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ABSTRACT. Minimality of the state space representation of a stochastic process places restrictions on the rank of certain matrices that show up in the leading coefficient of the principal part of the MA transfer functions implied by the system. When unit roots are allowed for, those restrictions and the reduced rank structure of the state process shape the integration and cointegration properties of the state and the observed processes. A characterization of cointegration is presented in the $I(d)$ case and it is further found that the present results lead to a construction of the canonical form in Bauer and Wagner (2012) *Econometric Theory*, 28, 1313-49.

1. INTRODUCTION

The structure of (A, B, C) in the state space system $x_{t+1} = Ax_t + B\varepsilon_t$, $y_t = Cx_t + \varepsilon_t$ plays a central role in determining the properties of x_t and y_t and given that the dynamics of the system is fully described by the state equation, the Jordan structure of A is crucial in this respect. Moreover, the triple (A, B, C) shapes the transfer functions which dictate how the state x_t, the observable y_t and the white noise process ε_t are connected, see Hannan and Deistler (1988) for the relations between various system representations in the stationary case.

The present paper provides a characterization of the link between the structure of (A, B, C) at each nonzero eigenvalue of A and the behaviour of the transfer functions around the corresponding singular points. The linear combinations that lead to a pole cancellation are fully described and when the results are specialized for unit modulus eigenvalues, these linear combinations are shown to characterize the (multiple-frequency) cointegration properties of x_t and y_t in the general $I(d)$ case. These results are achieved by combining the rank restriction implied by minimality of the

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state space representation with those in the extended local rank factorization (ELRF) developed in Franchi and Paruolo (2011, 2016), which is a recursive procedure that enables to construct the local Smith form and extended canonical systems of root functions of a generic matrix function, see Gohberg et al. (1993) for definitions and properties, and is here applied to the matrix pencil $I - Az$.

The present results are closely related to the ones in Bauer and Wagner (2012), who prove the existence of a canonical form for processes with unit roots with integer integration orders at arbitrary frequencies and use it to describe the cointegration properties of the system. These are characterized via orthogonality conditions in potentially large systems of equations defined by stacking the matrices of the canonical form. The connections with their analysis are investigated and it is shown that the ELRF provides to a construction of their canonical form. However, the characterization of cointegration presented here avoids stacking and it may thus be preferred when the number of observables is large, as in the case of Dynamic Factor Models considered in Barigozzi et al. (2016a,b).

The results in the present paper complement the literature on the representation theory of cointegration in the state space framework, see Aoki (1990) and Aoki and Havenner (1989, 1991) for the $I(1)$ case, and in the ARMA framework, which is far more numerous. The first result of the kind is the celebrated Granger Representation Theorem in Engle and Granger (1987), which allows to derive the infinite lag Error Correction form of an $I(1)$ process starting from its MA form. In the same MA framework Phillips (1991) introduced the Triangular Representation of $I(1)$ processes and this was subsequently generalised to $I(d)$ systems by Stock and Watson (1993). Still starting from the MA form of an $I(1)$ process an approach based on the Smith form was presented in Yoo (1986); this was further extended to the case of $I(2)$ systems in Engle and Yoo (1991) and Haldrup and Salmon (1998) and to seasonal roots in Hylleberg et al. (1990).

In a parallel strand of literature, the so-called cointegrated VAR literature, Johansen (1991) derived conditions under which a VAR process is $I(1)$ and the Granger Representation Theorem holds. A generalization to $I(2)$ and to seasonal roots was given in Johansen (1992) and Johansen and Schaumburg (1998) respectively, and the case of generic $I(d)$ processes in the AR framework was considered in D'Autume (1992), la Cour (1998) and Franchi (2007, 2010). See Franchi and Paruolo (2017) for a general inversion theorem that links the different approaches.

More recently, Barigozzi et al. (2016a,b) obtained a version of the Granger Representation Theorem for $I(1)$ Dynamic Factor Models with singular factors and Deistler and Wagner (2017) showed

that the cointegration properties of singular $I(1)$ ARMA processes only depend upon the autoregressive polynomial at one. The results presented here apply to Dynamic Factor Models with singular factors as well, since they can be written in state space form.

The rest of the paper is organized as follows: the remaining part of this introduction reports notational conventions and preliminaries, Section 2 presents rank restrictions implied by minimality and relates them to the order of integration of the system while Section 3 describes the ELRF and the subsequent pole cancellations that characterize cointegration. Section 4 provides a description of the cointegration structure and a numerical illustration of the results, Section 5 discusses the connections with Bauer and Wagner (2012) and Section 6 concludes. Appendix A contains proofs.

Notation. The techniques presented in the paper make repeated use of projections and rank factorizations, whose notation is introduced here. Let x be a $n \times r$ full column rank matrix; $x_\perp$ indicates a $n \times n-r$ full column rank matrix that forms a basis of $\operatorname{span}^\perp x$, the orthogonal complement of $\operatorname{span} x$. $P_x := \bar{x}x' = x\bar{x}'$, where $\bar{x} := x(x'x)^{-1}$, is the orthogonal projection onto $\operatorname{span} x$ and $P_{x_\perp} := I - P_x = \bar{x}_\perp x'_\perp = x_\perp \bar{x}'_\perp$ is the orthogonal projection onto $\operatorname{span}^\perp x$. Given a $n \times q$ matrix φ of rank $0 < r < \min(n, q)$, its rank factorization is written as $\varphi = c\alpha\beta'$, where α and β are $n \times r$, $q \times r$ full column rank matrices that respectively form a basis of the column space and of the row space of φ and $c \in \mathbb{R}$ is chosen for convenience in the calculations. When $r = 0$, i.e. $\varphi = 0$, set $\alpha = \beta = \bar{\alpha} = \bar{\beta} = 0$ and $\alpha_\perp = \beta_\perp = \bar{\alpha}_\perp = \bar{\beta}_\perp = I$ of the appropriate dimension. When $r = q$, i.e. φ has full column rank, set $\beta = I_q$ and $\beta_\perp = \bar{\beta}_\perp = 0$ and when $r = n$, i.e. φ has full row rank, set $\alpha = I_n$ and $\alpha_\perp = \bar{\alpha}_\perp = 0$.

2. Rank restrictions in minimal state space systems and order of integration

Consider the minimal state space system

$$\begin{aligned} x_{t+1} &= Ax_t + B\varepsilon_t, & A &\in \mathbb{R}^{n_x \times n_x}, & B &\in \mathbb{R}^{n_x \times n_y}, \\ y_t &= Cx_t + \varepsilon_t, & C &\in \mathbb{R}^{n_y \times n_x}, \end{aligned} \tag{2.1}$$

where the observed process y_t has dimension $n_y \times 1$, the state process x_t has dimension $n_x \times 1$ and the white noise process ε_t has dimension $n_y \times 1$ and positive definite covariance matrix Ω.

Remark that no restriction is involved in assuming that (2.1) is minimal; in fact, see Theorem 2.3.1 in Hannan and Deistler (1988), the class of observationally equivalent state space realizations of a given covariance structure always contains a representation with minimal dimension n_x and such a representation can always be constructed via the Kalman's decomposition theorem, see e.g. Antsaklis and Michel (2007, Section 6.2.3).

A well known necessary and sufficient condition for minimality of (2.1), see Theorem 6.2-3 in Kailath (1980), is that the couple (A, B) is controllable and the couple (A, C) is observable, i.e.

$$\text{rank}(B, AB, \dots, A^{n_x-1}B) = \text{rank}(C', A'C', \dots, (A^{n_x-1})'C') = n_x.$$

An equivalent characterization of controllability, observability and minimality is presented next. This is based on the Popov-Belovich-Hautus (PBH) rank tests, see Theorem 6.2-6 in Kailath (1980).

Theorem 2.1 (Rank restrictions in minimal state space systems)**.** *Let $\sigma(A)$ be the set of distinct eigenvalues of A; for each $\lambda_u \in \sigma(A)$, let $r_0 := \text{rank}(A - \lambda_u I)$ and consider the rank factorization $A - \lambda_u I = \alpha_0 \beta_0'$. Then*

i) λ_u is controllable if and only if $\alpha_{0\perp}' B$ of dimension $n_x - r_0 \times n_y$ has full row rank;

ii) λ_u is observable if and only if $C\beta_{0\perp}$ of dimension $n_y \times n_x - r_0$ has full column rank;

iii) (2.1) is minimal if and only if $\text{rank}(\alpha_{0\perp}' B) = \text{rank}(C\beta_{0\perp}) = n_x - r_0$ for any $\lambda_u \in \sigma(A)$.

The $n_x \times n_x - r_0$ matrices $\alpha_{0\perp}, \beta_{0\perp}$ are respectively bases of the left and right eigenspaces of A associated to the eigenvalue λ_u; in fact $\alpha_{0\perp}'(A - \lambda_u I) = \alpha_{0\perp}'\alpha_0\beta_0' = 0$ shows that any left eigenvector of A that corresponds to λ_u lies in the span of the columns of $\alpha_{0\perp}$ and similarly for $(A - \lambda_u I)\beta_{0\perp} = \alpha_0\beta_0'\beta_{0\perp} = 0$. Hence Theorem 2.1 highlights the relation between B and the left eigenspace of A for controllability and between C and the right eigenspace of A for observability.[1]

The MA representations of x_t and of y_t implied by (2.1) are

$$\begin{array}{ll} x_t = T_{x,\varepsilon}(L)\varepsilon_t & T_{x,\varepsilon}(z) := (I - Az)^{-1}Bz \\ y_t = T_{y,\varepsilon}(L)\varepsilon_t & T_{y,\varepsilon}(z) := I + C(I - Az)^{-1}Bz \end{array}, \qquad z \in \mathbb{C}, \tag{2.2}$$

and involve $(I - Az)^{-1}$, which has a pole of some order $d \in \mathbb{N}_+ := \{1, 2, \dots\}$ at the reciprocal of each nonzero eigenvalue of A, $z_u := \lambda_u^{-1}$, $0 \neq \lambda_u \in \sigma(A)$. In fact, for any $0 \neq \lambda_u \in \sigma(A)$, $A - \lambda_u I = -\lambda_u(I - Az_u)$ is non-invertible and hence $(I - Az)^{-1}$ has a singularity at z_u. Because $(I - Az)^{-1}$ is a rational function, the singularity is a pole and its Laurent representation around at z_u has the form

$$(I - Az)^{-1} =: (1 - \lambda_u z)^{-d}B(z) = \sum_{n=0}^{\infty} B_n(1 - \lambda_u z)^{n-d}, \qquad B_0 \neq 0, \qquad |B_0| = 0. \tag{2.3}$$

[1]For the Generalized Dynamic Factor Model $y_t = \Lambda f_t + \xi_t$, $A(L)f_t = Ru_t$, minimality places restrictions on the number n_u of shocks u_t and the number n_y of observables y_t. In fact Theorem 2.1 implies that an eigenvalue λ_u of the companion matrix A in $\widetilde{f}_t = A\widetilde{f}_{t-1} + \widetilde{R}u_t$ is controllable only if $n_u \geq n_f - q$ and it is observable only if $n_y \geq n_f - q$, where n_f is the number of factors f_t and $q = \text{rank}\, A(\lambda_u^{-1})$.

Note that $(I-Az)^{-1}$ is expanded around z_u, $B(z_u) = B_0 \neq 0$ is singular and the first d coefficients $\{B_n\}_{n=0}^{d-1}$ characterize the singularity of $(I - Az)^{-1}$ at $z = z_u$. In the following, $\{B_n\}_{n=0}^{\infty}$ are called the Laurent coefficients.

The next proposition states that the poles of the MA transfer functions in (2.2) coincide with those of $(I - Az)^{-1}$.

Theorem 2.2 (No pole cancellations in the MA transfer functions)**.** *Let $z_u := \lambda_u^{-1}$, $0 \neq \lambda_u \in \sigma(A)$, and consider the rank factorization $A - \lambda_u I = \alpha_0 \beta_0'$. Then $B(z_u) = \beta_{0\perp} \phi \alpha_{0\perp}' \neq 0$ for some ϕ and*

$$T_{x,\varepsilon}(z) = \frac{B(z)Bz}{(1-\lambda_u z)^d}, \qquad T_{y,\varepsilon}(z) = I + \frac{CB(z)Bz}{(1-\lambda_u z)^d}, \qquad CB(z_u)B = C\beta_{0\perp}\phi\alpha_{0\perp}'B \neq 0,$$

have a pole of some order $d \in \mathbb{N}_+$ at z_u.

Note that in principle the presence of B or C could imply $\phi\alpha_{0\perp}'B = 0$ or $C\beta_{0\perp}\phi = 0$ and hence change the order of the pole in the MA transfer functions. However, because in a minimal system $\alpha_{0\perp}'B$ and $C\beta_{0\perp}$ are full rank matrices, see Theorem 2.1, and $\phi \neq 0$ such pole cancellations arise only when the eigenvalue is non-controllable or non-observable, i.e. only in non-minimal representations.

As discussed in the next proposition, under the assumption that the largest eigenvalue of A is equal to 1, the order of the pole at $z = 1$ determines the order of integration of the state space system.[2]

Theorem 2.3 (Order of integration)**.** *Assume that $1 \in \sigma(A)$ and that $1 \neq \lambda_u \in \sigma(A)$ implies $|\lambda_u| < 1$ and let $d \in \mathbb{N}_+$ be the order of the pole of $(I - Az)^{-1}$ at $z = 1$. Then*

$$\Delta^d x_t = B(L)B\varepsilon_{t-1} \sim I(0), \qquad \Delta^d y_t = CB(L)B\varepsilon_{t-1} + \Delta^d \varepsilon_t \sim I(0), \qquad CB(1)B \neq 0,$$

i.e. $x_t \sim I(d)$ and $y_t \sim I(d)$.

Similarly, the order of integration of given transformations $\zeta_x(L)'x_t$ and $\zeta_y(L)'y_t$ of x_t and y_t coincides with the order of the pole of $\zeta_x(z)'T_{x,\varepsilon}(z)$ and $\zeta_y(z)'T_{y,\varepsilon}(z)$ at $z = 1$. For this reason, the characterization of the pole cancellations in the MA transfer functions presented in the next section leads to a full description of the cointegration structure of x_t and y_t, see Section 4 below.

[2]The definition of order of integration in Johansen (1996) is employed: a generic process u_t is integrated of order d if $\Delta^d u_t \sim I(0)$, i.e. $\Delta^d u_t = U(L)\varepsilon_t$, where ε_t is white noise, $U(z)$ is finite for all $z \in \mathbb{C} : |z| < 1 + \delta$ for some $\delta > 0$ and $U(1) \neq 0$.

3. Pole cancellations via extended local rank factorization

This section first describes the relation between the structure of the state matrix A and the order of the pole of $(I - Az)^{-1}$. This relation is fully characterized by the extended local rank factorization (ELRF) developed in Franchi and Paruolo (2011, 2016), which is a recursive procedure that enables to construct the Laurent coefficients, the local Smith form and extended canonical systems of root functions of a generic matrix function. These results are then used to characterize the pole cancellations in the MA transfer functions; when specialized for $z_u = 1$, these cancellations characterize the cointegration properties of x_t and y_t, see Section 4 below.

In the present context, the ELRF is performed on $I - Az$ at z_u; because $I - Az$ is a matrix polynomial of degree one, the results in Franchi and Paruolo (2016) simplify as follows.

Theorem 3.1 (ELRF and Laurent coefficients)**.** *Let $z_u := \lambda_u^{-1}$, $0 \neq \lambda_u \in \sigma(A)$, $r_0 := \operatorname{rank}(A - \lambda_u I)$, and consider the rank factorization $A - \lambda_u I = \alpha_0 \beta_0'$. Then $(I - Az)^{-1}$ has a pole of order $d \in \mathbb{N}_+$ at z_u if and only if*

$$\begin{cases} r_j < r_j^{\max} & (\textit{reduced rank condition}) \quad \textit{for } j = 1, \dots, d-1 \\ r_d = r_d^{\max} & (\textit{full rank condition}) \qquad \textit{for } j = d \end{cases},$$

where $r_j^{\max} := n_x - \sum_{i=0}^{j-1} r_i$, r_j is the rank of α_j and β_j in the rank factorization $P_{a_{j\perp}} Q_j P_{b_{j\perp}} = -z_u \alpha_j \beta_j'$ and

$$\begin{matrix} a_j := (\alpha_0, \dots, \alpha_{j-1}) \\ b_j := (\beta_0, \dots, \beta_{j-1}) \end{matrix}, \qquad Q_j := \begin{cases} Az_u & \textit{for } j = 1 \\ \lambda_u Q_{j-1} \sum_{i=0}^{j-2} \bar\beta_i \bar\alpha_i' Q_{i+1} & \textit{for } j = 2, \dots, d+1 \end{cases}.$$

Moreover, the Laurent coefficients $\{B_n\}_{n=0}^{\infty}$ in (2.3) satisfy

$$B_n = \begin{cases} -\lambda_u \bar\beta_d \bar\alpha_d' & \textit{for } n = 0 \\ \lambda_u (M B_{n-1} + N_n) & \textit{for } n = 1, \dots, d \\ \lambda_u M B_{n-1} & \textit{for } n = d+1, d+2, \dots \end{cases}, \qquad \begin{matrix} M := \sum_{j=0}^{d} \bar\beta_j \bar\alpha_j' Q_{j+1} \\ N_n := \sum_{j=d-n}^{d} \bar\beta_j \bar\alpha_j' C_{j+1,n} \end{matrix}, \quad (3.1)$$

where

$$C_{j+1,n} := \begin{cases} -1_{n=d} I & \textit{for } j = 0 \\ C_{j,n+1} + \lambda_u Q_j \sum_{i=d-n}^{j-1} \bar\beta_i \bar\alpha_i' C_{i+1,n} & \textit{for } j = 1, \dots, d \end{cases}, \qquad n = 0, \dots, d,$$

and $1_{\cdot}$ is the indicator function.

The ELRF is defined by the sequence of calculations in Theorem 3.1 and delivers the output[3]

$$d \qquad \text{and} \qquad \{\alpha_j, \beta_j, r_j, Q_{j+1}, C_{j+1,n}\}_{j=0,\dots,d,\, n=0,\dots,d}. \quad (3.2)$$

[3]When j is different from 0 or d, r_j (and thus α_j and β_j) can be equal to 0; in what follows, every statement implicitly assumes that they are nonzero, because the modifications required in the case $r_j = 0$ are straightforward.

While d and $\{r_j\}_{j=0}^{d}$ do not depend on the choice of bases in the rank factorizations and are thus uniquely defined, $\{\alpha_j, \beta_j, Q_{j+1}, C_{j+1,n}\}_{j=0,\ldots,d,\ n=0,1,\ldots}$ depend on such choices. In any case, i.e. for any choice of bases in the rank factorizations, the $n_x \times n_x$ matrices $(\alpha_0, \ldots, \alpha_d)$ and $(\beta_0, \ldots, \beta_d)$ have mutually orthogonal components, namely $\alpha_h'\alpha_j = \beta_h'\beta_j = 0$, $h \neq j$, and deliver bases of $\mathbb{C}^{n_x}$ (or of $\mathbb{R}^{n_x}$, when $\lambda_u \in \mathbb{R}$), so that the orthogonal projection identities $I_{n_x} = \sum_{j=0}^{d} P_{\alpha_j} = \sum_{j=0}^{d} P_{\beta_j}$ hold. Moreover, remark that minimality implies

$$\operatorname{rank}(\alpha_j'B) = \operatorname{rank}(C\beta_j) = r_j, \quad j = 1, \ldots, d, \tag{3.3}$$

i.e. that $\alpha_j'B$ has full row rank and $C\beta_j$ has full column rank for $j = 1, \ldots, d$. This follows from $\operatorname{rank}(\alpha_{0\perp}'B) = \operatorname{rank}(C\beta_{0\perp}) = n_x - r_0$, see Theorem 2.1, and from $\alpha_{0\perp} = (\alpha_1, \ldots, \alpha_d)$ and $\beta_{0\perp} = (\beta_1, \ldots, \beta_d)$. Furter note that minimality does not play a role in determining the rank of $\alpha_0'B$ and $C\beta_0$.

The necessary and sufficient condition for a pole of order d given in Theorem 3.1 is stated recursively in terms of $d+1$ rank restrictions on functions of sub-blocks of A, found by projecting its column and row spaces into appropriate subspaces, $P_{a_{j\perp}} Q_j P_{b_{j\perp}}$: the first $j = 1, \ldots, d$ conditions are reduced rank restrictions that establish that the order of the pole is greater than $j-1$ and the last one is the full rank condition that establishes that the order of the pole is exactly d. For $d = 1$ the ELRF coincides with the rank condition in Theorem 3 of Howlett (1982) and the $I(1)$ condition in Theorem 4.1 in Johansen (1991) and for $d = 2$ with the $I(2)$ condition in Theorem 3 in Johansen (1992). The ELRF is thus a generalization of the approach in Howlett (1982) and Johansen (1991, 1992), in which the order of the pole (1 or 2 only) is established recursively by checking the rank of a sequence of matrices until a full rank condition is satisfied.

The second part of Theorem 3.1 gives a recursive expression of the Laurent coefficients $\{B_n\}_{n=0}^{\infty}$ in terms of the output of the ELRF: the leading coefficient $B_0 = -\lambda_u \bar{\beta}_d \bar{\alpha}_d'$ is the Moore-Penrose inverse of the last rank factorization in the ELRF, $P_{a_{d\perp}} Q_d P_{b_{d\perp}} = -z_u \alpha_d \beta_d'$, and the remaining ones are calculated recursively via (3.1). Pre-multiplying (3.1) by β_j', one finds

$$\beta_d'B_0 = -\lambda_u \bar{\alpha}_d', \quad \beta_j'B_n = \begin{cases} 0 & \text{for } n = 0 \\ \lambda_u \bar{\alpha}_j' Q_{j+1} B_{n-1} & \text{for } n = 1, \ldots, d-j-1 \\ \lambda_u \bar{\alpha}_j' Q_{j+1} B_{n-1} - \lambda_u \bar{\alpha}_j' & \text{for } n = d-j \end{cases}, \quad j = 0, \ldots, d-1.$$

These relations dictate the reductions in the order of the pole of $(I - Az)^{-1}$ that can be achieved by linear combinations. These are described in the next proposition, which states that knowledge of $\{r_j\}_{j=0}^{d}$ fully characterizes the (unique) local Smith form $\Lambda(z)$ of $I - Az$ at z_u while

$\{\alpha_j, \beta_j, Q_{j+1}, C_{j+1,n}\}_{j=0,\dots,d,\, n=0,\dots,d}$ enable to construct two (non-unique) extended canonical systems of root functions $\Gamma(z)$ and $\Psi(z)$, see Gohberg et al. (1993) for definitions and properties.

Theorem 3.2 (Local Smith factorization via ELRF)**.** *Given the output of the* ELRF *in* (3.2)*, for* $j = 0, \dots, d$ *define the* $r_j \times n_x$ *matrix functions* $\gamma_j(z)'$ *and* $\psi_j(z)'$ *as*

$$\gamma_j(z)' := \begin{cases} \beta_j' - \lambda_u \bar{\alpha}_j' Q_{j+1}(1 - \lambda_u z) & \text{for } j = 0, \dots, d-2 \\ \beta_j' & \text{for } j = d-1, d \end{cases} \tag{3.4}$$

and $\psi_j(z)' := -\bar{\alpha}_j' + \sum_{k=1}^{j} \bar{\alpha}_j' C_{j+1,d-j+k}(1 - \lambda_u z)^k$ *and the* $n_x \times n_x$ *matrix functions* $\Gamma(z)$*,* $\Lambda(z)$ *and* $\Psi(z)$ *as*

$$\Gamma(z) := \begin{pmatrix} \gamma_0(z)' \\ \vdots \\ \gamma_d(z)' \end{pmatrix}, \quad \Lambda(z) := \begin{pmatrix} (1-\lambda_u z)^0 I_{r_0} & & \\ & \ddots & \\ & & (1-\lambda_u z)^d I_{r_d} \end{pmatrix}, \quad \Psi(z) := \lambda_u \begin{pmatrix} \psi_0(z)' \\ \vdots \\ \psi_d(z)' \end{pmatrix}.$$

Then

$$\Psi(z)(I - Az) = \Lambda(z)\Gamma(z), \qquad |\Psi(z_u)| \neq 0, \qquad |\Gamma(z_u)| \neq 0, \tag{3.5}$$

i.e. $\Lambda(z)$ *is the local Smith form of* $I - Az$ *at* z_u *and* $\Psi(z), \Gamma(z)$ *are extended canonical systems of root functions.*

This shows that the distinct partial multiplicities of $I - Az$ at z_u coincide with $\{j : r_j > 0\}$ in the ELRF and that there are exactly r_j partial multiplicities equal to a given j. This characterizes the local Smith form and thus, see Corollary 4.4 in Franchi and Paruolo (2016), the Jordan form of A results

$$J = \begin{pmatrix} J_{\lambda_u} & \\ & J_{\sigma(A)\setminus\lambda_u} \end{pmatrix}, \qquad J_{\lambda_u} := \begin{pmatrix} I_{r_d} \otimes J_{\lambda_u,d} & & \\ & \ddots & \\ & & I_{r_1} \otimes J_{\lambda_u,1} \end{pmatrix}, \tag{3.6}$$

where J_{λ_u} is the Jordan structure of A that corresponds to the eigenvalue λ_u, $J_{\sigma(A)\setminus\lambda_u}$ collects the Jordan structure of the remaining eigenvalues and $J_{\lambda_u,j}$ is a Jordan block of dimension j with eigenvalue λ_u. Eq. (3.6) shows that there are exactly r_j Jordan blocks of dimension $j = 1, \dots, d$ in J_{λ_u}, which has thus dimension $\sum_{j=1}^{d} j r_j$. Moreover, using $\Gamma(z)$ or $\Psi(z)$ one can construct similarity transformations X such that $AX = XJ$, see Corollary 4.4 in Franchi and Paruolo (2016).

Since (3.5) implies $\Gamma(z)(I - Az)^{-1} = \Lambda(z)^{-1}\Psi(z)$, one has

$$\gamma_j(z)'(I - Az)^{-1} = \frac{\lambda_u \psi_j(z)'}{(1 - \lambda_u z)^j}, \qquad \psi_j(z_u)' = -\bar{\alpha}_j', \qquad j = 0, \dots, d, \tag{3.7}$$

which shows that the blocks $\gamma_j(z)'$ of r_j rows in $\Gamma(z) = (\gamma_0(z), \cdots, \gamma_d(z))'$ are root functions of order $d-j$ of $(I - Az)^{-1}$; moreover, because $\psi_j(z_u)'$ has full row rank, $v'\gamma_j(z)'(I - Az)^{-1}$ has pole of order j for any $0 \neq v \in \mathbb{C}^{n_x}$, and in this case the pole is said to be irreducible.

It is next shown that knowledge of $\Gamma(z)$, $\Lambda(z)$ and $\Psi(z)$ fully describes the pole cancellations in the MA transfer functions. Substituting $\varepsilon_t = y_t - Cx_t$ in the state equation and rearranging one finds the associate (AS) state space form

$$\begin{aligned} x_{t+1} &= Fx_t + By_t, \qquad F := A - BC \in \mathbb{R}^{n_x \times n_x}, \\ y_t &= Cx_t + \varepsilon_t, \end{aligned}$$

from which one finds the representation of x_t in terms of y_t, i.e.

$$x_t = T_{x,y}(L)y_t, \qquad T_{x,y}(z) := (I - Fz)^{-1}Bz.$$

Because $T_{x,\varepsilon}(z) = T_{x,y}(z)T_{y,\varepsilon}(z)$, see Theorem 2.1 in Bart et al. (2008), it follows from (3.7) that

$$\gamma_j(z)'T_{x,\varepsilon}(z) = \tilde{\xi}_j(z)'T_{y,\varepsilon}(z) = \frac{\lambda_u \psi_j(z)'Bz}{(1-\lambda_u z)^j}, \qquad \psi_j(z_u)'B = -\bar{\alpha}_j'B, \qquad j = 0, \ldots, d, \tag{3.8}$$

where $\tilde{\xi}_j(z)' := \gamma_j(z)'T_{x,y}(z)$, so that $\gamma_j(z)'T_{x,\varepsilon}(z)$ and $\tilde{\xi}_j(z)'T_{y,\varepsilon}(z)$ have an irreducible pole of order $j = 1, \ldots, d$ and no pole for $j = 0$ at z_u. Truncation of $\tilde{\xi}_j(z)'$ up to degree $\max(0, d-j-1)$ leads to the following result.

Theorem 3.3 (Pole cancellations in the MA transfer functions)**.** *Consider $\gamma_j(z)'$ in* (3.4) *and let $\xi_j(z)' = \sum_{n=0}^{\max(0,d-j-1)} \xi_{j,n}'(1-\lambda_u z)^n$ be the truncation of $\tilde{\xi}_j(z)' := \gamma_j(z)'T_{x,y}(z)$ up to degree $\max(0, d-j-1)$. Then $\gamma_j(z)'T_{x,\varepsilon}(z)$ and $\xi_j(z)'T_{y,\varepsilon}(z)$ have an irreducible pole of order $j = 1, \ldots, d$ and no pole for $j = 0$ at z_u and*

$$\begin{array}{lll} \xi_{j,n}' = \beta_j'T_n - \lambda_u \bar{\alpha}_j'Q_{j+1}T_{n-1}, & T_{-1} := 0, & T_0 := G_0Bz_u, \\ T_n := -G_0(I - G_0)^{n-1}T_0, & G_0 := (I - Fz_u)^{-1}, & F := A - BC. \end{array} \tag{3.9}$$

This shows that the matrix polynomial $\gamma_j(z)'$ (of degree 0 or 1) is a root function of order $d-j$ of the MA transfer function $T_{x,\varepsilon}(z)$ and the matrix polynomial $\xi_j(z)'$ (of degree $\max(0, d-j-1)$) is a root function of order $d-j$ of the MA transfer function $T_{y,\varepsilon}(z)$. Because of (3.5), this fully describes the pole cancellations in the MA transfer functions.

As discussed in the next section, under the assumption that the largest eigenvalue of A is equal to 1, the order of integration of $\gamma_j(L)'x_t$ and $\xi_j(L)'y_t$ is given by the order of the pole of $\gamma_j(z)'T_{x,\varepsilon}(z)$ and $\xi_j(z)'T_{y,\varepsilon}(z)$ at $z = 1$ and hence Theorem 3.3 provides a full description of the cointegration structure of x_t and y_t.

4. Cointegration structure

The following notation is employed: $u_t \sim I_{nc}(d)$ indicates that u_t is integrated of order d and it is non-cointegrated, i.e. $\Delta^d u_t = U(L)\varepsilon_t$, where ε_t is white noise, $U(z)$ is finite for all $z \in \mathbb{C} : |z| < 1+\delta$ for some $\delta > 0$ and $U(1) \neq 0$ has full row rank.

Theorem 4.1 (Cointegration structure)**.** *Assume that* $1 \in \sigma(A)$ *and that* $1 \neq \lambda_u \in \sigma(A)$ *implies* $|\lambda_u| < 1$*, let* (3.2) *be the output of the* ELRF *of* $I - Az$ *at* $z_u = \lambda_u = 1$ *and define* φ, ζ *by the rank factorization* $-\bar{\alpha}_0' B = \varphi\zeta'$*. Then* $x_t \sim I(d)$ *and* $y_t \sim I(d)$ *if and only if* $r_d = n_x - \sum_{i=0}^{d-1} r_i > 0$*,* $\Delta^d x_t = B(L)B\varepsilon_{t-1} \sim I(0)$*,* $\Delta^d y_t = CB(L)B\varepsilon_{t-1} + \Delta^d \varepsilon_t \sim I(0)$*, where* $B(1) = -\bar{\beta}_d \bar{\alpha}_d'$*, and*

$$\begin{array}{lll} \varphi'\gamma_0(L)'x_t \sim I_{nc}(0), & \varphi_\perp'\gamma_0(L)'x_t = 0, & \gamma_j(L)'x_t \sim I_{nc}(j), \\ \varphi'\xi_0(L)'y_t \sim I_{nc}(0), & \varphi_\perp'\xi_0(L)' = 0, & \xi_j(L)'y_t \sim I_{nc}(j), \end{array}, \qquad j = 1, \ldots, d,$$

where $\gamma_j(z)'$ *and* $\xi_j(z)'$ *are found by setting* $z_u = \lambda_u = 1$ *in* (3.4) *and* (3.9) *respectively.*

Henceq (2.1) displays $s_0 := \operatorname{rank} \bar{\alpha}_0' B \leq r_0$ relations that are $I_{nc}(0)$ and r_j relations that are $I_{nc}(j)$, $j = 1, \ldots, d$. These have the following expressions

$$\begin{array}{r} \varphi'(\beta_0'x_t - \bar{\alpha}_0'Q_1\Delta x_t) \sim I_{nc}(0) \\ \beta_1'x_t - \bar{\alpha}_1'Q_2\Delta x_t \sim I_{nc}(1) \\ \vdots \\ \beta_{d-2}'x_t - \bar{\alpha}_{d-2}'Q_{d-1}\Delta x_t \sim I_{nc}(d-2) \\ \beta_{d-1}'x_t \sim I_{nc}(d-1) \\ \beta_d'x_t \sim I_{nc}(d) \end{array}, \qquad \begin{array}{r} \varphi'(\beta_0'T_0y_t + \xi_{0,1}'\Delta y_t + \cdots + \xi_{0,d-1}'\Delta^{d-1}y_t) \sim I_{nc}(0) \\ \beta_1'T_0y_t + \xi_{1,1}'\Delta y_t + \cdots + \xi_{1,d-1}'\Delta^{d-2}y_t \sim I_{nc}(1) \\ \vdots \\ \beta_{d-2}'T_0y_t + \xi_{d-2,1}'\Delta y_t \sim I_{nc}(d-2) \\ \beta_{d-1}'T_0y_t \sim I_{nc}(d-1) \\ \beta_d'T_0y_t \sim I_{nc}(d) \end{array},$$

which highlight how the properties of the processes change in the different directions provided by the basis $(\beta_0, \ldots, \beta_d)$ and how the $Q_\cdot$ and the $T_\cdot$ coefficients in the output of the ELRF and in the associate transfer function $T_{x,y}(z)$ can be used to find the $\xi_{\cdot,\cdot}$ that deliver polynomial cointegration. Observe that $\beta_0'x_t - \bar{\alpha}_0'Q_1\Delta x_t = \varphi\zeta'\varepsilon_{t-1}$ is white noise and that the $r_0 - s_0$ relations in x_t and in y_t that are equal to 0 are of different type, because $\varphi_\perp'\beta_0'x_t = \varphi_\perp'\bar{\alpha}_0'Q_1\Delta x_t$ while $\varphi_\perp'\xi_{0,n}' = 0$ follows from the fact that the column spaces of the $\xi_{0,n}'$ coefficients belong to $\operatorname{span}\varphi$.

Finally note that for $d = 1$ Theorem 4.1 reads $x_t \sim I(1)$ and $y_t \sim I(1)$ if and only if $r_1 = n_x - r_0 > 0$, $\Delta x_t = B(L)B\varepsilon_{t-1} \sim I(0)$, $\Delta y_t = CB(L)B\varepsilon_{t-1} + \Delta\varepsilon_t \sim I(0)$, where $B(1) = -\bar{\beta}_1\bar{\alpha}_1'$, and

$$\begin{array}{lll} \varphi'\beta_0'x_t \sim I_{nc}(0), & \varphi_\perp'\beta_0'x_t = 0, & \beta_1'x_t \sim I_{nc}(1), \\ \varphi'\beta_0'T_0y_t \sim I_{nc}(0), & \varphi_\perp'\beta_0'T_0 = 0, & \beta_1'T_0y_t \sim I_{nc}(1), \end{array}$$

while for $d = 2$ one has $x_t \sim I(2)$ and $y_t \sim I(2)$ if and only if $r_2 = n_x - r_0 - r_1 > 0$, $\Delta^2 x_t = B(L)B\varepsilon_{t-1} \sim I(0)$, $\Delta^2 y_t = CB(L)B\varepsilon_{t-1} + \Delta^2\varepsilon_t \sim I(0)$, where $B(1) = -\bar\beta_2\bar\alpha_2'$, and

$$\begin{array}{lll} \varphi'\left(\beta_0' x_t - \bar\alpha_0' A\Delta x_t\right) \sim I_{nc}(0), & \varphi_\perp'\left(\beta_0' x_t - \bar\alpha_0' A\Delta x_t\right) = 0, & \beta_j' x_t \sim I_{nc}(j), \\ \varphi'\left(\beta_0' T_0 y_t + \xi_{0,1}'\Delta y_t\right) \sim I_{nc}(0), & \varphi_\perp'\beta_0' T_0 = \varphi_\perp'\xi_{0,1}' = 0, & \beta_j' T_0 y_t \sim I_{nc}(j), \end{array} \qquad j = 1, 2,$$

where $\xi_{0,1}' = -(\beta_0' G_0 + \bar\alpha_0' A)T_0$.

4.1. **A numerical illustration.** This example is taken from Bauer and Wagner (2012) and it is used to illustrate the results; consider (2.1) with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 & 0.2 \\ 0 & 1 & 1 & -0.4 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \begin{array}{l} n_x = 4, \\ n_y = 3, \\ \sigma(A) = \{1, 0.5\}. \end{array}$$

The state matrix A is already in Jordan form (3.6), so that $d = 2$, $r_2 = r_1 = 1$ and $r_0 = n_x - r_1 - r_2 = 2$ and the ELRF of $I - Az$ at $z = 1$ delivers

$$\alpha_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_{0\perp} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} = (\alpha_1, \alpha_2), \quad \beta_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -0.5 \end{pmatrix}, \quad \beta_{0\perp} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = (\beta_1, \beta_2).$$

Hence

$$\alpha_{0\perp}' B = -\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad C\beta_{0\perp} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad -\bar\alpha_0' B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(-1 \quad 0 \quad 0) = \varphi\zeta',$$

so that $1 \in \sigma(A)$ is controllable and observable by Theorem 2.1, which implies $x_t \sim I(2)$ and $y_t \sim I(2)$, and $\bar\alpha_0' B$ has reduced rank $s_0 = 1$. The Laurent coefficients $\{B_n\}_{n=0}^\infty$ in $\Delta^2 x_t = B(L)B\varepsilon_{t-1} \sim I(0)$, $\Delta^2 y_t = CB(L)B\varepsilon_{t-1} + \Delta^2\varepsilon_t \sim I(0)$, where $B(z) = \sum_{n=0}^\infty B_n(1-z)^n$, are computed via (3.1) and result

$$B_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_n = (-1)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad n = 2, 3, \ldots.$$

The statements in Theorem 4.1 can thus be checked by direct computation as follows. Regarding x_t, from

$$\begin{pmatrix} \beta_2' \\ \beta_1' \\ \beta_0' \end{pmatrix} B_0 B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \beta_1' \\ \beta_0' \end{pmatrix} B_1 B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \beta_0' B_n B = (-1)^{n-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\bar\alpha_0' A B_0 B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \bar\alpha_0' A B_1 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar\alpha_0' A B_n B = (-1)^n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

for $n = 2, 3, \ldots$, one verifies that $\beta_2' B_0 B$ has full row rank and hence $\beta_2' x_t \sim I_{nc}(2)$, that $\beta_1' B_1 B - \delta' B_0 B$ has full row rank for any δ and hence $\beta_1' x_t \sim I_{nc}(1)$ and that $\beta_0' B_1 B - \bar\alpha_0' A B_0 B = 0$ and hence $\beta_0' x_t - \bar\alpha_0' A \Delta x_t \sim I(0)$. Moreover, $\beta_0' B_2 B - \bar\alpha_0' A B_1 B = \varphi \zeta'$ and $\beta_0' B_n B - \bar\alpha_0' A B_{n-1} B = 0$, $n = 3, 4, \ldots$, confirm that $\varphi'(\beta_0' x_t - \bar\alpha_0' A \Delta x_t) \sim I_{nc}(0)$ and $\varphi_\perp'(\beta_0' x_t - \bar\alpha_0' A \Delta x_t) = 0$.

Regarding y_t, from

$$\begin{pmatrix} \xi_{2,0}' \\ \xi_{1,0}' \\ \xi_{0,0}' \end{pmatrix} = \begin{pmatrix} \beta_2' T_0 \\ \beta_1' T_0 \\ \beta_0' T_0 \end{pmatrix} = c \begin{pmatrix} 1/c & 0.4 & -4.2 \\ 0 & 1 & 2.8 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \begin{array}{l} \xi_{0,1}' = -(\beta_0' G_0 + \bar\alpha_0' A) T_0 = \varphi(-c \quad c_1 c \quad -c_1 c) \\ \\ c = 0.2632, \\ c_1 = 0.3684 \end{array},$$

where the expressions of $\xi_{\cdot,\cdot}'$, $T_0 = G_0 B$, $G_0 = (I - F)^{-1}$, and $F = A - BC$ are found by setting $z_u = \lambda_u = 1$ in (3.9), and from

$$\begin{pmatrix} \xi_{2,0}' \\ \xi_{1,0}' \\ \xi_{0,0}' \end{pmatrix} C B_0 B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \xi_{1,0}' \\ \xi_{0,0}' \end{pmatrix} C B_1 B = c \begin{pmatrix} 4.8 & 1/c & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \xi_{0,0}' C B_n B = (-1)^n \varphi \zeta_0',$$

$$\xi_{0,1}' C B_0 B = -c \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \xi_{0,1}' C B_1 B = \varphi \zeta_1', \qquad \xi_{0,1}' C B_n B = (-1)^{n-1} \varphi \zeta_2',$$

for $n = 2, 3, \ldots$, where $\zeta_0' = 2 c_1 \zeta'$, $\zeta_1' = c(-0.6216, 1, 1.3684)$ and $\zeta_2' = (0.3767, 0, 0)$, one verifies that $\xi_{2,0}' C B_0 B$ has full row rank and hence $\xi_{2,0}' y_t \sim I_{nc}(2)$, that $\xi_{1,0}' C B_1 B - \delta' C B_0 B$ has full row rank for any δ and hence $\xi_{1,0}' y_t \sim I_{nc}(1)$ and that $\xi_{0,0}' C B_1 B + \xi_{0,1}' C B_0 B = 0$ and hence $\xi_{0,0}' y_t + \xi_{0,1}' \Delta y_t \sim I(0)$. Moreover, the above expressions confirm that the column spaces of the $\xi_{0,\cdot}'$ coefficients belong to span φ and hence $\varphi'(\xi_{0,0}' y_t + \xi_{0,1}' \Delta y_t) \sim I_{nc}(0)$ and $\varphi_\perp' \xi_{0,0}' = \varphi_\perp' \xi_{0,1}' = 0$.

5. A characterization of the canonical form in Bauer and Wagner (2012)

Assume that $e^{\pm i\omega_u} \in \sigma(A)$, where $u = 1, \dots, \ell$ and $0 \leq \omega_1 < \cdots < \omega_\ell \leq \pi$, and that $e^{\pm i\omega_u} \neq \lambda_u \in \sigma(A)$ implies $|\lambda_u| < 1$. In this case, Theorem 2 in Bauer and Wagner (2012) proves the existence of a canonical form that highlights the unit root structure of the system, defined in that paper as[4]

$$(\omega_u, (g_1, \dots, g_h)), \qquad 1 \leq g_1 \leq \cdots \leq g_h, \qquad u = 1, \dots, \ell, \tag{5.1}$$

where h is the smallest integer such that $(H_u - e^{-i\omega_u} I)^h = 0$, $g_k := \operatorname{rank}(H_u - e^{-i\omega_u} I)^{h-k} - \operatorname{rank}(H_u - e^{-i\omega_u} I)^{h-k+1}$ and

$$H_u := e^{-i\omega_u} \begin{pmatrix} I_{g_1} & Q_1 & & \\ & I_{g_2} & \ddots & \\ & & \ddots & Q_{h-1} \\ & & & I_{g_h} \end{pmatrix}, \qquad Q_k := e^{i\omega_u} (I_{g_k} \quad 0_{g_k \times g_{k+1} - g_k}).$$

Because H_u is a reordering of the Jordan structure of A that corresponds to the eigenvalue $e^{-i\omega_u}$ and the latter is characterized by the ELRF in (3.6), the following result holds.

Theorem 5.1 (State space unit root structure via ELRF)**.** *The state space unit root structure in* (5.1) *is such that* $h = d$ *and* $g_k = \sum_{j=d-k+1}^{d} r_j$*, where* d *and* $\{r_j\}_{j=0,\dots,d}$ *are defined by the* ELRF *of* $I - Az$ *at* $z_u = e^{i\omega_u}$*.*

Thus the ELRF provides a characterization of the canonical form in Bauer and Wagner (2012), which is a realization of the state space unit root structure.

6. Conclusions

The rank restrictions implied by minimality and those related to the Jordan structure of the state matrix fully characterize the integration and cointegration properties of a state space system in the general $I(d)$ case. The ELRF has been shown to deliver a full description of all the relevant quantities and because it can be performed at any (stationary, unit, explosive) eigenvalue, the present results are not specific to the unit root case.

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[4]The present notation is different from the original one: g_j and H_u replace d_j^k and J_k in Bauer and Wagner (2012).

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Appendix A. Proofs

Proof of Theorem 2.1. For the PBH rank test, see e.g. Theorem 6.2-6 in Kailath (1980), (2.1) is controllable if and only if $\text{rank}(A-\lambda_u I, B) = n_x$ for all $\lambda_u \in \mathbb{C}$ and it is observable if and only if $\text{rank}(A'-\lambda_u I, C') = n_x$ for all $\lambda_u \in \mathbb{C}$. Because $\text{rank}(A-\lambda_u I) = n_x$ implies $\text{rank}(A-\lambda_u I, B) = \text{rank}(A'-\lambda_u I, C') = n_x$, such rank conditions are not informative unless λ_u is an eigenvalue of A. Let λ_u be an eigenvalue of A, use $A-\lambda_u I = \alpha_0\beta_0'$ and the projection identities $I = \alpha_0\bar{\alpha}_0' + \bar{\alpha}_{0\perp}\alpha_{0\perp}' = \bar{\beta}_0\beta_0' + \beta_{0\perp}\bar{\beta}_{0\perp}'$ to write

$$\begin{pmatrix} A-\lambda_u I & B \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0' & \alpha_0\bar{\alpha}_0'B + \bar{\alpha}_{0\perp}\alpha_{0\perp}'B \end{pmatrix} = \begin{pmatrix} \alpha_0 & \bar{\alpha}_{0\perp} \end{pmatrix} \begin{pmatrix} \beta_0' & \bar{\alpha}_0'B \\ 0 & \alpha_{0\perp}'B \end{pmatrix},$$

$$\begin{pmatrix} A-\lambda_u I \\ C \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0' \\ C\bar{\beta}_0\beta_0' + C\beta_{0\perp}\bar{\beta}_{0\perp}' \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 \\ C\bar{\beta}_0 & C\beta_{0\perp} \end{pmatrix} \begin{pmatrix} \beta_0' \\ \bar{\beta}_{0\perp}' \end{pmatrix}.$$

This shows that λ_u is controllable if and only if $\text{rank}(\alpha_{0\perp}'B) = n_x - r_0$ and it is observable if and only if $\text{rank}(C\beta_{0\perp}) = n_x - r_0$. Because (2.1) is minimal if and only if each eigenvalue is controllable and observable, minimality is equivalent to $\text{rank}(\alpha_{0\perp}'B) = \text{rank}(C\beta_{0\perp}) = n_x - r_0$ for each $\lambda_u \in \sigma(A)$. ■

Proof of Theorem 2.2. Substituting $I - Az = (I - Az_u) + (1-\lambda_u z)Az_u$ and $(I-Az)^{-1} = (1-\lambda_u z)^{-d}B(z)$ in $(I-Az)(I-Az)^{-1} = I$, one finds

$$(I - Az_u)\frac{B(z)}{(1-\lambda_u z)^d} + Az_u\frac{B(z)}{(1-\lambda_u z)^{d-1}} = I.$$

Because $(I-Az_u)B(z_u)$ is the only term that loads $(1-\lambda_u z)^{-d}$ in $(I-Az)(I-Az)^{-1} = I$, then $(I-Az_u)B(z_u) = 0$. Similarly, starting from $(I-Az)^{-1}(I-Az) = I$ one finds $B(z_u)(I-Az_u) = 0$ and hence $(I-Az_u)B(z_u) = B(z_u)(I-Az_u) = 0$. Because $I - Az_u = -z_u(A-\lambda_u I) = -z_u\alpha_0\beta_0'$ and $B(z_u) \neq 0$, it follows that $B(z_u) = \beta_{0\perp}\phi\alpha_{0\perp}'$ for some non-zero ϕ. Then $C\beta_{0\perp}\phi\alpha_{0\perp}'Bz_u$ is the leading coefficient of $T_{y,\varepsilon}(z) = I + (1-\lambda_u z)^{-d}CB(z)Bz$ at z_u. By Theorem 2.1, λ_u is controllable if and only if $\text{rank}(\alpha_{0\perp}'B) = n_x - r_0$ and it is observable if and only if $\text{rank}(C\beta_{0\perp}) = n_x - r_0$; because $C\beta_{0\perp}\phi\alpha_{0\perp}'B = 0$ and $\text{rank}(C\beta_{0\perp}) = \text{rank}(\alpha_{0\perp}'B) = n_x - r_0$ imply $\phi = 0$, one reaches a contradiction and hence it follows that $C\beta_{0\perp}\phi\alpha_{0\perp}'B \neq 0$. This shows that the poles of $T_{x,\varepsilon}(z) = \frac{B(z)Bz}{(1-\lambda_u z)^d}$ and $T_{y,\varepsilon}(z) = I + \frac{CB(z)Bz}{(1-\lambda_u z)^d}$ coincide with those of $(I-Az)^{-1}$. ■

Proof of Theorem 2.3. The assumption $1 \in \sigma(A)$ and $|\lambda_u| < 1$ for any $1 \neq \lambda_u \in \sigma(A)$ implies $(I-Az)^{-1} = (1-z)^{-d}B(z)$, where $B(z)$ is finite for all $|z| \leq 1+\delta$ for some $\delta > 0$ and $B(1) \neq 0$; then, see Theorem 2.2, the same holds for the MA transfer functions $T_{x,\varepsilon}(z) = (1-z)^{-d}B(z)Bz$

and $T_{y,\varepsilon}(z) = I + (1-z)^{-d}CB(z)Bz$, where $CB(1)B \neq 0$ by minimality, and thus one finds $\Delta^d x_t = B(L)B\varepsilon_{t-1} \sim I(0)$ and $\Delta^d y_t = CB(L)B\varepsilon_{t-1} + \Delta^d \varepsilon_t \sim I(0)$, which show that $x_t \sim I(d)$ and $y_t \sim I(d)$. ■

Proof of Theorem 3.1. Substituting $I - Az = A_0 + A_1(1-\lambda_u z)$, where $A_0 := I - Az_u$ and $A_1 := Az_u$, and $(I-Az)^{-1} = \sum_{n=0}^{\infty} B_n(1-\lambda_u z)^{n-d}$ in $(I-Az)(I-Az)^{-1} = I$, one finds

$$A_0B_0 = 0, \qquad A_0B_n + A_1B_{n-1} = 1_{n=d}I, \qquad n = 1, 2, \ldots, \tag{A.1}$$

where $1.$ is the indicator function. In the following, equations in system (A.1) are indexed according to the highest value of the subscript of B_n; for instance $A_0B_0 = 0$ is referred to as equation 0. Note that the identity appears in equation d, which is the order of the pole. Because $I - Az_u = -z_u\alpha_0\beta_0'$, equation 0 implies $\beta_0'B_0 = 0$ so that $B_0 = P_{\beta_0}B_0 + P_{\beta_{0\perp}}B_0 = P_{\beta_{0\perp}}B_0$ and equation 1 reads $A_0B_1 + A_1P_{\beta_{0\perp}}B_0 = 1_{1=d}I$. Moreover, because $A_1 = Az_u = I + z_u\alpha_0\beta_0'$, one has $P_{\alpha_{0\perp}}A_1 = P_{\alpha_{0\perp}}$ and $A_1P_{\beta_{0\perp}} = P_{\beta_{0\perp}}$ and hence equation 1 becomes $-z_u\alpha_0\beta_0'B_1 + P_{\beta_{0\perp}}B_0 = 1_{1=d}I$.

The proof of Theorem 3.1 is based on rewriting equation $n \geq j = 0, 1, \ldots$ in system (A.1) as

$$z_u\alpha_j\beta_j'B_{n-j} = P_{a_{j\perp}}Q_{j+1}B_{n-j-1} + P_{a_{j\perp}}C_{j+1,n-j}, \tag{A.2}$$

where α_j, β_j are defined by the rank factorization

$$P_{a_{j\perp}}Q_jP_{b_{j\perp}} = -z_u\alpha_j\beta_j', \qquad \begin{matrix} a_j := (\alpha_0, \ldots, \alpha_{j-1}) \\ b_j := (\beta_0, \ldots, \beta_{j-1}) \end{matrix},$$

Q_j is defined by the recursions

$$Q_j := \begin{cases} Az_u & \text{for } j = 1 \\ \lambda_u Q_{j-1}\sum_{i=0}^{j-2}\bar{\beta}_i\bar{\alpha}_i'Q_{i+1} & \text{for } j = 2, \ldots, d+1 \end{cases}, \tag{A.3}$$

and $C_{j+1,n}$ is defined by the recursions

$$C_{j+1,n} := \begin{cases} -1_{n=d}I & \text{for } j = 0 \\ C_{j,n+1} + \lambda_u Q_j\sum_{i=0}^{j-1}\bar{\beta}_i\bar{\alpha}_i'C_{i+1,n} & \text{for } j = 1, \ldots, d \end{cases}. \tag{A.4}$$

The proof of (A.2) is by induction. For $j = 0$, (A.2) reads $z_u\alpha_0\beta_0'B_n = P_{a_{0\perp}}Q_1B_{n-1} + P_{a_{0\perp}}C_{1,n}$,; by definition, $Q_1 = Az_u$, $C_{1,n} = -1_{n=d}$ and $a_0 = b_0 = 0$, which implies $P_{a_{0\perp}} = P_{b_{0\perp}} = I$. Hence (A.2) for $j = 0$ coincides with equation n in (A.1). Next assume that (A.2) holds for $j = 0, \ldots, \ell-1$ for some $\ell > 1$; one wishes to show that it also holds for $j = \ell$. Write (A.2) for $j = \ell - 1$, $z_u\alpha_{\ell-1}\beta_{\ell-1}'B_{n-\ell+1} = P_{a_{\ell-1\perp}}Q_\ell B_{n-\ell} + P_{a_{\ell-1\perp}}C_{\ell,n-\ell+1}$, pre-multiply by $P_{a_{\ell\perp}}$ and rearrange terms to find

$$0 = P_{a_{\ell\perp}}Q_\ell B_{n-\ell} + P_{a_{\ell\perp}}C_{\ell,n-\ell+1} := U + V. \tag{A.5}$$

Inserting $I = P_{b_\ell} + P_{b_{\ell\perp}}$ between Q_ℓ and $B_{n-\ell}$ in U one finds

$$U = P_{a_{\ell\perp}} Q_\ell P_{b_{\ell\perp}} B_{n-\ell} + P_{a_{\ell\perp}} Q_\ell P_{b_\ell} B_{n-\ell} =: U_1 + U_2.$$

Substituting $P_{b_\ell} = P_{\beta_0} + \cdots + P_{\beta_{\ell-1}}$, one has $U_2 = P_{a_{\ell\perp}} Q_\ell \sum_{i=0}^{\ell-1} P_{\beta_i} B_{n-\ell}$, where

$$P_{\beta_i} B_{n-\ell} = \lambda_u \bar{\beta}_i \bar{\alpha}_i' Q_{i+1} B_{n-\ell-1} + \lambda_u \bar{\beta}_i \bar{\alpha}_i' C_{i+1,n-\ell},$$

is derived using the induction assumption and replacing n with $n - \ell + j$ and j with i in (A.2). Substituting in U_2, one finds

$$U_2 = P_{a_{\ell\perp}} \left(\lambda_u Q_\ell \sum_{i=0}^{\ell-1} \bar{\beta}_i \bar{\alpha}_i' Q_{i+1} \right) B_{n-\ell-1} + P_{a_{\ell\perp}} \left(\lambda_u Q_\ell \sum_{i=0}^{\ell-1} \bar{\beta}_i \bar{\alpha}_i' C_{i+1,n-\ell} \right);$$

hence using (A.3) and (A.4), one has $U_2 + V = P_{a_{\ell\perp}} Q_{\ell+1} B_{n-\ell-1} + P_{a_{\ell\perp}} C_{\ell+1,n-\ell}$ so that substituting the rank factorization $P_{a_{\ell\perp}} Q_\ell P_{b_{\ell\perp}} = -z_u \alpha_\ell \beta_\ell'$ in U_1 and rearranging terms, (A.5) is rewritten as

$$z_u \alpha_\ell \beta_\ell' B_{n-\ell} = P_{a_{\ell\perp}} Q_{\ell+1} B_{n-\ell-1} + P_{a_{\ell\perp}} C_{\ell+1,n-\ell}.$$

This shows that (A.2) holds for $j = \ell$ and completes the proof by induction. Replacing $n - j$ with n one rewrites (A.2) as

$$z_u \alpha_j \beta_j' B_n = P_{a_{j\perp}} Q_{j+1} B_{n-1} + P_{a_{j\perp}} C_{j+1,n}, \qquad j = 0, \ldots, d, \qquad n = 0, 1, \ldots, \qquad B_{-1} := 0, \tag{A.6}$$

where

$$C_{j+1,n} = \begin{cases} 0 & \text{for } n < d - j \\ -I & \text{for } n = d - j \\ 0 & \text{for } n > d \end{cases} \tag{A.7}$$

follows from the definition of $C_{j+1,n}$ in (A.4). Setting $n = 0$ in (A.6) and substituting $P_{a_{j\perp}} Q_j P_{b_{j\perp}} = -z_u \alpha_j \beta_j'$ one then finds $P_{a_{j\perp}} Q_j P_{b_{j\perp}} B_0 = 0$, $j = 0, \ldots, d-1$, and $P_{a_{d\perp}} Q_d P_{b_{d\perp}} B_0 = P_{a_{d\perp}}$, because the identity is in equation d in (A.1). This shows that $(I - Az)^{-1}$ has a pole of order $d \in \mathbb{N}_+$ at z_u if and only if $r_j := \operatorname{rank}(P_{a_{j\perp}} Q_j P_{b_{j\perp}}) < \operatorname{rank} P_{a_{j\perp}} =: r_j^{\max}$, $j = 1, \ldots, d-1$, and $r_d := \operatorname{rank}(P_{a_{d\perp}} Q_d P_{b_{d\perp}}) = \operatorname{rank} P_{a_{d\perp}} =: r_d^{\max}$. This completes the proof of the first part of the statement of Theorem 3.1. The recursion for the Laurent coefficients in (3.1) is found as follows: pre-multiplying (A.6) by $\bar{\beta}_j \bar{\alpha}_j'$ to find

$$z_u P_{\beta_j} B_n = \bar{\beta}_j \bar{\alpha}_j' Q_{j+1} B_{n-1} + \bar{\beta}_j \bar{\alpha}_j' C_{j+1,n}, \tag{A.8}$$

sum over j, use the projection identity $\sum_{j=0}^{d} P_{\beta_j} = I$ and rearrange to find

$$B_n = \left(\lambda_u \sum_{j=0}^{d} \bar{\beta}_j \bar{\alpha}_j' Q_{j+1} \right) B_{n-1} + \left(\lambda_u \sum_{j=0}^{d} \bar{\beta}_j \bar{\alpha}_j' C_{j+1,n} \right). \tag{A.9}$$

This completes the proof. ■

Proof of Theorem 3.2. Pre-multiplying (A.9) by β_j' and rearranging one finds

$$\beta_j' B_n - \lambda_u \bar{\alpha}_j' Q_{j+1} B_{n-1} = \lambda_u \bar{\alpha}_j' C_{j+1,n}. \tag{A.10}$$

Next consider

$$(I - Az)^{-1} = \sum_{n=0}^{\infty} B_n (1 - \lambda_u z)^{n-d}$$

and pre-multiply respectively by β_j' and by $(1 - \lambda_u z)\lambda_u \bar{\alpha}_j' Q_{j+1}$ to find

$$\beta_j'(I - Az)^{-1} = \sum_{n=0}^{\infty} \beta_j' B_n (1 - \lambda_u z)^{n-d}, \qquad (1 - \lambda_u z)\lambda_u \bar{\alpha}_j' Q_{j+1}(I - Az)^{-1} = \sum_{n=1}^{\infty} \lambda_u \bar{\alpha}_j' Q_{j+1} B_{n-1} (1 - \lambda_u z)^{n-d}.$$

Subtracting the two expressions and rearranging one has

$$\left(\beta_j' - (1 - \lambda_u z)\lambda_u \bar{\alpha}_j' Q_{j+1}\right)(I - Az)^{-1} = \sum_{n=0}^{\infty} \left(\beta_j' B_n - \lambda_u \bar{\alpha}_j' Q_{j+1} B_{n-1}\right)(1 - \lambda_u z)^{n-d},$$

and hence, substituing (A.10) in the rhs of the equation, one finds

$$\gamma_j(z)'(I - Az)^{-1} = \lambda_u \sum_{n=0}^{\infty} \bar{\alpha}_j' C_{j+1,n} (1 - \lambda_u z)^{n-d},$$

where $\gamma_j(z)' := \beta_j' - (1 - \lambda_u z)\lambda_u \bar{\alpha}_j' Q_{j+1}$. Because $C_{j+1,n} = 0$ for $n + j < d$ and $C_{j+1,n} = 0$ for $n > d$, see (A.7), one finds

$$\gamma_j(z)'(I - Az)^{-1} = \lambda_u \sum_{n=d-j}^{d} \bar{\alpha}_j' C_{j+1,n} (1 - \lambda_u z)^{n-d} = (1 - \lambda_u z)^{-j} \lambda_u \sum_{n=0}^{j} \bar{\alpha}_j' C_{j+1,d-j+n} (1 - \lambda_u z)^n$$

i.e.

$$\gamma_j(z)'(I - Az)^{-1} = \frac{\lambda_u \psi_j(z)'}{(1 - \lambda_u z)^j}, \qquad j = 0, \ldots, d, \tag{A.11}$$

where $\psi_j(z)' := \sum_{n=0}^{j} \bar{\alpha}_j' C_{j+1,d-j+n} (1 - \lambda_u z)^n$. Note that $\gamma_j(z_u)' = \beta_j'$ and because $C_{j+1,d-j} = -I$ for $j = 0, \ldots, d$, see (A.7), $\psi_j(z_u)' = -\bar{\alpha}_j' C_{j+1,d-j} = -\bar{\alpha}_j'$, so that $\Gamma(z) := (\gamma_0(z), \ldots, \gamma_d(z))'$ and $\Psi(z) := \lambda_u(\psi_0(z), \ldots, \psi_d(z))'$ are non-singular at z_u. Stacking (A.11) one thus finds

$$\Gamma(z)(I - Az)^{-1} = \Lambda(z)^{-1} \Psi(z), \qquad |\Gamma(z_u)| \neq 0, \qquad |\Psi(z_u)| \neq 0,$$

which shows that $\Lambda(z) := \mathrm{diag}((1 - \lambda_u z)^0 I_{r_0}, \ldots, (1 - \lambda_u z)^d I_{r_d})$ is the local Smith form of $I - Az$ at z_u and $\Gamma(z)$, $\Psi(z)$ are extended canonical systems of root functions. ■

The proof of Theorem 3.3 is based on the following lemma.

Lemma A.1 (T_n coefficients)**.** *Let $F := A - BC$ and $z_u := \lambda_u^{-1}$, $0 \neq \lambda_u \in \sigma(A)$. Then $I - Fz_u$ is non-singular and $T_{x,y}(z_u) = (I - Fz_u)^{-1}Bz_u \neq 0$, i.e. the poles of the AS transfer function do not coincide with those of $(I - Az)^{-1}$. Moreover,*

$$T_{x,y}(z) = \sum_{n=0}^{\infty} T_n(1 - \lambda_u z)^n, \qquad T_n = \begin{cases} G_0 B z_u \neq 0 & \text{for } n = 0 \\ -G_0(I - G_0)^{n-1}T_0 & \text{for } n = 1, 2, \dots \end{cases}, \tag{A.12}$$

where $G_0 := (I - Fz_u)^{-1}$.

Proof of Lemma A.1. It is first shown that if $\lambda_u \in \sigma(A)$ is controllable and observable then $\lambda_u \notin \sigma(F)$ and hence $I - Fz_u = -z_u(F - \lambda_u I)$ is non-singular at z_u. Use the projection identities $I = \alpha_0\bar{\alpha}_0' + \bar{\alpha}_{0\perp}\alpha_{0\perp}' = \bar{\beta}_0\beta_0' + \beta_{0\perp}\bar{\beta}_{0\perp}'$ to write

$$\begin{aligned} F - \lambda_u I = A - \lambda_u I - BC = \alpha_0\beta_0' - BC &= \begin{pmatrix} \alpha_0 & -B \end{pmatrix} \begin{pmatrix} \beta_0' \\ C \end{pmatrix} \\ &= \begin{pmatrix} \alpha_0 & \bar{\alpha}_{0\perp} \end{pmatrix} \begin{pmatrix} I_{r_0} & -\bar{\alpha}_0' B \\ 0 & -\alpha_{0\perp}' B \end{pmatrix} \begin{pmatrix} I_{r_0} & 0 \\ C\bar{\beta}_0 & C\beta_{0\perp} \end{pmatrix} \begin{pmatrix} \beta_0' \\ \bar{\beta}_{0\perp}' \end{pmatrix}. \end{aligned}$$

Because $\operatorname{rank}(\alpha_{0\perp}' B) = \operatorname{rank}(C\beta_{0\perp}) = n_x - r_0$, one has that $F - \lambda_u I$ is non-singular and hence λ_u is not an eigenvalue of F. Because $F - \lambda_u I = -\lambda_u(I - Fz_u)$, this shows that $I - Fz_u$ is invertible and thus z_u is not a pole of $T_{x,y}(z) := (I - Fz)^{-1}Bz$. Moreover, because $T_{x,y}(z_u) = (I - Fz_u)^{-1}Bz_u = 0$ implies $B = 0$, this proves the first part of the statement. Next it is proved that

$$T_{x,y}(z) = \sum_{n=0}^{\infty} T_n(1 - \lambda_u z)^n, \quad T_n = \begin{cases} G_0 B z_u \neq 0 & \text{for } n = 0 \\ -G_0(I - G_0)^{n-1}T_0 & \text{for } n = 1, 2, \dots \end{cases}, \quad G_0 := (I - Fz_u)^{-1}.$$

Let $(I - Fz)^{-1} = \sum_{n=0}^{\infty} G_n(1 - \lambda_u z)^n$ and write $I - Fz = F_0 - F_1(1 - \lambda_u z)$, where $F_0 = I - Fz_u$ and $F_1 = -Fz_u$. Then $(I - Fz)(I - Fz)^{-1} = I$ implies $G_0F_0 = I$ and $G_nF_0 = G_{n-1}F_1$ for $n = 1, 2, \dots$; hence $G_0 = F_0^{-1}$ and $G_n = G_{n-1}F_1G_0$ for $n = 1, 2, \dots$, i.e. $G_n = (G_0F_1)^nG_0$ for $n = 0, 1, \dots$. Because $F_1 = F_0 - I$ one has $G_0F_1 = I - G_0$ and hence $G_n = (I - G_0)^nG_0$, $n = 0, 1, \dots$. Summing and subtracting $(I - Fz)^{-1}Bz_u$ in $T_{x,y}(z) := (I - Fz)^{-1}Bz$ one finds $T_{x,y}(z) = (I - Fz)^{-1}Bz_u - (I - Fz)^{-1}Bz_u(1 - \lambda_u z)$; substituting $(I - Fz)^{-1} = \sum_{n=0}^{\infty} G_n(1 - \lambda_u z)^n$ and rearranging one has $T_{x,y}(z) = G_0Bz_u + \sum_{n=1}^{\infty} G_nBz_u(1 - \lambda_u z)^n - \sum_{n=1}^{\infty} G_{n-1}Bz_u(1 - \lambda_u z)^n$ and hence

$$T_{x,y}(z) = G_0Bz_u + \sum_{n=1}^{\infty}(G_n - G_{n-1})Bz_u(1 - \lambda_u z)^n.$$

Using $G_n = (I - G_0)^nG_0$, $n = 0, 1, \dots$, one finds $G_n - G_{n-1} = -G_0(I - G_0)^{n-1}G_0$, $n = 1, 2, \dots$, and hence $T_n = -G_0(I - G_0)^{n-1}T_0$ for $n = 1, 2, \dots$. This completes the proof. ∎

Proof of Theorem 3.3. Eq. (3.8) shows that $\gamma_j(z)'T_{x,\varepsilon}(z)$ has an irreducible pole of order $j = 1,\dots,d$ and no pole for $j = 0$ at z_u. From the convolution of $\gamma_j(z)' := \beta_j' - \lambda_u\bar\alpha_j'Q_{j+1}(1-\lambda_u z)$ in (3.4) and $T_{x,y}(z) = \sum_{n=0}^{\infty} T_n(1-\lambda_u z)^n$ in (A.12) one finds $\tilde\xi_j(z)' := \gamma_j(z)'T_{x,y}(z) = \sum_{n=0}^{\infty}\xi_{j,n}'(1-\lambda_u z)^n$, where

$$\xi_{j,0}' = \beta_j'T_0, \qquad \xi_{j,n}' = \begin{cases} \beta_j'T_n - \lambda_u\bar\alpha_j'Q_{j+1}T_{n-1} & \text{for } j = 0,\dots,d-2 \\ \beta_j'T_n & \text{for } j = d-1, d \end{cases}, \quad n = 1,2,\dots. \tag{A.13}$$

Writing $\tilde\xi_j(z)' = \xi_j(z)' + (1-\lambda_u z)^{d-j}\rho_j(z)'$, where

$$\xi_j(z)' := \begin{cases} \sum_{n=0}^{d-j-1}\xi_{j,n}'(1-\lambda_u z)^n & \text{for } j = 0,\dots,d-2 \\ \xi_{j,0}' & \text{for } j = d-1, d \end{cases} \tag{A.14}$$

is the truncation of $\tilde\xi_j(z)'$ up to degree $\max(0, d-j-1)$ and $\rho_j(z)'$ is defined accordingly, and using $T_{y,\varepsilon}(z) = I + (1-\lambda_u z)^{-d}CB(z)Bz$, one finds

$$\tilde\xi_j(z)'T_{y,\varepsilon}(z) = \xi_j(z)'T_{y,\varepsilon}(z) + (1-\lambda_u z)^{d-j}\rho_j(z)' + (1-\lambda_u z)^{-j}\rho_j(z)'CB(z)Bz.$$

Substituting into (3.8) and rearranging, one has

$$\xi_j(z)'T_{y,\varepsilon}(z) = \frac{\lambda_u\psi_j(z)'Bz - \rho_j(z)'CB(z)Bz}{(1-\lambda_u z)^j} - (1-\lambda_u z)^{d-j}\rho_j(z)', \tag{A.15}$$

where

$$\lambda_u\psi_j(z_u)'B - \rho_j(z_u)'CB(z_u)B = -(\lambda_u I_{r_j} \quad -\xi_{j,d-j}'C\bar\beta_d)\begin{pmatrix}\bar\alpha_j'B \\ \bar\alpha_d'B\end{pmatrix} \tag{A.16}$$

has full row rank for $j = 1,\dots,d$ by minimality. This shows that $\xi_j(z)'T_{y,\varepsilon}(z)$ has an irreducible pole of order $j = 1,\dots,d$ and no pole for $j = 0$ at z_u. This completes the proof. ■

Proof of Theorem 4.1. The first part of the statement follows directly from Theorems 2.3 and 3.1. Next set $z_u = \lambda_u = 1$ in $\gamma_j(z)'$ in (3.4) and $\xi_j(z)'$ in (3.9) and consider $\gamma_j(L)'x_t$ and $\xi_j(L)'y_t$ for $j = 1,\dots,d$. Because $\gamma_j(z)'T_{x,\varepsilon}(z)$ and $\xi_j(z)'T_{y,\varepsilon}(z)$ have an irreducible pole of order $j = 1,\dots,d$ at $z = 1$, see Theorem 3.3, one has $\gamma_j(L)'x_t \sim I_{nc}(j)$ and $\xi_j(L)'y_t \sim I_{nc}(j)$ for $j = 1,\dots,d$. Next set $j = 0$ and consider $\gamma_0(L)'x_t$ and $\xi_0(L)'y_t$. Regarding $\xi_0(L)'y_t$, from (3.8) one has $\gamma_0(z)'T_{x,\varepsilon}(z) = -\bar\alpha_0'Bz$, where the rank of $-\bar\alpha_0'B$ is unrestricted, see the discussion below (3.3); from the rank factorization $-\bar\alpha_0'B = \varphi\zeta'$ one then has $\varphi'\gamma_0(z)'T_{x,\varepsilon}(z) = \zeta'z$ and $\varphi_\perp'\gamma_0(z)'T_{x,\varepsilon}(z) = 0$ and hence

$$\bar\varphi'\gamma_0(L)'x_t = \zeta'\varepsilon_{t-1} \sim I_{nc}(0), \qquad \varphi_\perp'\gamma_0(L)'x_t = 0.$$

Finally consider $\xi_0(L)'y_t$; it is first proved that $\varphi'\xi_0(L)'y_t \sim I_{nc}(0)$. From (A.15) and (A.16) one has $\xi_0(z)'T_{y,\varepsilon}(z) = \psi_0(z)'Bz - \rho_0(z)'CB(z)Bz - (1-z)^d\rho_0(z)'$, where

$$\psi_0(1)'B - \rho_0(1)'CB(1)B = (I_{r_0} \quad \xi'_{0,d}C\bar{\beta}_d)\begin{pmatrix} -\bar{\alpha}'_0B \\ \bar{\alpha}'_dB \end{pmatrix} = (\bar{\varphi} \quad \xi'_{0,d}C\bar{\beta}_d)\begin{pmatrix} \zeta' \\ \bar{\alpha}'_dB \end{pmatrix}.$$

The last matrix has full row rank because $B = \alpha_0\bar{\alpha}'_0B + \bar{\alpha}_{0\perp}\alpha'_{0\perp}B = (-\alpha_0\varphi \quad \bar{\alpha}_{0\perp})\begin{pmatrix} \zeta' \\ \alpha'_{0\perp}B \end{pmatrix}$, so that $\operatorname{rank} B = n_x - r_0 + s_0$, where $s_0 := \operatorname{rank}(\bar{\alpha}'_0B)$; hence

$$\varphi'(\psi_0(1)'B - \rho_0(1)'CB(1)B) = (I_{s_0} \quad \varphi'\xi'_{0,d}C\bar{\beta}_d)\begin{pmatrix} \zeta' \\ \bar{\alpha}'_dB \end{pmatrix}$$

has full row rank and thus $\varphi'\xi_0(L)'y_t \sim I_{nc}(0)$.

It is next proved that $\varphi'_\perp\xi_0(z)' = 0$; setting $j = 0$ and $\lambda_u = 1$ in (A.13) and (A.14) one finds $\xi_0(z)' = \sum_{n=0}^{d-1}\xi'_{0,n}(1-z)^n$, where

$$\xi'_{0,n} = \begin{cases} \beta'_0T_0 & \text{for } n = 0 \\ \beta'_0T_n - \bar{\alpha}'_0Q_1T_{n-1} & \text{for } n = 1,2,\dots \end{cases}, \quad T_n = \begin{cases} G_0B & \text{for } n = 0 \\ -G_0(I-G_0)^{n-1}T_0 & \text{for } n = 1,2,\dots \end{cases}$$

and $G_0 = (I-F)^{-1}$. It is next shown that

$$\xi'_{0,n} = \begin{cases} \varphi\zeta'(I - CT_0) & \text{for } n = 0 \\ \varphi\zeta'CG_0T_0 - \xi'_{0,0} & \text{for } n = 1 \\ \varphi\zeta'CG_0T_{n-1} & \text{for } n = 2,3,\dots \end{cases}, \tag{A.17}$$

which implies $\varphi'_\perp\xi'_{0,n} = 0$ for $n = 0,1,\dots$ and hence $\varphi'_\perp\xi_0(z)' = 0$.

First consider $\xi'_{0,0} = \beta'_0G_0B$; from $F = A - BC$ and $A - I = \alpha_0\beta'_0$, one has $I - F = -\alpha_0\beta'_0 + BC$ and hence post-multiplying by G_0 one has $I = -\alpha_0\beta'_0G_0 + BCG_0$. Pre-multiplying by $\bar{\alpha}'_0$ one finds

$$\bar{\alpha}'_0 = -\beta'_0G_0 + \bar{\alpha}'_0BCG_0 \tag{A.18}$$

and rearranging one has $\beta'_0G_0 = -\bar{\alpha}'_0 - \varphi\zeta'CG_0$, where the rank factorization $-\bar{\alpha}'_0B = \varphi\zeta'$ has been employed. Post-multiplying by B one thus has $\beta'_0G_0B = \varphi\zeta'(I - CG_0B)$. This shows that (A.17) holds for $n = 0$.

Next consider $\xi'_{0,n} = \beta'_0T_n - \bar{\alpha}'_0Q_1T_{n-1}$, $n = 1,2,\dots$; from $Q_1 = A$ and $A - I = \alpha_0\beta'_0$, one has $\bar{\alpha}'_0Q_1 = \bar{\alpha}'_0(I + \alpha_0\beta'_0) = \bar{\alpha}'_0 + \beta'_0$ so that

$$\xi'_{0,n} = \beta'_0(T_n - T_{n-1}) - \bar{\alpha}'_0T_{n-1}, \quad n = 1,2,\dots, \quad T_n - T_{n-1} = \begin{cases} -G_0T_0 - T_0 & \text{for } n = 1 \\ -G_0T_{n-1} & \text{for } n = 2,3,\dots \end{cases},$$

where the expression of $T_n - T_{n-1}$ follows from (A.12). Pre-multiplying $T_n - T_{n-1}$ by β_0' one finds

$$\beta_0'(T_n - T_{n-1}) = \begin{cases} -\beta_0' G_0 T_0 - \beta_0' T_0 & \text{for } n = 1 \\ -\beta_0' G_0 T_{n-1} & \text{for } n = 2, 3, \dots \end{cases}$$

and substituing $-\beta_0' G_0 = \bar{\alpha}_0' + \varphi\zeta' C G_0$, which follows from (A.18), and $-\beta_0' T_0 = -\beta_0' G_0 B = -\xi_{0,0}'$ one has

$$\beta_0'(T_n - T_{n-1}) = \begin{cases} \bar{\alpha}_0' T_0 + \varphi\zeta' C G_0 T_0 - \xi_{0,0}' & \text{for } n = 1 \\ \bar{\alpha}_0' T_{n-1} + \varphi\zeta' C G_0 T_{n-1} & \text{for } n = 2, 3, \dots \end{cases},$$

which shows that (A.17) holds for $n = 1, 2, \dots$ and completes the proof. ■

Proof of Theorem 5.1. The Jordan structure J_{λ_u} of A that corresponds to the eigenvalue $\lambda_u = e^{-i\omega_u}$ has exactly r_j Jordan blocks of dimension $j = 1, \dots, d$, see (3.6). Because h is by definition the index of λ_u as an eigenvalue of A, and this is equal to the size of the largest Jordan block, one has $h = d$. Moreover, because H_u and J_{λ_u} are similar, $g_k := \operatorname{rank}(H_u - \lambda_u I)^{d-k} - \operatorname{rank}(H_u - \lambda_u I)^{d-k+1} = \operatorname{rank}(J_u - \lambda_u I)^{d-k} - \operatorname{rank}(J_u - \lambda_u I)^{d-k+1} =: w_{d-k+1}$, so that $(g_d, \dots, g_1) = (w_1, \dots, w_d)$ is the Weyr characteristic of A associated to λ_u and hence $g_{d-j+1} - g_{d-j} = w_j - w_{j+1}$ is the number of Jordan blocks of dimension j, see Section 3.1 in Horn and Johnson (2013). Solving $g_{d-j+1} - g_{d-j} = r_j$, where $g_0 := 0$, one finds $g_k = \sum_{j=d-k+1}^{d} r_j$ for $k = 1, \dots, d$. ■