identification.

Remark that a minimal representation always exists, i.e. one can always transform a non-minimal state space form into an equivalent minimal representation; further note that if the state space representation of the solution is minimal, then the unimodularity condition is necessary for the existence of a finite order VAR, as shown in Franchi and Vidotto (2013). Because the unimodularity condition is sufficient in the square case, i.e. both in minimal and non-minimal square systems, this condition provides a characterization of the existence of a finite order VAR representation in minimal square systems. Thus, before applying it to determine if a finite order VAR representation exists, one simply needs to check whether the state space form of the solution is minimal and if not transform it into an equivalent minimal representation. A discussion of how to perform such reduction is presented. The transformation consists in eliminating irrelevant states, i.e. those that are non-controllable, non-observable or both. Given that the observed variables and the economic shocks are not affected by the reduction, the economic assumptions that define the square case are unchanged. Moreover, because impulse response functions (IRFs) of equivalent non-minimal and minimal systems are the same, the economic implications of the solution are also invariant with respect to this transformation. Finally, also the coefficients of the VAR representation of equivalent non-minimal and minimal systems are equal. Hence the economic interpretation and the implications of the model derived from equivalent non-minimal and minimal systems are exactly the same.

The results have a neat interpretation in terms of the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) for the poor man's invertibility condition. When the state space representation of the

solution is non-minimal, the eigenvalues are divided into two groups: those in the first group are associated to the minimal state representation and are relevant in determining whether the observables admit a VAR representation, whereas those in the second group are associated to irrelevant states and are cancelled by the transformation that reduces the system to a minimal form. Relevant and irrelevant eigenvalues do not have the same role in shaping the properties of the system; identifying them is thus crucial to determine whether the observables of a DSGE model admit a VAR representation. The reduction of the system to a minimal form allows one to do that.

The analysis developed in the paper is then applied to the version of the Smets and Wouters (2007) model implemented in Iskrev (2010). For standard choices of observable variables and parameters values, the state space representations of the solutions turn out to be non-minimal; the sets of relevant and irrelevant eigenvalues are then characterized. The analysis reveals the presence of both zero and non-zero irrelevant eigenvalues, which reflects additional linear dependence among equations that causes non-identificability of the model, as discussed in the supplementary material of Komunjer and Ng (2011).

The rest of the paper is organized as follows. Section 2 introduces the square case and provides a counterexample to Proposition 2.1 and Corollary 2.2 in Ravenna (2007). Section 3 discusses non-minimality of state space representations of DSGE models and shows that if the state space is minimal the unimodularity condition provides a characterization of the existence of a finite order VAR representation. Section 4 discusses how to transform a non-minimal system to an equivalent minimal representation and Section 5 provides an interpretation of the results in terms of the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) for the poor man's invertibility condition. Section 6 applies the results to the Smets and Wouters (2007) model and Section 7 concludes. All proofs are collected in the Appendix.

2. The square case and the unimodularity condition

Consider an equilibrium of an economic model with representation

$$\begin{aligned} y_t &= Px_{t-1} + Qz_t \\ x_t &= Rx_{t-1} + Sz_t \\ z_t &= \textstyle\sum_{i=1}^{q} Z_i z_{t-i} + \varepsilon_t, \end{aligned} \tag{1}$$

where y_t is an $n_y \times 1$ vector of endogenous variables, x_t is an $n_x \times 1$ vector of endogenous state variables, z_t is an $n_\varepsilon \times 1$ vector of exogenous state variables, and ε_t is an $n_\varepsilon \times 1$ vector of economic shocks.[1] The following assumption is maintained throughout the paper.

Assumption 2.1 (Square case)**.** *Assume y_t is observed, $n_y = n_\varepsilon$ and Q is invertible.*

This is called square case, see Fernández-Villaverde et al. (2007); in this case (1) is such that the number of observed variables is equal to the number of economic shocks and the matrix that loads the exogenous state

[1] As shown in Ravenna (2007), by defining a new state vector x_t that includes both endogenous and exogenous states in (1), one can rewrite the system as $x_t = Ax_{t-1} + B\varepsilon_t$, $y_t = Cx_{t-1} + D\varepsilon_t$, as e.g. done in Fernández-Villaverde et al. (2007).

variables into the observed variables is invertible. Rearranging the first equation one has $z_t = Q^{-1}(y_t - Px_{t-1})$ and one can thus rewrite (1) as

$$\begin{aligned} y_t &= Px_{t-1} + Qz_t \\ x_t &= (R - SQ^{-1}P)x_{t-1} + SQ^{-1}y_t \\ z_t &= \textstyle\sum_{i=1}^{q} Z_i z_{t-i} + \varepsilon_t. \end{aligned} \tag{2}$$

Proposition 2.1 in Ravenna (2007) states that a finite order VAR representation of y_t exists if and only if the determinant of the $n_y \times n_y$ matrix polynomial

$$|G(L)|I_{n_y} + PD_G(L)SQ^{-1}L,$$

where $|G(L)|$ and $D_G(L)$ are respectively determinant and adjoint of $G(L) = I_{n_x} - RL$, is of degree zero in the lag operator L. Equivalently, Corollary 2.2 in Ravenna (2007) states that a finite order VAR representation of y_t exists if and only if the determinant of the matrix $n_x \times n_x$ polynomial

$$I_{n_x} - (R - SQ^{-1}P)L$$

is of degree zero in the lag operator L. A matrix polynomial is called unimodular if its determinant is a non-zero constant, see e.g. Antsaklis and Michel (2007, p.283), and hence, given that the two conditions above are equivalent, in the following we refer to either of them as the unimodularity condition.

The nilpotency condition in Franchi and Vidotto (2013) provides an equivalent and more direct formulation of the unimodularity condition.

Proposition 2.2. *The unimodularity condition holds if and only if $R - SQ^{-1}P$ is nilpotent, i.e. its eigenvalues are all equal to zero.*

The present formulation of the unimodularity condition involves the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) for the poor man's invertibility condition. The latter consists in the stability of $R - SQ^{-1}P$, i.e. its eigenvalues being all less than one in modulus. From Proposition 2.2 one immediately sees why the unimodularity condition is stronger than the poor man's invertibility condition and thus it is able to eliminate the infinitely many lags from the autoregressive representation. This is because a nilpotent matrix is stable, but a stable matrix may not be nilpotent. Given that the unimodularity condition and the nilpotency condition are equivalent, whatever stated regarding one of the two also applies to the other. Hence in the following we refer to either of the three formulations above as the unimodularity condition. Finally observe that the matrix polynomial $Z(L) = I_{n_\varepsilon} - \sum_{i=1}^{q} Z_i L^i$ in $Z(L)z_t = \varepsilon_t$ plays no role in the unimodularity condition.

2.1. **A counterexample to Proposition 2.1 and Corollary 2.2 in Ravenna (2007).** This example shows that in the square case a finite order VAR representation of y_t may exist even though the unimodularity condition fails, unlike stated in Proposition 2.1 and Corollary 2.2 in Ravenna (2007). Let $n_x = n_y = n_\varepsilon = 2$

and take

$$R = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad S = I_2, \qquad P = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad Q = I_2.$$

This is a square case in which R is stable, because its eigenvalues are $\{0, 2/3\}$. Further note that the unimodularity condition is violated because

$$R - SQ^{-1}P = R - P = \frac{2}{3}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\{0, 2/3\}$ and it is not nilpotent, i.e.

$$I_2 - (R - SQ^{-1}P)L = \begin{pmatrix} 1 - \frac{2}{3}L & 0 \\ -\frac{2}{3}L & 1 \end{pmatrix}$$

has determinant $1 - \frac{2}{3}L$ and it is not unimodular. Nevertheless, contrarily to what stated in Proposition 2.1 and Corollary 2.2 in Ravenna (2007), y_t admits a VAR representation; in fact, because $P(R - P) = 0$, lagging the second equation in (2) and substituting it into the first one, one finds

$$y_t = P(R - SQ^{-1}P)x_{t-2} + PSQ^{-1}y_{t-1} + Qz_t = Py_{t-1} + z_t$$

and hence the finite order VAR representation $A(L)y_t = \varepsilon_t$, where $A(L) = (I_2 - \sum_{i=1}^{q} Z_i L^i)(I_2 - PL)$. This shows that a failure of the unimodularity condition does not imply that a finite order VAR representation of the observables does not exist. That is, in the square case the unimodularity condition is not necessary for the existence of a finite order VAR representation.

3. Non-minimal state space representations of DSGE models

In the square case, the unimodularity condition is a sufficient condition for the existence of a finite order VAR representation of y_t. This is shown in Ravenna (2007). Because minimality of the state space representation, i.e. the fact that the dimension of the state vector x_t is non-reducibile, is irrelevant in this respect, the unimodularity condition is sufficient both in minimal and non-minimal square systems. On the contrary, the counterexample shows that minimality matters for the necessity of the unimodularity condition. Indeed, inspection of the counterexample reveals that the given state space form is non-minimal. In Franchi and Vidotto (2013) it is shown that if the state space representation of the equilibrium is minimal, then nilpotency of $R - SQ^{-1}P$ is a necessary condition for the existence of a finite order VAR representation of y_t. Hence this condition provides a characterization of finite order VAR representations in minimal square systems. Proposition 3.1 below collects these results. Given that a minimal representation always exists, before applying the unimodularity condition to economic models of interest one simply needs to check whether the state space form of the solution is minimal and if not transform it into an equivalent minimal representation. A discussion of how to perform such reduction is presented in Section 4.

Proposition 3.1. *If the state space representation of a solution of a DSGE is square and non-minimal, then the unimodularity condition is sufficient but not necessary for the existence of a finite order VAR*

representation of y_t. If the state space representation is square and minimal, then it is a necessary and sufficient condition.

In the square case, the unimodularity condition is a sufficient condition for the existence of a finite order VAR representation of y_t. This is shown in Ravenna (2007). If in addition the square state space form is minimal, nilpotency of $R - SQ^{-1}P$ provides a characterization of the existence of a finite order VAR representation of y_t. This is shown in Franchi and Vidotto (2013). The last result implies that a violation of the unimodularity condition may wrongly indicate the non-existence of a finite order VAR only if the system is non-minimal, as indeed happens in the counterexample.

Non-minimal representations are natural outcomes of standard solution methods, such as Sims (2002). This is illustrated in Komunjer and Ng (2011), where it is shown that the state space representations of standard economic models such as those in Christiano et al. (2005), An and Schorfheide (2007), Smets and Wouters (2007), García-Cicco et al. (2010) are all non-minimal. In Section 6 below, the Smets and Wouters (2007) model is analysed.

It is interesting to observe that minimality is not a property of the economic model but of the given representation of its solution, which depends on how one defines states and observable variables. This fact can be illustrated via the permanent income model in Fernández-Villaverde et al. (2007),

$$\begin{aligned} c_t &= c_{t-1} + \sigma_w(1 - R^{-1})\varepsilon_t \\ \widetilde{y}_t &= \sigma_w \varepsilon_t, \end{aligned} \tag{3}$$

where c_t is consumption, $\widetilde{y}_t$ is labour income and $R > 1$ is the gross interest rate. Recall that minimality holds if and only if (1) is both controllable and observable, see e.g. Ch.8.3.2 in Antsaklis and Michel (2007), i.e. if $\operatorname{rank}\mathcal{C} = \operatorname{rank}\mathcal{O} = n_x$, where

$$\mathcal{C} = \begin{pmatrix} S & RS & \cdots & R^{n_x-1}S \end{pmatrix}, \qquad \mathcal{O} = \begin{pmatrix} P \\ PR \\ \vdots \\ PR^{n_x-1} \end{pmatrix} \tag{4}$$

are respectively called controllability and observability matrix.

Consider two cases, one in which labour income is observable and one in which savings are observable. First suppose labour income is observable; by letting $x_t = c_t$, $y_t = \widetilde{y}_t$ one writes (3) as a square state space in which $n_x = n_y = n_\varepsilon = 1$ and $R = 1$, $S = \sigma_w(1 - R^{-1})$, $P = 0$, $Q = \sigma_w$. Note that because $P = 0$ this state space representation is non-observable and hence non-minimal. Again here, because $\widetilde{y}_t = \sigma_w \varepsilon_t$, y_t has a finite order VAR representation and the failure of the unimodularity condition wrongly indicates the non-existence of a finite order VAR; in fact $R - SQ^{-1}P = R = 1$ is not nilpotent, i.e. $1 - (R - SQ^{-1}P)L = 1 - L$ has not degree 0 in L. Note also that the poor man's invertibility condition fails, because $R - SQ^{-1}P = 1$, and wrongly indicates non-invertibility.

Next, as in Fernández-Villaverde et al. (2007), assume savings are observable, sum and subtract c_t in the second equation and rearrange terms to find

$$\begin{aligned} c_t &= c_{t-1} + \sigma_w(1 - R^{-1})\varepsilon_t \\ \widetilde{y}_t - c_t &= -c_{t-1} + \sigma_w R^{-1}\varepsilon_t. \end{aligned} \tag{5}$$

Letting $x_t = c_t$, $y_t = \widetilde{y}_t - c_t$ one writes (5) as a square state space in which $n_x = n_y = n_\varepsilon = 1$ and $R = 1$, $S = \sigma_w(1 - R^{-1})$, $P = -1$, $Q = \sigma_w R^{-1}$. This state space representation is controllable and observable, and hence minimal. The unimodularity condition is not satisfied and its failure correctly signals that y_t does not admit a finite order VAR representation. A similar reasoning applies to the poor man's invertibility condition. Remark that the economic model behind the two state space representations is just the same; it is only the assumption on what is observed that makes one being minimal and the other not, hence transforming the conditions from being necessary and sufficient to being only sufficient.

Finally note that similar situations arise when dropping observable variables in order to make a singular system become square, as proposed in Canova et al. (2013). This is illustrated in the supplement to Komunjer and Ng (2011) when discussing identification of the An and Schorfheide (2007) model.

4. Reduction to a minimal state space representation

A minimal representation of (1) always exists. In Proposition 4.1 it is shown how to find it. This result is a consequence of the Kalman's decomposition theorem, see e.g. Section 6.2.3 in Antsaklis and Michel (2007). It is important to note that the reduction does not affect the observed variables and the economic shocks and hence the economic assumptions that define the square case are unchanged. Furthermore, the impulse response functions (IRFs) of equivalent non-minimal and minimal systems are the same and also the coefficients of the VAR representation are uniquely defined. Hence the economic interpretation of the solution is invariant with respect to this transformation. In the following, span A indicates the space spanned by the columns of A and by rank decomposition of A we mean finding ϑ and φ bases of $\operatorname{span} A$ and $\operatorname{span} A'$ so that $A = \vartheta\varphi'$; $\varphi_\perp$ indicates a basis of the orthogonal complement of $\operatorname{span}\varphi$, i.e. $\varphi'\varphi_\perp = 0$ and $(\varphi, \varphi_\perp)$ is square and invertible.

Proposition 4.1. *Let $\mathcal{C} = \alpha\beta'$ and $\mathcal{O} = \delta\gamma'$ be rank decompositions of the controllability and observability matrices in* (4) *and let ξ be a basis of the intersection of* $\operatorname{span}\alpha$ *and* $\operatorname{span}\gamma_\perp$*. Then there exists μ, orthogonal to ξ and normalized so that $\mu'\mu = I_{n_m}$, such that (μ, ξ) is a basis the space spanned by α; moreover, $P_m = P\mu$, $R_m = \mu' R\mu$, $S_m = \mu' S$, and $x_{m,t} = \mu' x_t$ are such that*

$$\begin{aligned} y_t &= P_m x_{m,t-1} + Q z_t \\ x_{m,t} &= R_m x_{m,t-1} + S_m z_t \\ z_t &= \textstyle\sum_{i=1}^q Z_i z_{t-i} + \varepsilon_t \end{aligned} \tag{6}$$

is a minimal representation of (1)*. Furthermore for all $h \geq 0$, one has*

$$PR^h S = P_m R_m^h S_m, \qquad P(R - SQ^{-1}P)^h S = P_m(R_m - S_m Q^{-1} P_m)^h S_m.$$

It is important to note that the state space representations (1) and (6) share the same observable variables y_t, the same economic shocks ε_t and the same matrix Q, i.e. the reduction to a minimal representation does not affect the economic assumptions that define the square case. Furthermore, because $PR^hS = P_mR_m^hS_m$ and z_t is the same in the two systems, the impulse response functions of (1) and (6) are the same. Also note that $P(R - SQ^{-1}P)^hS = P_m(R_m - S_mQ^{-1}P_m)^hS_m$ implies that the coefficients of the VAR representation of (1) and (6) are the same. This shows that irrelevant states, i.e. those that are non-controllable, non-observable or both and are cancelled by the transformation $x_{m,t} = \mu' x_t$, do not affect the definition of the square case, the VAR representation of y_t and the IRFs of the solution of the DSGE. Hence the economic interpretation of (1) and (6) is the same.

Next we analyse how the transformation $x_{m,t} = \mu' x_t$ acts on x_t in order to eliminate the possibile irrelevant states. There are four possibilities: first suppose $\operatorname{rank}\mathcal{C} = \operatorname{rank}\mathcal{O} = n_x$, so that (1) is both controllable and observable. Then one can take $\alpha = I_{n_x}$, $\beta = \mathcal{C}'$, $\delta = \mathcal{O}$, $\gamma = I_{n_x}$; because $\gamma_\perp = 0$, $\xi = 0$ is the only vector in the intersection of $\operatorname{span}\alpha$ and $\operatorname{span}\gamma_\perp$, and hence one has $\mu = \alpha = I_{n_x}$. This shows that $x_{m,t} = x_t$, i.e. the dimension n_m of the minimal state vector is equal to n_x and no reduction in the dimension of x_t is possibile. That is, x_t does not contain irrelevant states and (1) is already a minimal representation. Next suppose that the system is non-controllable and observable, i.e. $r_\mathcal{C} = \operatorname{rank}\mathcal{C} < n_x$ and $\operatorname{rank}\mathcal{O} = n_x$; then α is an $n_x \times r_\mathcal{C}$ full column rank matrix and, because $\gamma_\perp = 0$, $\xi = 0$ is the only vector in the intersection of $\operatorname{span}\alpha$ and $\operatorname{span}\gamma_\perp$. Hence one can take $\mu = \alpha(\alpha'\alpha)^{-1/2}$ and $x_{m,t} = \mu' x_t$ reduces the dimension of the state vector from n_x to $n_m = r_\mathcal{C}$ by eliminating $n_x - r_\mathcal{C}$ non-controllable states. The third possibility is that the system is controllable and non-observable, i.e. $\operatorname{rank}\mathcal{C} = n_x$ and $r_\mathcal{O} = \operatorname{rank}\mathcal{O} < n_x$; then γ is an $n_x \times r_\mathcal{O}$ full column rank matrix and, because $\alpha = I_{n_x}$, $\xi = \gamma_\perp$ is a basis of intersection of $\operatorname{span}\alpha$ and $\operatorname{span}\gamma_\perp$. Hence one can take $\mu = \gamma(\gamma'\gamma)^{-1/2}$ and $x_{m,t} = \mu' x_t$ reduces the dimension of the state vector from n_x to $n_m = r_\mathcal{O}$ by eliminating $n_x - r_\mathcal{O}$ non-observable states. The fourth and final case arises when the system is non-controllable and non-observable, $\operatorname{rank}\mathcal{C} < n_x$ and $\operatorname{rank}\mathcal{O} < n_x$, and α and $\gamma_\perp$ are full column rank matrices of dimension $n_x \times r_\mathcal{C}$ and $n_x \times n_x - r_\mathcal{O}$ respectively. If $\operatorname{span}\alpha$ and $\operatorname{span}\gamma_\perp$ have only the zero vector in common, $\xi = 0$ is a basis of their intersection and there are no states that are controllable and non-observable, i.e. all the states that are controllable are also observable. Hence one can take $\mu = \alpha(\alpha'\alpha)^{-1/2}$ and $x_{m,t} = \mu' x_t$ reduces the dimension of the state vector from n_x to $n_m = r_\mathcal{C}$ by eliminating $n_x - r_\mathcal{C}$ non-controllable states. Otherwise, if $\xi \neq 0$ is a basis of the intersection of $\operatorname{span}\alpha$ and $\operatorname{span}\gamma_\perp$, then $\xi' x_t$ contains $r_\xi = \operatorname{rank}\xi$ states that are controllable and non-observable while $\mu' x_t$ contains $r_\mathcal{C} - r_\xi$ states that are controllable and observable. Hence $x_{m,t} = \mu' x_t$ reduces the dimension of the state vector from n_x to $n_m = r_\mathcal{C} - r_\xi$ by eliminating $n_x - r_\mathcal{C} + r_\xi$ non-controllable and non-observable states. This exhausts all possible cases and illustrates that Proposition 4.1 covers all of them.

4.1. **Illustration via the example in Subsection** 2.1. The state space form in the example in Subsection 2.1 is controllable but not observable, i.e. it is non-minimal, because $n_x = 2$ and

$$\mathcal{C} = \begin{pmatrix} I_2 & R \end{pmatrix}, \qquad \mathcal{O} = \begin{pmatrix} P \\ PR \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

imply $\operatorname{rank}\mathcal{C} = 2$ and $\operatorname{rank}\mathcal{O} = 1$. This falls into the third case described below Proposition 4.1. Apply Proposition 4.1 to find a minimal representation: take $\alpha = I_2$, $\beta' = (I_2, R)$, $\delta = \frac{1}{3}(-1,-1,0,0)'$, $\gamma = (1,-1)'$, and $\gamma_\perp = (1,1)'$; then $\xi = \gamma_\perp$, $\mu = \gamma(\gamma'\gamma)^{-1/2} = \frac{1}{\sqrt{2}}(1,-1)'$ and one finds

$$P_m = P\mu = \frac{\sqrt{2}}{3}\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \qquad R_m = \mu' R \mu = 0, \qquad S_m = \mu' S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}.$$

Because $x_{m,t} = \mu' x_t$ is a scalar, one has $n_m = 1$; the corresponding controllability and the observability matrices are thus $\mathcal{C}_m = S_m \neq 0$, $\mathcal{O}_m = P_m \neq 0$ so that $\operatorname{rank}\mathcal{C}_m = \operatorname{rank}\mathcal{O}_m = 1 = n_m$, i.e. (6) is a minimal representation of (1). Moreover, one has that

$$R_m - S_m Q^{-1} P_m = \frac{1}{3}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

is nilpotent, i.e. $1-(R_m - S_m Q^{-1} P_m)L = 1$ is unimodular. The unimodularity condition holds in the minimal representation and correctly signals that y_t admits a finite order VAR representation. Note that the economic shocks and the observed variables are the same in the two representations, i.e. Q is not affected by the transformation. Moreover, note that $PR^hS = P_m R_m^h S_m$ and $P(R - SQ^{-1}P)^h S = P_m(R_m - S_m Q^{-1} P_m)^h S_m$ so that (1) and (6) have the same IRFs and the same VAR representation. The reduction of the system to a minimal form allows one to use the unimodularity condition in order to determine the existence of a finite order VAR.

5. Irrelevant eigenvalues

Propositions 2.2, 3.1 and 4.1 show that, when (1) is non-minimal, it is the eigenvalues of $R_m - S_m Q^{-1} P_m$ and not those of $R - SQ^{-1}P$ that determine whether the observables of the DSGE admit a VAR representation. The following proposition illustrates the relationship between the eigenvalues of those two matrices.

Proposition 5.1. *Consider the state space representations* (1) *and* (6)*; if λ_i is an eigenvalue of $R_m - S_m Q^{-1} P_m$, then it is also an eigenvalue of $R - SQ^{-1}P$. The converse holds if and only if* (1) *is minimal.*

When (1) is non-minimal, the eigenvalues of $R - SQ^{-1}P$ are divided into two groups: the first group contains the eigenvalues of $R_m - S_m Q^{-1} P_m$ and the second one all the remaining eigenvalues. The eigenvalues in the first group are associated to the minimal state representation and determine whether y_t admits a VAR representation, whereas those in the second are associated to irrelevant states, i.e. those that are

non-controllable, non-observable or both, and are cancelled by the transformation $x_{m,t} = \mu' x_t$. These eigenvalues neither affect the VAR representation nor the IRFs of the solution of the DSGE and hence they are irrelevant. Relating the non-existence of a finite order VAR representation to the presence of non-zero irrelevant eigenvalues leads to a wrong conclusion because those eigenvalues play no role. This explains why the failure of the unimodularity condition does not necessarily rule out the existence of a finite order VAR representation: its failure may as well be due to non-zero irrelevant eigenvalues. When (1) is minimal, the eigenvalues of $R - SQ^{-1}P$ and those of $R_m - S_m Q^{-1} P_m$ coincide, i.e. all the eigenvalues of $R - SQ^{-1}P$ are relevant; thus, in order for a finite order VAR representation to exist, each of them must be zero. In this case the unimodularity condition becomes necessary. A similar reasoning applies to the poor man's invertibility condition: the failure of this condition does not necessarily imply non-invertibility because it may as well be due to unstable irrelevant eigenvalues. Relevant and irrelevant eigenvalues do not have the same role in shaping the properties of the system and hence distinguishing them is crucial. The reduction of the system to a minimal form allows one to do that.

5.1. **Illustration via the example in Subsection** 2.1**.** The eigenvalues of $R - SQ^{-1}P$ are $\{0, 2/3\}$; the eigenvalue at 0 is due to $R_m - S_m Q^{-1} P_m = 0$, see subsection 4.1 and determines the existence of the finite order VAR representation while the eigenvalue at $2/3$ is associated to the non-observable state and it is cancelled by the transformation $x_{m,t} = \mu' x_t$. Relating the non-existence of a finite order VAR representation to the irrelevant eigenvalue at $2/3$ leads to a wrong conclusion because that eigenvalue plays no role, i.e. all the relevant information is contained in $R_m - S_m Q^{-1} P_m = 0$.

It is interesting to observe how a slight modification of the example leads to the opposite situation. Let R, S and Q be as above and define P to be the transpose of P above, i.e. let

$$P = \frac{1}{3} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$R - SQ^{-1}P = R - P = \frac{2}{3} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

is the transpose of the matrix $R - SQ^{-1}P$ above and has the same set of eigenvalues $\{0, 2/3\}$. As in the previous case, this state space form is controllable but not observable, i.e. it is non-minimal, because $n_x = 2$, $\mathcal{C}$ is the same as above and

$$\mathcal{O} = \begin{pmatrix} P \\ PR \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

implies $\operatorname{rank} \mathcal{O} = 1$. Applying Proposition 4.1 one finds a minimal representation: take $\alpha = I_2$, $\beta' = (I_2, R)$, $\delta = \frac{1}{3}(-1, 1, -2/3, 2/3)'$, $\gamma = (1, 1)'$, and $\gamma_\perp = (1, -1)'$; then $\xi = \gamma_\perp$, $\mu = \gamma(\gamma'\gamma)^{-1/2} = \frac{1}{\sqrt{2}}(1, 1)'$ and one

finds

$$P_m = P\mu = \frac{\sqrt{2}}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad R_m = \mu' R\mu = \frac{2}{3}, \qquad S_m = \mu' S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The corresponding controllability and the observability matrices are $\mathcal{C}_m = S_m \neq 0$, $\mathcal{O}_m = P_m \neq 0$ so that $\operatorname{rank}\mathcal{C}_m = \operatorname{rank}\mathcal{O}_m = 1 = n_m$, i.e. (6) is a minimal representation of (1). Finally,

$$R_m - S_m Q^{-1} P_m = \frac{2}{3} - \frac{1}{3}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{2}{3}$$

is not nilpotent. Again here the unimodularity condition is violated because of the eigenvalue at 2/3. However, it is now this eigenvalue, which is due to $R_m - S_m Q^{-1} P_m = 2/3$, to be relevant and hence to determine the non-existence of the finite order VAR representation. The irrelevant eigenvalue is now at 0 and it is cancelled by the transformation $x_{m,t} = \mu' x_t$. Because the eigenvalues of $R - SQ^{-1}P$ are the same in the two cases one cannot disentangle one from the other, and hence conclude that a VAR representation does not exist because the unimodularity condition fails, unless he identifies which eigenvalues are relevant and which are irrelevant. Transforming the system to a minimal representation allows one to do that.

6. Application to the Smets and Wouters (2007) model

This section applies the analysis developed above to the Smets and Wouters (2007) model.[2] We consider the version of the model implemented in Iskrev (2010), to which we refer for a detailed description of the equations.[3] The model has 14 endogenous variables and 7 economic shocks; the endogenous variables are output (y_t), consumption (c_t), investment (i_t), utilized and installed capital (k_t^s, k_t), capacity utilization (z_t), rental rate of capital (r_t^k), Tobin's q (q_t), price and wage markup (μ_t^p, μ_t^w), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). The economic shocks are innovations to total factor productivity (η_t^a), investment-specific technology (η_t^I), government purchases (η_t^g), risk premium (η_t^b), price and wage markup (η_t^p, η_t^w), and monetary policy (η_t^r).

The MATLAB implementation of the model is as in the replication programs associated to Iskrev (2010) and consists of a system of 40 equations framed into the Sims (2002) setup. For a given value of the parameters, the model is solved via GENSYS.M;[4] this delivers the $n_x \times n_x$ matrix A and the $n_x \times n_\varepsilon$ matrix B in $x_t = Ax_{t-1} + B\varepsilon_t$, where $n_x = 40$ and $n_\varepsilon = 7$. The number of observable variables n_y is set equal to the number of economic shocks, $n_y = n_\varepsilon = 7$, and the vector y_t is defined by selecting the entries of x_t that correspond to output (y_t), consumption (c_t), investment (i_t), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). This is done by letting $y_t = Hx_t$, where H is an $n_y \times n_x$

[2]The replication programs can be downloaded from http://w3.uniroma1.it/mfranchi/.

[3]The log-linearized equilibrium conditions of the model can be found in Table 1 and Table 2 in Iskrev (2010).

[4]The replication programs associated to Iskrev (2010) are available at the JME Science Direct web page. The same code is employed for the analysis of the Smets and Wouters (2007) model in the supplementary material of Komunjer and Ng (2011). GENSYS.M can be downloaded from http://sims.princeton.edu/yftp/gensys/.

selection matrix of 0s and 1s. In this way one finds $y_t = Cx_{t-1} + D\varepsilon_t$, where $C = HA$ and $D = HB$ have respectively dimension $n_y \times n_x$ and $n_\varepsilon \times n_\varepsilon$. Thus the solution of the model takes the ABCD form

$$\begin{aligned} x_t &= Ax_{t-1} + B\varepsilon_t \\ y_t &= Cx_{t-1} + D\varepsilon_t, \end{aligned} \tag{7}$$

where $n_x = 40$ and $n_y = n_\varepsilon = 7$, and fits into the framework of (1).

First we set the values of the parameters at their prior means, see Table 3 in Iskrev (2010). The resulting system (7) is non-minimal because it is non-controllable and non-observable; indeed, the controllability and observability matrices $\mathcal{C} = (B, AB, \cdots, A^{n_x-1}B)$ and $\mathcal{O} = (C', A'C', \cdots, A'^{n_x-1}C')'$ are such that $\operatorname{rank}\mathcal{C} = 17 < 40 = n_x$ and $\operatorname{rank}\mathcal{O} = 16 < 40 = n_x$. Reduction to minimality leaves unaffected y_t and ε_t and delivers A_m, B_m and C_m such that $x_{m,t} = A_m x_{m,t-1} + B_m\varepsilon_t$, $y_t = C_m x_{m,t-1} + D\varepsilon_t$ is a minimal representation of original ABCD form.[5] The matrices A_m, B_m and C_m have respectively dimension $n_m \times n_m$, $n_m \times n_\varepsilon$ and $n_y \times n_m$, where $n_m = 14$ is the dimension of the minimal state vector. When setting the values of the parameters at their posterior means, see Table 3 in Iskrev (2010), again one finds that the corresponding system (7) is non-controllable and non-observable and hence non-minimal. In this case, $\operatorname{rank}\mathcal{C} = 19 < 40 = n_x$, $\operatorname{rank}\mathcal{O} = 16 < 40 = n_x$ and the dimension of the minimal state vector turns out to be $n_m = 16$. These results are reported in the next table.

Parameters set at	Prior means	Posterior means
Dimension of the original state	40	40
Rank of the controllability matrix	17	19
Rank of the observability matrix	16	16
Original state minimal	No	No
Dimension of the minimal state	14	16

TABLE 1. Prior and posterior means as in Table 3 in Iskrev (2010).

For both choices of parameters values, the system is non-minimal because it is both non-controllable and non-observable and hence both situations are instances of the fourth case described below Proposition 4.1. Moreover, the dimension of the minimal state is in both cases less than the rank of the controllability matrix, $n_m < r_{\mathcal{C}}$, so that both situations present states that are non-controllable and non-observable, see the $\xi \neq 0$ subcase described below Proposition 4.1. Indeed, when the parameters are set at the prior means there are $r_\xi = r_{\mathcal{C}} - n_m = 17 - 14 = 3$ states that are controllable and non-observable and one can reduce the dimension of the state vector from $n_x = 40$ to $n_m = 14$ by eliminating $n_x - n_m = 40 - 14 = 26$ non-controllable and non-observable states. Similarly, when the parameters are set at the posterior means there are $r_\xi = r_{\mathcal{C}} - n_m = 19 - 16 = 3$ states that are non-controllable and non-observable and one can reduce the dimension of the state vector from $n_x = 40$ to $n_m = 16$ by eliminating $n_x - n_m = 40 - 16 = 24$ non-controllable

[5]Reduction to minimality is performed using the MATLAB CONTROL SYSTEM TOOLBOX.

and non-observable states. This additional linear dependence among equations causes non-identificability of the model, as discussed in the supplementary material of Komunjer and Ng (2011).

Next we compute $A - BD^{-1}C$ and $A_m - B_m D^{-1} C_m$ for both choices of parameters values; the next table reports the eigenvalues of the two matrices for each choice of parameters values.

Parameters set at	Prior means		Posterior means	
	$A - BD^{-1}C$	$A_m - B_m D^{-1} C_m$	$A - BD^{-1}C$	$A_m - B_m D^{-1} C_m$
Dimension of the matrix	40×40	14×14	40×40	16×16
# of non-zero eigenvalues	6	4	6	6
# of zero eigenvalues	34	10	34	10
$\lvert\lambda_1\rvert$	0.97728	0.97728	0.97716	0.97716
$\lvert\lambda_2\rvert$	0.97728	0.97728	0.96358	0.96358
$\lvert\lambda_3\rvert$	0.77493	0.77493	0.84080	0.84080
$\lvert\lambda_4\rvert$	0.50000		0.54316	0.54316
$\lvert\lambda_5\rvert$	0.50000		0.54316	0.54316
$\lvert\lambda_6\rvert$	0.41080	0.41080	0.42763	0.42763

TABLE 2. Prior and posterior means as in Table 3 in Iskrev (2010). Zero eigenvalues have absolute value less than $1e - 07$.

The $A - BD^{-1}C$ matrices have a total of 40 eigenvalues; for both choices of parameters values, 34 of them are equal to zero and 6 are non-zero. The moduli of the non-zero eigenvalues are reported in decreasing order in Table 2. Because the original state space representations are non-minimal, one needs to identify which eigenvalues are relevant and which are irrelevant before concluding for the non-existence of a VAR representation. As illustrated in Section 5, this can be done by computing the eigenvalues of the matrices $A_m - B_m D^{-1} C_m$ in the minimal representations. The results are reported in Table 2. First consider the case in which parameters are set at the prior means; the 40 eigenvalues of $A - BD^{-1}C$ are divided into a group of 14 relevant eigenvalues, those that belong to $A_m - B_m D^{-1} C_m$, and a group of 26 irrelevant eigenvalues, all the remaining ones, see Proposition 5.1. Because not each relevant eigenvalue is equal to 0, one concludes that the observables do not admit a finite order VAR representation. It is interesting to observe that $\lambda_4 = \lambda_5 = 0.5$ are absent from the set of eigenvalues of $A_m - B_m D^{-1} C_m$ and are thus irrelevant. Hence the set of irrelevant eigenvalues of $A - BD^{-1}C$ is composed of $34 - 10 = 24$ zero eigenvalues and $6 - 4 = 2$ non-zero eigenvalues.

When parameters are set at the posterior means, $A - BD^{-1}C$ has 16 relevant eigenvalues, those that belong to $A_m - B_m D^{-1} C_m$, and 24 irrelevant eigenvalues, all the remaining ones. Again here not each relevant eigenvalue is equal to 0, and hence one concludes that the observables do not admit a finite order VAR representation. Finally note that in this case the set of irrelevant eigenvalues of $A - BD^{-1}C$ is composed of $34 - 10 = 24$ zero eigenvalues and $6 - 6 = 0$ non-zero eigenvalues.

7. Conclusion

In the present paper we have shown that minimality of the state space representation of a solution of a DSGE matters if one wants to determine whether the observables of the DSGE admit a VAR representation. In particular, we have first provided a counterexample which shows that, unlike stated in Proposition 2.1 and Corollary 2.2 in Ravenna (2007), if the state space representation of the solution is non-minimal, the observables of a DSGE may admit a finite order VAR representation even though the unimodularity condition fails. This implies that the class of DSGE models that admit a finite order VAR representation is larger than the one described by that condition. It is further shown that if the state space representation of the solution is minimal, then the unimodularity condition is necessary. Given that a minimal state space representation always exists, before applying the unimodularity condition one simply needs to check whether the state space representation of the solution of the DSGE is minimal and if not transform it into an equivalent minimal form. A discussion of how to perform such reduction is presented. The results are illustrated via the counterexample and then applied to the Smets and Wouters (2007) model. Similar results apply to the poor man's invertibility condition in Fernández-Villaverde et al. (2007).

Appendix A. Proofs

Proof of Proposition 2.2. Let $F = R - SQ^{-1}P$; the proof consists in showing that $F(z) = I_{n_x} - Fz$ is unimodular if and only if F is nilpotent. Observe that $\lambda_0 \neq 0$ is an eigenvalue of F if and only if $z_0 = \lambda_0^{-1}$ is a root of $|I_{n_x} - Fz| = 0$. We next show that if the eigenvalues of F are all equal to zero, then $F(z)$ is unimodular. Suppose that this is not the case, namely that there exists $z_0 \neq 0$ such that $|F(z_0)| = 0$. Because $I_{n_x} - Fz_0 = (-z_0)(F - z_0^{-1}I_{n_x})$, one has $|F - z_0^{-1}I_{n_x}| = 0$, i.e. $\lambda_0 = z_0^{-1} \neq 0$ is an eigenvalue of F. This contradicts the hypothesis and hence $F(z)$ must be unimodular. Similarly one proves that if $F(z)$ is unimodular, then the eigenvalues of F are all equal to zero. ■

Proof of Proposition 3.1. Write (2) as

$$\begin{aligned} y_t &= Px_{t-1} + u_t, & u_t &= Qz_t, \\ x_t &= Fx_{t-1} + SQ^{-1}y_t, & F &= R - SQ^{-1}P, \\ u_t &= \textstyle\sum_{i=1}^{q} U_i u_{t-i} + Q\varepsilon_t, & U_i &= QZ_iQ^{-1}. \end{aligned}$$

Write the second equation as $x_t = (I_{n_x} - FL)^{-1}SQ^{-1}y_t$, lag it and substitute it into the first equation to find $(I_{n_y} - P(I_{n_x} - FL)^{-1}SQ^{-1}L)y_t = u_t$; from the third equation one has $u_t = U(L)^{-1}Q\varepsilon_t$, where $U(L) = I_{n_\varepsilon} - \sum_{i=1}^{q} U_i L^i$, and one then finds

$$U(L)V(L)y_t = Q\varepsilon_t, \qquad V(L) = (I_{n_y} - P(I_{n_x} - FL)^{-1}SQ^{-1}L).$$

Because $U(L)$ is a polynomial, this shows that y_t admits a finite order VAR representation if and only if $V(L)$ is a polynomial. (*Suff.*) If $|I_{n_x} - FL| = c \neq 0$, then $(I_{n_x} - FL)^{-1}$ is a polynomial and hence the same holds for $V(L)$ irrespectively of minimality of (1). (*Nec.*) If $V(L)$ is a polynomial, then the same must hold for

$P(I_{n_x} - FL)^{-1}S$. Because in minimal systems P and S cannot cancel the poles of $(I_{n_x} - FL)^{-1}$, see Franchi and Vidotto (2013) for the proof, it must be that $(I_{n_x} - FL)^{-1}$ is a polynomial, i.e. $|I_{n_x} - FL| = c \neq 0$. ■

Proof of Proposition 4.1. The result is found by applying Kalman's decomposition theorem, see e.g. Section 6.2.3 in Antsaklis and Michel (2007). In order to do so, rank decompose $\mathcal{C}$ and $\mathcal{O}$ in (4) as $\mathcal{C} = \alpha\beta'$ and $\mathcal{O} = \delta\gamma'$, where α and γ are respectively $n_x \times r_{\mathcal{C}} = \operatorname{rank}\mathcal{C} \leq n_x$ and $n_x \times r_{\mathcal{O}} = \operatorname{rank}\mathcal{O} \leq n_x$ full column rank matrices that span the column and the row spaces of $\mathcal{C}$ and $\mathcal{O}$ respectively. Let $\gamma_\perp$ be a basis of the orthogonal complement of the space spanned by γ, i.e. $\gamma_\perp$ is a $n_x \times (n_x - r_{\mathcal{O}})$ full column rank matrix such that $\gamma_\perp'\gamma = 0$. If $r_{\mathcal{O}} = n_x$, let $\gamma_\perp = 0$. Next consider the intersection of the space spanned by α and that spanned by $\gamma_\perp$ and let ξ be a basis of this intersection. Because ξ belongs to the intersection, it belongs to $\operatorname{span}\alpha$ and to $\operatorname{span}\gamma_\perp$. Hence one can find μ orthogonal to ξ so that (μ, ξ) is a basis of $\operatorname{span}\alpha$ and ζ orthogonal to ξ so that (ζ, ξ) is a basis of $\operatorname{span}\gamma_\perp$. Further one can normalize μ, ξ and ζ so that $\mu'\mu = I_{n_m}$, $\xi'\xi = I_{n_\xi}$ and $\zeta'\zeta = I_{n_\zeta}$. Finally define ψ such that $M = (\mu, \xi, \psi, \zeta)$ is square and non-singular and normalize ψ so that $\psi'\psi = I_{n_\psi}$. Observe that $M'M = I_{n_x}$, i.e. $M^{-1} = M'$; then, see Theorem 6.6 in Antsaklis and Michel (2007, p.245),

$$\widehat{R} = M'RM = \begin{pmatrix} \mu'R\mu & 0 & \mu'R\psi & 0 \\ \xi'R\mu & \xi'R\xi & \xi'R\psi & \xi'R\zeta \\ 0 & 0 & \psi'R\psi & 0 \\ 0 & 0 & \zeta'R\psi & \zeta'R\zeta \end{pmatrix}, \qquad \widehat{S} = M'S = \begin{pmatrix} \mu'S \\ \xi'S \\ 0 \\ 0 \end{pmatrix},$$

$$\widehat{P} = PM = \begin{pmatrix} P\mu & 0 & P\psi & 0 \end{pmatrix};$$

hence $P_m = P\mu$, $R_m = \mu'R\mu$, $S_m = \mu'S$, and $x_{m,t} = \mu'x_t$ are such that (6) is a minimal representation of (1). Finally note that $PR^hS = P_mR_m^hS_m$; in fact $PR^hS = PMM'R^hMM'S = \widehat{P}M'R^hM\widehat{S}$ and $R^h = RR\cdots R = RMM'RMM'\cdots MM'R$ imply $PR^hS = \widehat{P}\widehat{R}^h\widehat{S}$ and

$$\widehat{P}\widehat{R}^h\widehat{S} = \begin{pmatrix} P_mR_m^h & 0 & * & 0 \end{pmatrix} \begin{pmatrix} S_m \\ \xi'S \\ 0 \\ 0 \end{pmatrix} = P_mR_m^hS_m.$$

Similarly, because $PF^hS = \widehat{P}\widehat{F}^h\widehat{S}$, where $F = R - SQ^{-1}P$, and

$$\widehat{F} = M'FM = \widehat{R} - \widehat{S}Q^{-1}\widehat{P} = \begin{pmatrix} F_{\mu\mu} & 0 & F_{\mu\psi} & 0 \\ F_{\xi\mu} & R_{\xi\xi} & F_{\xi\psi} & R_{\xi\zeta} \\ 0 & 0 & R_{\psi\psi} & 0 \\ 0 & 0 & R_{\zeta\psi} & R_{\zeta\zeta} \end{pmatrix}, \tag{8}$$

where the shorthand notation $F_{jk} = j'Fk$ and $R_{jk} = j'Rk$ for $j, k = \mu, \xi, \psi, \zeta$ is employed, one finds $PF^hS = P_mF_m^hS_m$, having defined $F_m = F_{\mu\mu} = \mu'(R - SQ^{-1}P)\mu = R_m - S_mQ^{-1}P_m$. ■

Proof of Proposition 5.1. Let M, $\widehat{R}$, $\widehat{S}$, $\widehat{P}$, F, and $\widehat{F}$ be as in the proof of Proposition 4.1 and observe that $F - \lambda I_{n_x} = MM'(F - \lambda I_{n_x})MM' = M(\widehat{F} - \lambda I_{n_x})M'$ implies $|F - \lambda I_{n_x}| = |\widehat{F} - \lambda I_{n_x}|$. Hence, see (8),

one has

$$|F - \lambda I_{n_x}| = |\widehat{F} - \lambda I_{n_x}| = |F_m - \lambda I_{n_m}||R_{\xi\xi} - \lambda I_{n_\xi}||R_{\psi\psi} - \lambda I_{n_\psi}||R_{\zeta\zeta} - \lambda I_{n_\zeta}|.$$

This shows that the eigenvalues of F are the union of the eigenvalues of F_m and those of R_{jj}, $j \neq \mu$, so that if λ_i is an eigenvalue of F_m then it is also an eigenvalue of F. The second part of the statement follows from the fact that (1) is minimal if and only if $\mu = I_{n_x}$, see Proposition 4.1. ■

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