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## Cointegration, root functions and minimal bases

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# COINTEGRATION, ROOT FUNCTIONS AND MINIMAL BASES ${ }^{1}$ 

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#### Abstract

This paper discusses the concept of cointegrating space for systems integrated of order higher than 1. It is first observed that the notions of (polynomial) cointegrating vectors and of root functions coincide. Second, the cointegrating space is defined as a subspace of the space of rational vectors. Third, it is shown that canonical sets of root functions can be used to generate a basis of the cointegrating space. Fourth, results on how to reduce bases of rational vector spaces to polynomial bases with minimal order - i.e. minimal bases - are shown to imply the separation of polynomial cointegrating vectors that potentially do not involve differences of the process from the ones that require them. Finally, it is argued that minimality of polynomial bases and economic identification of cointegrating vectors can be properly combined.


Keywords: VAR, Cointegration, I(d), Vector spaces.

## 1. INTRODUCTION

In their seminar paper, Engle and Granger (1987) introduced the notion of cointegration and of cointegrating rank for processes integrated of order 1 , or $\mathrm{I}(1)$. They did this in the following way: ${ }^{1}$

Definition: The components of the vector $x_{t}$, are said to be co-integrated of order $d, b$, denoted $x_{t} \sim C I(d, b)$, if (i) all components of $x_{t}$, are $I(d)$; (ii) there exists a vector $\beta(\neq 0)$ so that $z_{t}=$ $\beta^{\prime} x_{t} \sim I(d-b), b>0$. The vector $\beta$ is called the co-integrating vector.
[...]
If $x_{t}$ has $p$ components, then there may be more than one co-integrating vector $\beta$. It is clearly possible for several equilibrium relations to govern the joint behavior of the variables. In what follows, it will be assumed that there are exactly $r$ linearly independent co-integrating vectors, with $r \leq p-1$, which are gathered together into the $p \times r$ array $\beta$. By construction the rank of $\beta$ will be $r$ which will be called the "co-integrating rank" of $x_{t}$.
Engle and Granger (1987) did not define explicitly the notion of cointegrating space, but just the cointegrating rank, which corresponds to its dimension; the explicit mention of the cointegrating space was first made in Johansen (1988).
The Granger representation theorem in Engle and Granger (1987) showed that the cointegration matrix $\beta$ needs to be orthogonal to the Moving Average (MA) impact matrix of $\Delta x_{t}$. More precisely, for $\Delta x_{t}=C(L) \varepsilon_{t}$, the MA impact matrix $C(1)$ has rank equal to $p-r$ and representation $C(1)=\beta_{\perp} a^{\prime}$, where $\beta_{\perp}$ is a basis of the orthogonal complement of the space spanned by $\beta$ and $a$ is full column rank.

[^0]Johansen $(1991,1992)$ stated the appropriate conditions under which the Granger representation theorem holds for $\mathrm{I}(1)$ and $\mathrm{I}(2)$ vector AutoRegressive processes (VAR) $A(L) x_{t}=\varepsilon_{t}$, where the AR impact matrix $A(1)$ has rank equal to $r<p$ and representation $A(1)=-\alpha \beta^{\prime}$, with $\alpha$ and $\beta$ of full column rank. He defined the cointegrating space as $\mathcal{B}=\operatorname{span}\left(\beta^{\prime}\right)$, i.e. as the vector space generated by the row vectors $\beta_{j}^{\prime}$ in $\beta^{\prime}$ over the field of reals $\mathbb{R} .^{2}$
Johansen (1991) noted that $\mathcal{B}$ is uniquely defined by the rank factorization $A(1)=-\alpha \beta^{\prime}$, but the choice of basis $\beta^{\prime}$ is arbitrary, i.e. $\beta^{\prime}$ is not identified. Hypotheses that do not constraint $\mathcal{B}$ are hence untestable. He proposed likelihood ratio tests on $\mathcal{B}$ and described asymptotic properties of a just-identified version of $\beta^{\prime}$. Later Johansen (1995) discussed the choice of basis $\beta^{\prime}$ as an econometric identification problem of a system of cointegrating relations describing the long-run, along the lines of the classical identification problem of system of equation studied in econometrics since the early days of the Cowles Commission.

The observation in Johansen (1988) that the cointegrating vectors formed a vector space $\mathcal{B}$ of $1 \times p$ real vectors (the rows in $\beta^{\prime}$ ) over the field of reals $\mathbb{R}$, a subspace of $\mathbb{R}^{p}$, was an important breakthrough. For instance, it provided the answer to the question: 'how many cointegrating vectors should one estimate in a given system of dimension $p$ ?'. A proper answer is in fact: 'a set of $r$ linearly independent vectors, spanning the cointegrating space $\mathcal{B}$, i.e. a basis of $\mathcal{B}$ '.
Similarly, when assuming that a set of $p$ interest rates followed an $\mathrm{I}(1)$ process, the notion of cointegrating space $\mathcal{B}$ enables to answer questions like 'How should one test that all interest rates differentials are stationary?'. In fact, if all $\binom{p}{2}=p(p-1) / 2$ interest rates differentials were stationary, then one should have cointegrating rank $r=p-1$, which gives a first testable hypothesis on the cointegrating rank; moreover one does not need to test all possible interest rates differentials to be stationary, but, if the cointegrating rank has been found to be $p-1$, one needs to test that the cointegrating space is spanned by any set of linearly independent $r$ contrasts between pairs of interest rates. If the cointegrating rank is found to be $0<r<p-1$, one may still want to test the latter implication, since this remains a restriction on the cointegrating space $\mathcal{B}$. These questions, and many more, found clear answers thanks to the introduction of the notion of cointegrating space.

The notion of cointegrating space, together with the complementary notion of attractor space, has been recently discussed in the context of functional time series for infinite dimensional Hilbert space valued AR processes with unit roots, see Beare and Seo (2019); Franchi and Paruolo (2019a), and for infinite dimensional Banach space valued AR processes with unit roots, see Seo (2019).
For systems with variables integrated of order $d, I(d)$, Granger and Lee (1989) and Engle and Yoo (1991) introduced the related notions of multicointegration and polynomial cointegration; see also Engsted and Johansen (2000). However, no proper discussion of cointegrating spaces has been proposed in the literature for higher order systems.
The present paper closes this gap, making use of classical concepts in local spectral theory, see Gohberg et al. (1993). A central role is played by canonical system of root functions, which have already been exploited in Franchi and Paruolo (2011, 2016) to characterize the inversion of a matrix function, and used in Franchi and Paruolo (2019b) to derive the generalization of the Granger-Johansen representation theorem for $\mathrm{I}(d)$ processes. These tools are employed here for the first time to discuss the definition of coin-

[^1]tegrating space for $I(d)$ processes, and to reduce bases to polynomial order to minimal degree - i.e. minimal bases - along the lines of Forney (1975). Finally the paper discusses how minimality in the basis can be combined with economic restrictions to obtain identification of the basis, using the results in Mosconi and Paruolo (2017).

The rest of the paper is organised as follows. Section 2 reports definitions of integration and cointegration in $I(d)$ systems. Section 3 defines root functions, whose properties are discussed in Section 4. Section 5 defines the cointegration space as a subspace of the space of rational vectors. Section 6 shows how canonical system of root functions provide a polynomial basis for the cointegrating space. Section 7 discusses minimal bases and Section 8 shows how to obtain minimal bases in the $I(2)$ case. Section 9 discusses ways to combine minimality with economic identification in the $I(2)$ case. Section 10 concludes; the Appendix reports some additional proofs.

## 2. SETUP AND DEFINITIONS

This section introduces notation and basic definitions of integrated and cointegrated processes. Consider a white noise sequence $\varepsilon_{t}$ with positive-definite covariance matrix $\Omega$; special cases are when $\varepsilon_{t}$ is a martingale difference sequence with second moments, or when $\varepsilon_{t}$ is i.i.d. For simplicity, it is assumed here that $\left\{\varepsilon_{t}, t \in \mathbb{Z}\right\}$ is a $p \times 1$ i.i.d. sequence with $\mathrm{E}\left(\varepsilon_{t}\right)=0$ and $\mathrm{E}\left(\varepsilon_{t} \varepsilon_{s}^{\prime}\right)=0, s \neq t$.

Consider next the linear process $u_{t}=\mathrm{E}\left(u_{t}\right)+C(L) \varepsilon_{t}$, where $C(z)$ be a $p \times n$ matrix function with coefficient matrices with elements in $\mathbb{R}$ or $\mathbb{C}$, analytic on $D(0, \rho), \rho>1$, with all minors not identically equal to 0 , where $D\left(z_{\omega}, \rho\right)$ denotes the open $\operatorname{disc}\{z \in \mathbb{C}$ : $\left.\left|z-z_{\omega}\right|<\rho\right\}$ with center $z_{\omega} \in \mathbb{C}$ and radius $\rho>0$.

The matrix function $C(z)$ can be expanded around any interior point $z_{\omega} \in D(0, \rho)$ because of analyticity of $C(z)$ on it. In particular consider the point $z_{\omega}=e^{i \omega}$ on the unit circle at frequency $\omega$, which lies inside $D(0, \rho)$ because $\rho>1$.

The following definition, which parallels Johansen (1996) Chapter 3, specifies the $I_{\omega}(0)$ class of processes as a subset of all linear processes built from the white noise sequence $\varepsilon_{t}$, and introduces the notion of $I_{\omega}(d)$ processes using the difference operator at frequency $\omega, \Delta_{\omega}:=1-e^{-i \omega} L=1-z_{\omega}^{-1} L$.

Definition 2.1 (Integrated processes at frequency $\omega$ ) Let $C(z)$ be analytic on $D(0, \rho)$, $\rho>1$, and let $\varepsilon_{t}$ be a white noise process. If $\left\{u_{t}, t \in \mathbb{Z}\right\}$, satisfies $u_{t}=\mathrm{E}\left(u_{t}\right)+C(L) \varepsilon_{t}$, then $u_{t}$ is called a linear process; if, in addition,

$$
\begin{equation*}
C\left(z_{\omega}\right) \neq 0 \tag{2.1}
\end{equation*}
$$

then $u_{t}$ is said to be integrated of order zero at frequency $\omega$, indicated $u_{t} \sim I_{\omega}(0)$.
Let $d_{1}, d_{2}$ be finite non-negative integers; if $\Delta_{\omega}^{d_{1}}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\Delta_{\omega}^{d_{2}}\left(u_{t}-\mathrm{E}\left(u_{t}\right)\right)$ where $u_{t} \sim I_{\omega}(0)$, then $x_{t}$ is said to be integrated of order $d:=d_{1}-d_{2}$ at frequency $\omega$, indicated $x_{t} \sim I_{\omega}(d)$; in this case $x_{t}$ has representation

$$
\begin{equation*}
\Delta_{\omega}^{d_{1}}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\Delta_{\omega}^{d_{2}} C(L) \varepsilon_{t} \tag{2.2}
\end{equation*}
$$

where $C\left(z_{\omega}\right) \neq 0$.
Remark 2.2 (Entries in $C(z)$ ) When $\omega$ differs from 0 or $\pi$, the point $z_{\omega}=e^{i \omega}$ has a nonzero complex part, and hence the matrix $C\left(z_{\omega}\right)$ in (2.1) has complex entries, even when the matrix coefficients in the expansion of $C\left(z_{\omega}\right)$ around $z=0$ are real. In the
following, coefficient matrices are therefore understood to be complex unless otherwise noted. In this context, the notation $A^{\prime}$ indicates the conjugate transpose of $A$.

Following Gregoir (1999), the summation operator at frequency $\omega$, is defined as

$$
\begin{equation*}
\mathcal{S}_{\omega} u_{t}:=1_{t>0} \sum_{j=1}^{t} u_{j} e^{-i \omega(t-j)}-1_{t<0} \sum_{j=0}^{t+1} u_{j} e^{-i \omega(t-j)} \tag{2.3}
\end{equation*}
$$

Basic properties of the operator are proved in Gregoir (1999); these include

$$
\begin{equation*}
\Delta_{\omega} \mathcal{S}_{\omega} u_{t}=u_{t}, \quad \mathcal{S}_{\omega} \Delta_{\omega} u_{t}=u_{t}-u_{0} e^{-i \omega t} \tag{2.4}
\end{equation*}
$$

Remark 2.3 (Cancellations of $\Delta_{\omega}$ ) Take $d_{1}=d_{2}=1$ in (2.2), which in this case reads $\Delta_{\omega} x_{t}=\Delta_{\omega} u_{t}$ with $u_{t} \sim I_{\omega}(0)$. Applying the $\mathcal{S}_{\omega}$ operator on both sides one obtains $x_{t}-x_{0} e^{-i \omega t}=u_{t}-v_{0} e^{-i \omega t}$. ${ }^{3}$ If one assigns the initial value of $x_{0}$ equal to $v_{0}$, one obtains $x_{t}=u_{t}$, which corresponds to the cancellation of $\Delta_{\omega}$ from both sides of (2.2). The same reasoning applies for generic $d_{1}, d_{2}>0$ to the cancellation of $\Delta^{\min \left(d_{1}, d_{2}\right)}$ from both sides of (2.2).

REmARK 2.4 (Initial values) Remark 2.3 shows that one can simplify powers of $\Delta_{\omega}$ from both sides of (2.2) by properly assigning initial values; this cancellation is implicitly assumed in the following, and deterministic terms arising from initial values are not made explicit. The cancellation of powers of $\Delta_{\omega}$ is implicitly incorporated in the Definition 2.1 of $I_{\omega}(d)$ processes.

Remark 2.5 (Negative orders) Note that $d_{2}>0$ allows to define also negative orders of integration. In the following, expression of the type $x_{t} \sim I_{\omega}(-h)$ for positive $h$ are understood to mean $x_{t}=\Delta_{\omega}^{h} u_{t}$ for $u_{t} \sim I_{\omega}(0)$. Definition 3.3 in Johansen (1996) of an $I(d)$ process is found by setting $\omega=d_{2}=0$.

Cointegration is the property of (possibly polynomial) linear combinations of $x_{t}$ to have a lower order of integration with respect to the original order of integration of $x_{t}$.

Definition 2.6 (Cointegrating vectors at frequency $\omega$ ) Let $x_{t} \sim I_{\omega}(d)$ be as in Definition 2.1 and let $b(z)^{\prime}=\sum_{j=0}^{\infty} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}$ be a $1 \times p$ row vector function, analytic on $D\left(z_{\omega}, \eta\right)$ for some $\eta>0$ with $b\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$; then $b(L)^{\prime}$ is called a cointegrating vector at frequency $\omega$ if $b(L)^{\prime} x_{t} \sim I_{\omega}(d-s)$, for some $s>0$, i.e.

$$
\begin{equation*}
b(L)^{\prime} \Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\Delta_{\omega}^{s} g(L)^{\prime} \varepsilon_{t}, \quad g\left(z_{\omega}\right)^{\prime} \neq 0^{\prime} \tag{2.5}
\end{equation*}
$$

The integer $s$ is called the order of the cointegrating vector.
REmARK 2.7 (Entries in cointegrating vectors) Similarly to Remark 2.2, the coefficient vectors $b_{j}^{\prime}$ in the expansion of $b(z)^{\prime}$ around $z_{\omega}$ are in general complex.

Remark 2.8 (Relation to definitions in the literature) When $s=1$ and $\omega=0$, Definition 2.6 reduces to the one in Engle and Granger (1987), Johansen (1988). When $s>1$ and $\omega=0$, Definition 2.6 covers the definitions of multicointegration and polynomial cointegration and in Granger and Lee (1989), Engle and Yoo (1991), Johansen (1996). The present definitions are the ones used also in Franchi and Paruolo (2019b).

[^2]Example $2.9(\mathrm{I}(1))$ Following Johansen (1988), consider $A(L) x_{t}=\varepsilon_{t}$ with $A(z)=$ $I+\sum_{j=1}^{k} A_{j}(z-1)^{j}$ analytic on $\mathbb{C}$. Assume also that $\operatorname{det} A(z)=0$ has only solutions outside $D(0, \rho), \rho>1$, or at $z=1$. Johansen (1988) showed that for $x_{t}$ to be $\mathrm{I}(1)$ at frequency $\omega=0$, a set of necessary and sufficient conditions are:
i) $A(1)=-\alpha_{0} \beta_{0}^{\prime}$ with $\alpha_{0}, \beta_{0}$ full column rank matrices of dimension $p \times r_{0}, r_{0}<p$,
ii) $P_{\alpha_{0 \perp}} A_{1} P_{\beta_{0 \perp}}=-\alpha_{1} \beta_{1}^{\prime}$ of maximal rank $r_{1}=p-r_{0}$.

In this case $x_{t}$ satisfies (2.2) for $d_{1}=1, d_{2}=0$.
Example 2.10 (I(2)) Following Johansen (1992), consider the same VAR process as in Example 2.9. Johansen (1992) showed that for $x_{t}$ to be $\mathrm{I}(2)$ at frequency $\omega=0$, a set of necessary and sufficient conditions are:
i) $A(1)=-\alpha_{0} \beta_{0}^{\prime}$ with $\alpha_{0}, \beta_{0}$ full column rank matrices of dimension $p \times r_{0}, r_{0}<p$,
ii) $P_{\alpha_{0 \perp}} A_{1} P_{\beta_{0 \perp}}=-\alpha_{1} \beta_{1}^{\prime}$ with $\alpha_{1}, \beta_{1}$ full column rank matrices of dimension $p \times r_{1}$, $r_{1}<p-r_{0}$,
iii) $P_{\left(\alpha_{0}, \alpha_{1}\right)_{\perp}}\left(A_{2}+A_{1} \bar{\beta}_{0} \bar{\alpha}_{0}^{\prime} A_{1}\right) P_{\left(\beta_{0}, \beta_{1}\right)_{\perp}}=-\alpha_{2} \beta_{2}^{\prime}$ of maximal rank $r_{2}=p-r_{0}-r_{1}$.

In this case $x_{t}$ satisfies (2.2) for $d_{1}=2, d_{2}=0$.

## 3. CANONICAL SYSTEM OF ROOT FUNCTIONS

Assume that $x_{t} \sim I_{\omega}(d)$ with $d>0$, i.e.

$$
\begin{equation*}
\Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=C(L) \varepsilon_{t}, \quad C\left(z_{\omega}\right) \neq 0 \tag{3.1}
\end{equation*}
$$

This section discusses the connection between cointegrating vectors, as defined in Definition 2.6, and (left) root functions, as defined below following the definition in Gohberg et al. (1993).

Definition 3.1 (Root function) Let $G(z)$ be a $q \times p$ matrix function, $q \leq p$, analytic on $D\left(z_{\omega}, \eta\right), \eta>0$, and of full rank except at the point $z=z_{\omega}$. A $1 \times q$ row vector function $\phi(z)^{\prime}$ analytic on $D\left(z_{\omega}, \eta\right)$ is called a root function of $G(z)$ at $z_{\omega}$ if $\phi\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$ and

$$
\phi(z)^{\prime} G(z)=\left(z-z_{\omega}\right)^{s} \widetilde{\phi}(z)^{\prime}, \quad s>0, \quad \widetilde{\phi}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}
$$

The positive integer $s$ is called the order of the root function $\phi(z)^{\prime}$ at $z_{\omega}$.
It is immediate to see that a cointegrating vector is a root function of $C(z)$ and vice versa, i.e. that the two notions coincide, as stated in the following proposition.

Proposition 3.2 (Cointegrating vectors and root functions) $b(L)^{\prime}$ is a cointegrating vector at frequency $\omega$ if and only if $b(z)^{\prime}$ is a root function of $C(z)$ at $z_{\omega}=e^{i \omega}$.

Proof: Pre-multiplying (3.1) by $b(L)^{\prime}$ one finds

$$
\begin{equation*}
b(L)^{\prime} \Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=b(L)^{\prime} C(L) \varepsilon_{t}, \tag{3.2}
\end{equation*}
$$

One can have cancellation of some power of $\Delta_{\omega}$ from both sides of this equation, and hence have cointegration, if and only if one can factor $\Delta_{\omega}^{s}$ from the r.h.s. of (3.2) for some $s>0$, i.e. if and only if

$$
\begin{equation*}
b(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{s} \widetilde{b}(z)^{\prime} \quad \text { with } \quad s>0 \quad \text { and } \quad \widetilde{b}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime} \tag{3.3}
\end{equation*}
$$

where $\left(-z_{\omega}\right)^{s} \widetilde{b}(z)^{\prime}=g(z)^{\prime}$ in (2.5). Note that (3.2) together with (3.3) is equivalent to $b(L)^{\prime} x_{t} \sim I_{\omega}(d-s)$.
Q.E.D.

An extension of the results in Gohberg et al. (1993) shows that the order of a root functions is finite, because it is bounded by the order of $z_{\omega}$ as a zero of all minors of $G(z)$ of order $m \leq p$. This is reported in the next proposition.

Proposition 3.3 (Bound on the order of a root function) The order of a root function of $G(z)$ at $z_{\omega}$ is at most equal to the minimum among the orders of $z_{\omega}$ as a zero of all possible minors of $G(z)$ of order $m \leq p$.

Proof: In case of a square matrix $G(z)$, i.e. $p=m$, Gohberg et al. (1993) prove that the order of root functions is less or equal to $z_{\omega}$ as a zero of $\operatorname{det} G(z)$; observe that the statement in the proposition reduces to the same statement in the square case. Assume now $m<p$. The root functions need to factorize $\left(z-z_{\omega}\right)^{s}$ from all selections of $m$ columns from $G(z)$. Applying the argument for the square matrix case to all these selections, one obtains the statement.
Q.E.D.

Next, canonical system of root functions for the matrix function $C(z)$ in (3.1) at $z_{\omega}$ are introduced, see Gohberg et al. (1993). Let $\mathcal{G}$ indicate the set of root functions. Choose from $\mathcal{G}$ a root function $\phi_{1}(z)^{\prime}$ of the highest order $s_{1}$. Since the orders of the root functions are bounded by Proposition 3.3, such a function exists. Next proceed by induction over $j=2, \ldots$, choosing $\phi_{j}(z)^{\prime} \in \mathcal{G}$ to be of the highest order $s_{j}$ such that $\phi_{j}\left(z_{\omega}\right)^{\prime}$ is linearly independent from $\phi_{1}\left(z_{\omega}\right)^{\prime}, \ldots, \phi_{j-1}\left(z_{\omega}\right)^{\prime}$. Because $q:=\operatorname{dim}\left(\operatorname{Im} C\left(z_{\omega}\right)\right)^{\perp}<\infty$, this process ends with $q$ root functions $\phi_{1}(z)^{\prime}, \ldots, \phi_{q}(z)^{\prime}$.
Note that the columns in $a:=\left(\phi_{1}\left(z_{\omega}\right), \ldots, \phi_{q}\left(z_{\omega}\right)\right)$ span the finite dimensional space $\left(\operatorname{Im} C\left(z_{\omega}\right)\right)^{\perp}$, so that one can choose vectors $\left(\phi_{q+1}, \ldots, \phi_{p}\right)=a_{\perp}$ that span its orthogonal complement. This construction leads to the following definition.

Definition 3.4 ((Extended) canonical system of root functions) Let $\phi_{1}(z)^{\prime}, \ldots, \phi_{q}(z)^{\prime}$ and $\phi_{q+1}^{\prime}, \ldots, \phi_{p}^{\prime}$ be constructed as above; then

$$
\left(\begin{array}{c}
\phi_{1}(z)^{\prime} \\
\vdots \\
\phi_{q}(z)^{\prime}
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{c}
\phi_{1}(z)^{\prime} \\
\vdots \\
\phi_{q}(z)^{\prime} \\
\underline{\phi}_{q+1}^{\prime}- \\
\vdots \\
\phi_{p}^{\prime}
\end{array}\right)
$$

are called a canonical system of root functions (respectively an extended canonical system of root functions) of $C(z)$ at $z_{\omega}$ of orders $\left(s_{1}, s_{2}, \ldots, s_{q}\right)$ (respectively $\left(s_{1}, s_{2}, \ldots, s_{q}, s_{q+1}, \ldots, s_{p}\right)$ ) with $\infty>s_{1} \geq s_{2} \geq \cdots \geq s_{q}>0=s_{q+1}=\cdots=s_{p}$.

Such a canonical system of root functions is not unique; in fact one could replace $\phi_{1}(z)$ with $\phi_{1}^{\star}(z)^{\prime}:=\phi_{1}(z)^{\prime}+z^{s_{1}-s_{2}} \phi_{2}(z)^{\prime}$ obtaining, see (3.3),

$$
\phi_{1}^{\star}(z)^{\prime} C(z)=z^{s_{1}} \widetilde{\phi}_{1}^{\prime}(z)+z^{s_{1}-s_{2}+s_{2}} \widetilde{\phi}_{2}^{\prime}(z)=z^{s_{1}} \widetilde{\phi}^{\star \prime}(z)
$$

where $\widetilde{\phi}^{\prime \star}(z):=\widetilde{\phi}_{1}^{\prime \prime}(z)+\widetilde{\phi}_{2}^{\prime \star}(z)$ with $\widetilde{\phi}^{\prime \star}\left(z_{\omega}\right) \neq 0^{\prime}$, because $\widetilde{\phi}^{\prime *}\left(z_{\omega}\right)=0$ would contradict the fact that $s_{1}$ is maximal. This shows that $\phi_{1}^{\prime \prime}(z)$ can replace $\phi_{1}^{\prime}(z) .{ }^{4}$

[^3]While a canonical system of root functions (and also a canonical system of root functions) is not unique, the orders $s_{1} \geq s_{2} \geq \cdots \geq s_{q}>0=s_{q+1}=\cdots=s_{p}$ are uniquely determined by $C(z)$, see Lemma 1.1 in Gohberg et al. (1993); they are called partial multiplicities.
Finally, consider the local Smith factorization of $A(z)$ at $z=z_{\omega}$, see Gohberg et al. (1993), i.e. the factorization

$$
\begin{equation*}
A(z)=E(z) M(z) H(z), \tag{3.4}
\end{equation*}
$$

where $M(z)=\operatorname{diag}\left(\left(z-z_{\omega}\right)^{\kappa_{h}}\right)_{h=1, \ldots, p}$ is uniquely defined and contains the partial multiplicities $\kappa_{1} \leq \cdots \leq \kappa_{p}$ of $A(z)$ at $z=z_{\omega}$, where $\kappa_{h}=s_{p+1-h}$, and the matrices $E(z), H(z)$ are analytic and invertible in a neighbourhood of $z=z_{\omega}$ and are non-unique. $M(z)$ and $E(z), H(z)$ are respectively called the local Smith form and the canonical system of root functions of $A(z)$ at $z=z_{\omega} \cdot{ }^{5}$

REmark 3.5 (Extended canonical system of root functions in the $\mathrm{I}(1)$ case) In the $I(1)$ case, see Example 2.9, the orders of an extended canonical system of root functions of $C(z)$ at 1 are $\left(s_{1}, \ldots, s_{r_{0}}, s_{r_{0}+1}, \ldots, s_{p}\right)=(1, \ldots, 1,0, \ldots, 0)$ and a possible choice of canonical system of root functions corresponding to these unique orders is given by the $p$ rows in $\left(\beta_{0}, \beta_{1}\right)^{\prime}$.

REmark 3.6 (Extended canonical system of root functions in the $\mathrm{I}(2)$ case) In the $\mathrm{I}(2)$ case, see Example 2.10, the orders of a canonical system of root functions of $C(z)$ at 0 are $\left(s_{1}, \ldots, s_{r_{0}}, s_{r_{0}+1}, \ldots, s_{r_{0}+r_{1}}, s_{r_{0}+r_{1}+1}, \ldots, s_{p}\right)=(2, \ldots, 2,1, \ldots, 1,0, \ldots, 0)$ and a possible choice of canonical system of root functions corresponding to these unique orders is given by the $p$ rows in $\left(\beta_{0}+\Delta \beta_{2} \delta^{\prime}, \beta_{1}, \beta_{2}\right)^{\prime}$, where $\delta=\bar{\alpha}_{0}^{\prime} A_{1}$.

## 4. PROPERTIES OF ROOT FUNCTIONS

This section discusses properties of root function; in particular it shows that cointegrating vectors can be truncated to a polynomial without loss of generality. Consider the set $\mathcal{A}$ of $1 \times p$ row vectors $b(z)^{\prime}=\left(f_{1}(z), \ldots, f_{p}(z)\right)$ with elements $f_{j}(z) j=1, \ldots, p$ that are scalar analytic functions of $z$ for $z \in D\left(z_{\omega}, \eta\right), \eta>0, f_{j}(z)=\sum_{h=0}^{\infty} f_{j h}\left(z-z_{\omega}\right)^{h}$ and $f_{j h} \in F$, where $F=\mathbb{R}$ or $\mathbb{C}$. Next denote by $\mathcal{U}$ the set of $1 \times p$ cointegrating vectors, i.e. root functions, and observe that $\mathcal{U} \subset \mathcal{A}$.

Proposition 4.1 (Linear combinations with polynomial coefficients) Let $b(z)^{\prime} \in \mathcal{U}$ be a cointegrating vector of order $s$ and let $c(z)^{\prime} \in \mathcal{A}$ be any analytic vector function; then, for $m \geq 1$, the $1 \times p$ row vector $a(z)^{\prime}$ with

$$
a(z)^{\prime}=b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}
$$

is still a cointegrating vector in $\mathcal{U}$ of order at least $n=\min (m, s)$. In particular, when $m=s$, this implies that the truncation of $b(z)^{\prime}$ to the polynomial $b^{(s-1)}(z)^{\prime}$ consisting of the terms up to order $s-1$ is still a root function of order at least $s$.

Proof: By definition, see (3.3), one has $b(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{s} \widetilde{b}(z)^{\prime}$ with $\widetilde{b}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$. Hence

$$
a(z)^{\prime} C(z)=b(z)^{\prime} C(z)+\left(z-z_{\omega}\right)^{m} c(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{n} \widetilde{a}(z)^{\prime} .
$$

[^4]where $\widetilde{a}(z)^{\prime}=\left(z-z_{\omega}\right)^{s-n} \widetilde{b}(z)^{\prime}+\left(z-z_{\omega}\right)^{m-n} c(z)^{\prime} C(z)$.
If $\widetilde{a}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$, then $a(z)^{\prime}$ is a root function of order $n$. If, instead, $\widetilde{a}\left(z_{\omega}\right)^{\prime}=0^{\prime}$, then $a(z)^{\prime}$ is a root function of order greater than $n$. This shows that $a(z)$ is a root function of order at least equal to $n$.
Next consider the special case $m=s$; expand $b(z)^{\prime}$ around $z_{\omega}$ as $b(z)^{\prime}=\sum_{j=0}^{\infty} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}$ and select $c(z)^{\prime}=-\sum_{j=s+1}^{\infty} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j-s}$. This shows that $a(z)^{\prime}=b(z)^{\prime}+\left(z-z_{\omega}\right)^{s} c(z)^{\prime}=$ $\sum_{j=0}^{s-1} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}$, the polynomial $b^{(s-1)}(z)^{\prime}$ of order $s-1$ obtained by truncation of the tail of $b(z)^{\prime}$.
Q.E.D.

The properties discussed in the previous proposition include several types of effects, illustrated in the following examples.

Example 4.2 (Adding the tail) Take $b(z)^{\prime}=\sum_{j=0}^{2} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}, m=3, c(z)^{\prime}=\sum_{j=3}^{\infty} c_{j}^{\prime}(z-$ $\left.z_{\omega}\right)^{j-3}$. One can see that $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}=\sum_{j=0}^{\infty} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}$ with $b_{j}^{\prime}:=c_{j}^{\prime}$ for $j=3, \ldots, \infty$; in this case the operation $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}$ is 'adding a tail' to the cointegrating vector $b(z)^{\prime}$, where the 'tail' is identified as the highest powers in $z$ of the polynomial. This operation preserves the fact that $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}$ is still a cointegrating vector, possibly of higher order.

Example 4.3 (Mixing the tail) Take $b(z)^{\prime}=\sum_{j=0}^{2} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}, m=1, c(z)^{\prime}=\sum_{j=0}^{1} c_{j}^{\prime}(z-$ $\left.z_{\omega}\right)^{j}$; one obtains $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}=b_{0}^{\prime}+\left(b_{1}^{\prime}+c_{0}^{\prime}\right) z+\left(b_{2}^{\prime}+c_{1}^{\prime}\right) z^{2}$, where the coefficients to the powers 1 and 2 of the resulting polynomial are mixed. This operation preserves the fact that $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}$ is still a cointegrating vector, possibly of lower order.

Example 4.4 (Cutting the tail) Use Example 4.3 above, choosing $c_{0}^{\prime}=-b_{1}^{\prime}$ and $c_{1}^{\prime}=$ $-b_{2}^{\prime}$; one obtains $b(z)^{\prime}+\left(z-z_{\omega}\right)^{m} c(z)^{\prime}=b_{0}^{\prime}$, so as to eliminate the powers 1 and 2 of the resulting polynomial. This operation preserves the fact that $b(z)^{\prime}=b_{0}^{\prime}$ is still a cointegrating vector, with no polynomial part.

As the examples above illustrate, there is a close relation between a root function $b(z)^{\prime}:=\sum_{j=0}^{\infty} b_{j}^{\prime}\left(z-z_{w}\right)^{j}$ and its truncation to a polynomial of degree $s-1, b^{(s-1)}(z)^{\prime}:=$ $\sum_{j=0}^{s-1} b_{j}^{\prime}\left(z-z_{\omega}\right)^{j}$, as described in the following proposition.

Proposition 4.5 (Truncation) One has the following properties:
(i) $b(z)^{\prime}$ is a root function of $C(z)$ at $z_{\omega}$ of order at least $s$ if and only if $b^{(s-1)}(z)^{\prime}$ is a root function of $C(z)$ at $z_{\omega}$ of order at least $s$;
(ii) $b(z)^{\prime}$ is a root function of $C(z)$ at $z_{\omega}$ of order $s$ if and only if $b^{(s)}(z)^{\prime}$ is a root function of $C(z)$ at $z_{\omega}$ of order $s$.

Proof: (i) Let $C(z):=\sum_{j=0}^{\infty} C_{j}\left(z-z_{\omega}\right)^{j}$. By (3.3), one has

$$
\begin{equation*}
b(z)^{\prime} C(z)=\sum_{h=0}^{\infty} b_{h}^{\prime}\left(z-z_{\omega}\right)^{h} \sum_{j=0}^{\infty} C_{j}\left(z-z_{\omega}\right)^{j}=\sum_{j=0}^{\infty} W_{j}\left(z-z_{\omega}\right)^{j} \tag{4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{n}=\sum_{j=0}^{n} b_{h}^{\prime} C_{n-h}, \quad n=0,1, \ldots \tag{4.2}
\end{equation*}
$$

One can hence factor at least $\left(z-z_{\omega}\right)^{s}$ from the r.h.s. if and only if $W_{0}=\cdots=W_{s-1}=0$. The statement follows from the fact that $W_{h}$ for $h=0, \ldots, s-1$ in (4.2) depend only on $b_{0}, \ldots, b_{s-1}$ that are common to $b(z)$ and $b^{(s-1)}(z)$.
(ii) In (4.1) one can factor $\left(z-z_{\omega}\right)^{s} \widetilde{b}(z)^{\prime}$ with $\widetilde{b}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$ from the r.h.s. if and only if $W_{0}=\cdots=W_{s-1}=0$ and $W_{s} \neq 0$.
Q.E.D.

Remark 4.6 (Irrelevance of higher order terms) Proposition 4.5 shows that the root function $b(z)$ of order $s$ and its truncated version $b^{(s)}(z)$ are exangeable; all coefficients in $b(z)$ past order $s$ are irrelevant.

Proposition 4.7 (Degree of root functions in canonical system of root functions) Let $b(z)^{\prime}:=\sum_{j=0}^{s} b_{j}^{\prime} z^{j} \in \mathcal{G}$ be a root function in a canonical system of root functions of $C(z)$ at $z_{\omega}$ of order $s$; then its truncation $b^{(s-1)}(z)^{\prime}:=\sum_{j=0}^{s-1} b_{j}^{\prime} z^{j}$ to a polynomial of degree $s-1$ is still a root function of $C(z)$ at 0 of the same order $s$.

Proof: The only addition to the proof of Proposition 4.5 is that the orders of $b(z)^{\prime}$ and $b^{(s-1)}(z)^{\prime}$ are the same, which follows from the fact that $s$ is maximal, and hence no further cancellations are possible for $b^{(s-1)}(z)^{\prime}$.
Q.E.D.

Note that $b(z)^{\prime}$ and $b^{(s-1)}(z)^{\prime}$ share the same coefficients $W_{h}$ for $h=0, \ldots, s-1$ in representation (4.1)-(4.2) and no further cancellations are possible in this direction. This shows that $b^{(s-1)}(z)^{\prime}$ contains the relevant part of the same cointegrating relation.

Remark 4.8 (Equivalence classes - Part 1) Proposition 4.5 states that $b^{(s)}(z)^{\prime}$ can be taken to represent all root function $b(z)^{\prime}$ with the same coefficients up to order $s$ and a different tail. This set is an equivalence class, with representative element $b^{(s)}(z)^{\prime}$. This observation suggests that, despite cointegrating vectors and root functions are analytic functions, attention can be restricted to the polynomials $b^{(s)}(z)^{\prime}$ to represent all equivalent analytic cointegrating vectors with different tails.

## 5. THE COINTEGRATING SPACE

This section defines the cointegrating space as a subspace of the space of rational vectors in $z$ over the field of rational scalars in $z$, where linear (sub-) spaces require the set of scalar to be a field.
Denote by $F$ the field of reals $\mathbb{R}$ or complex numbers $\mathbb{C}$. Let $F[z]$ indicate the polynomial ring formed as the set of polynomials in $z$ with coefficients in $F$. As it is well known, $F[z]$ is a ring but not a field, see e.g. Hungerford (1980) because polynomials, unlike rational functions, lack the multiplication inverse.
Let $F(z)$ denote the field of fractions of the polynomial ring $F[z]$, i.e. the smallest field containing all elements $c(z)=a(z) / d(z)$, with $d(z) \neq 0$, where $c(z)$ and $d(z)$ are polynomials in $F[z]{ }^{6}$

Remark 5.1 (Equivalence classes - Part 2) Observe that the element $a(z) / d(z) \in F(z)$ with $d(z) \neq 0$ represent an equivalence class of all ratios of polynomials $f(z) / g(z) \in F(z)$ such that $a(z) g(z)=d(z) f(z)$. Because of this, the representative element $a(z) / d(z)$ of

[^5]this equivalence class can be chosen with $a(z)$ and $d(z)$ relatively prime and $d(z)$ monic; this is assumed in the following.

Recall that a vector space involves a pair of sets $(\mathcal{V}, F)$, where $\mathcal{V}$ is the set of vectors and $F$ is the field of scalars, equipped with the operations of vector addition and multiplication by a scalar. In the present context, the field of scalars $F$ is chosen equal to $F(z)$, the field of fractions of the polynomial ring $F[z],{ }^{7}$ and the set of vectors is selected as the set of $1 \times p$ row vectors with elements in $F(z)$, i.e. $\mathcal{Q}=F(z)^{p}=F(z) \times F(z) \times \cdots \times F(z)(p$ times).
The pair $(\mathcal{Q}, F(z))$ is a vector space with the usual vector addition and multiplication operations and it is the ambient space for the cointegrating space defined below; the Appendix reports a verification of the closure of $(\mathcal{Q}, F(z))$ with respect to these operations.

Remark 5.2 (Rational vectors with a common denominator) Note that the denominators in the elements of $a(z)^{\prime} \in \mathcal{Q}, a(z)^{\prime}=\left(f_{1}(z), \ldots, f_{p}(z)\right)$, with $f_{j}(z)=c_{j}(z) / d_{j}(z)$, can be grouped together considering the least common multiple $d(z)$ of $d_{1}(z), \ldots, d_{p}(z)$, for which $d(z)=d_{j}(z) h_{j}(z)$, with $h_{j}(z)$ is a polynomial in $F[z]$, so that

$$
\begin{equation*}
a(z)^{\prime}=\frac{1}{d(z)} b(z)^{\prime} \quad b(z)^{\prime}=\left(f_{1}(z) h_{1}(z), \ldots, f_{p}(z) h_{p}(z)\right) \tag{5.1}
\end{equation*}
$$

Note that $d(z)$ and $b(z)^{\prime}$ in (5.1) are relatively prime in the sense that no common factor can be simplified between $d(z)$ and all the elements in $b(z)^{\prime}$.

One next needs to discuss cointegrating properties for rational row vectors $a(z)^{\prime}=$ $b(z)^{\prime} / d(z)$, see Definition 2.6. Consider some vector polynomial $b(L)^{\prime} \in F[z]^{p}$ satisfying (3.1) in Definition 2.6 of cointegrating vector, i.e.

$$
\begin{equation*}
b(L)^{\prime} \Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\left(-z_{\omega} \Delta_{\omega}\right)^{s} \widetilde{b}(L)^{\prime} \varepsilon_{t} \tag{5.2}
\end{equation*}
$$

with $\widetilde{b}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$ and $s>0$.
Next consider some scalar polynomial $d(z) \in F[z]$ as denominator in $a(z)^{\prime}=b(z)^{\prime} / d(z)$. In case $d\left(z_{\omega}\right)=0$, one can express $d(z)$ as $d(z)=\left(z-z_{\omega}\right)^{n} \widetilde{d}(z)$ where $\widetilde{d}\left(z_{\omega}\right) \neq 0$; note that this expression includes the case of $d\left(z_{\omega}\right) \neq 0$ when setting $n=0$. Multiplying both sides of (5.2) by $1 / d(L)$ for $d(z) \in F[z]$, one finds that $a(L)^{\prime} \Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)$ equals

$$
\begin{equation*}
\frac{b(L)^{\prime}}{d(L)} \Delta_{\omega}^{d}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\frac{b(L)^{\prime}}{\widetilde{d}(L)} \Delta_{\omega}^{d-n}\left(x_{t}-\mathrm{E}\left(x_{t}\right)\right)=\left(-z_{\omega}\right)^{s} \Delta_{\omega}^{s-n} \frac{\widetilde{b}(L)^{\prime}}{\widetilde{d}(L)} \varepsilon_{t}, \quad \frac{\widetilde{b}\left(z_{\omega}\right)^{\prime}}{\widetilde{d}\left(z_{\omega}\right)} \neq 0^{\prime} \tag{5.3}
\end{equation*}
$$

Note that the difference between the orders of integration on the two sides of equation (5.3) is still $(d-n)-(s-n)=d-s$, so that $a(L)^{\prime}$ satisfies the requirements of a cointegrating vector in Definition 2.6, and $a(z)^{\prime} x_{t} \sim I(d-s)$.

REmARK 5.3 (Attention can be restricted to numerators) From the above, one hence concludes that (5.3) holds iff (5.2) holds, i.e. the cointegration property for a rational row vector $a(z)^{\prime}$ is associated with the cointegration property for its row vector numerator $b(z)^{\prime}$.

[^6]Remark 5.4 (Equivalence classes - Part 3) Another consequence of (5.2)-(5.3) is that all rational vectors that have the same row vector numerator $b(z)^{\prime}$ have the same cointegrating properties; these form an equivalence class. Any vector with $b(z)^{\prime}$ as numerator can be taken to represent this equivalence class.

Because of (5.2)-(5.3), $\mathcal{Q}$ contains all rational $1 \times p$ row vectors $a(z)^{\prime}=b(z)^{\prime} / d(z)$ where $b(z)^{\prime}$ is a root function of $C(z)$. Next define $\mathcal{G}$ as the set of root function of $C(z)$ in (3.1) and denote by $\mathcal{B}$ the smallest subset of $\mathcal{Q}$ containing $\mathcal{G}$. One can prove the following proposition.

Proposition $5.5((\mathcal{B}, F(z))$ is a subspace of $(\mathcal{Q}, F(z)))$ The pair $(\mathcal{B}, F(z))$ forms a linear subspace of the vector space $(\mathcal{Q}, F(z))$.

Proof: Let $a(z)^{\prime}=b(z)^{\prime} / d(z)$ and $g(z)^{\prime}=k(z)^{\prime} / h(z)$ be vectors in $\mathcal{B}$ and let $c(z)=$ $n(z) / m(z)$ be a scalar in $F(z)$. Because $a(z)^{\prime}, g(z)^{\prime} \in \mathcal{B}$, one has that $b(z)^{\prime}$ and $k(z)^{\prime}$ are root functions for $C(z)$, respectively of order $s_{b}$ and $s_{k}$. Because $(\mathcal{Q}, F(z))$ is the ambient vector space, one has that $a(z)^{\prime}+g(z)^{\prime} \in \mathcal{Q}$; more precisely one finds

$$
a(z)^{\prime}+g(z)^{\prime}=\frac{b(z)^{\prime}}{d(z)}+\frac{k(z)^{\prime}}{h(z)}=\frac{h(z) b(z)^{\prime}+d(z) k(z)^{\prime}}{d(z) h(z)}=: \frac{f(z)^{\prime}}{\ell(z)}, \quad \text { say }
$$

where $f(z)^{\prime}:=h(z) b(z)^{\prime}+d(z) k(z)^{\prime}$ and $\ell(z):=d(z) h(z) \neq 0$ because both $d(z), h(z) \neq 0$. Thanks to (5.2)-(5.3), in order to prove that $a(z)^{\prime}+g(z)^{\prime} \in \mathcal{B}$ it remains to show that $f(z)^{\prime}$ is a root function. One finds, letting $s:=\min \left(s_{b}, s_{k}\right)$

$$
f(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{s_{b}} h(z) \widetilde{b}(z)^{\prime}+\left(z-z_{\omega}\right)^{s_{k}} d(z) \widetilde{k}(z)^{\prime}=\left(z-z_{\omega}\right)^{s} \widetilde{f}(z)^{\prime}
$$

where $\widetilde{f}(z)^{\prime}=\left(z-z_{\omega}\right)^{s_{b}-s} h(z) \widetilde{b}(z)^{\prime}+\left(z-z_{\omega}\right)^{s_{k}-s} d(z) \widetilde{k}(z)^{\prime}$. This shows that $f(z)^{\prime}$ is a root function at least of order $s:=\min \left(s_{b}, s_{k}\right)$, and hence $a(z)^{\prime}+g(z)^{\prime} \in \mathcal{B}$. This shows that $\mathcal{B}$ is a vector space, and a proper subspace of $\mathcal{Q}$ because $\mathcal{B} \subset \mathcal{Q}$; this completes the proof. Q.E.D.

The previous proposition hence allows to formulate the following definition of cointegrating space.

Definition 5.6 (Cointegrating space at frequency $\omega$ ) All nonzero row vectors in $\mathcal{B}$ are cointegrating vectors, and the linear subspace $(\mathcal{B}, F(z))$ of $(\mathcal{Q}, F(z))$, is called the 'cointegrating space'.

Remark that in the $I(1)$ case there is no need to consider polynomials, and the ambient space is $\left(\mathbb{R}^{p}, \mathbb{R}\right)$ or $\left(\mathbb{C}^{p}, \mathbb{C}\right)$. In the $I(2)$ case one needs to consider polynomials, and the ambient space is $(\mathcal{Q}, F(z))$.

## 6. THE LOCAL RANK FACTORIZATION PROVIDES A BASIS

This section shows that a canonical system of root functions $\varphi(z)^{\prime}$ provides a basis for the cointegrating space $\mathcal{B}$, and it shows how to explicitly obtain a canonical system of root functions for a generic AR process

$$
\begin{equation*}
A(L) X_{t}=\varepsilon_{t}, \quad A_{0} \neq 0, \quad\left|A_{0}\right|=0 \tag{6.1}
\end{equation*}
$$

with $A(z)$ analytic for all $z \in D(0, \delta), \delta>1$, having roots at $z=z_{\omega}$ and at $|z|>1$.
The first theorem states that $\varphi(z)^{\prime}$ is a basis of the cointegrating space.

Theorem 6.1 (A canonical system of root functions gives a basis of $\mathcal{B}$ ) A canonical system of root functions $\varphi(z)^{\prime}$ gives a basis of the cointegrating space $(\mathcal{B}, F(z)$ ).

Proof: One needs to prove that any root function can be obtained by taking linear combinations of the rows in $\varphi(z)^{\prime}$ with coefficients in $F(z)$. Consider a root function $b(z)^{\prime}$, and observe that $b(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{\widetilde{b}} \widetilde{b}(z)^{\prime}$ for some $s>0$ with $\widetilde{b}\left(z_{\omega}\right)^{\prime} \neq 0^{\prime}$. This implies that $b\left(z_{\omega}\right)^{\prime} C\left(z_{\omega}\right)=0$, i.e. that $b\left(z_{\omega}\right) \in\left(\operatorname{Im} C\left(z_{\omega}\right)\right)^{\perp}$, where $q:=\operatorname{dim}\left(\operatorname{Im} C\left(z_{\omega}\right)\right)^{\perp}$.
Proceed by contradiction, assuming that $b(z)^{\prime}$ cannot be obtained by taking linear combinations of the rows in $\varphi(z)^{\prime}$ with coefficients in $F(z)$, i.e. that $v(z)^{\prime}(\varphi(z), b(z))^{\prime}=0$ implies $v(z)^{\prime}=0$, where $v(z)^{\prime} \in F(z)^{q+1}$. Hence $v\left(z_{\omega}\right)^{\prime}\left(\varphi\left(z_{\omega}\right), b\left(z_{\omega}\right)\right)^{\prime}=0$ implies $v\left(z_{\omega}\right)^{\prime}=0$, with $v\left(z_{\omega}\right)^{\prime} \in F^{q+1}$. This implies that $b\left(z_{\omega}\right)^{\prime}$ is linearly independent from $\varphi\left(z_{\omega}\right)^{\prime}$, and this contradicts the fact that $q:=\operatorname{dim}\left(\operatorname{Im} C\left(z_{\omega}\right)\right)^{\perp}$. Thus no root function $b(z)^{\prime}$ exists that is linearly independent form the canonical system of root functions $\varphi(z)^{\prime}$. This completes the proof.
Q.E.D.

The rest of the section is devoted to the explicit construction of a canonical system of root functions $\varphi(z)^{\prime}$ for the AR process in (6.1). The derivation of the Granger representation theorem involves the inversion of the matrix function

$$
\begin{equation*}
A(z)=\sum_{n=0}^{\infty} A_{n}\left(z-z_{\omega}\right)^{n}, \quad A_{n} \in \mathbb{C}^{p \times p}, \quad A_{0} \neq 0, \quad\left|A_{0}\right|=0 \tag{6.2}
\end{equation*}
$$

around the singular point $z=z_{\omega} \in D(0, \delta)$. This includes the case of matrix polynomials $A(z)$, in which the degree of $A(z)$ is finite, $k$ say, with $A_{n}=0$ for $n>k$, and $A(z)$ is analytic for all $z \in \mathbb{C}$.
The inversion of $A(z)$ around the singular point $z=z_{\omega}$ yields an inverse with a pole of some order $d=1,2, \ldots$ at $z=z_{\omega}$; an explicit condition on the coefficients $\left\{A_{n}\right\}_{n=0}^{\infty}$ in (6.2) for $A(z)^{-1}$ to have a pole of given order $d$ is described in Theorem 6.2 below; this is indicated as the POLE $(d)$ condition in the following. Under the POLE $(d)$ condition, $A(z)^{-1}$ has Laurent expansion around $z=z_{\omega}$ given by

$$
\begin{equation*}
A(z)^{-1}=:\left(z-z_{\omega}\right)^{-d} C(z)=\sum_{n=0}^{\infty} C_{n}\left(z-z_{\omega}\right)^{n-d}, \quad C_{0} \neq 0, \quad\left|C_{0}\right|=0 \tag{6.3}
\end{equation*}
$$

Note that $C\left(z_{\omega}\right)=C_{0} \neq 0$ and $C(z)$ is expanded around $z=z_{\omega}$. In the following, the coefficients $\left\{C_{n}\right\}_{n=0}^{\infty}$ are called the Laurent coefficients. The first $d$ of them, $\left\{C_{n}\right\}_{n=0}^{d-1}$, make up the principal part and characterize the singularity of $A(z)^{-1}$ at $z=z_{\omega}$.
The following result is taken from Franchi and Paruolo (2019b).
Theorem $6.2\left(\operatorname{POLE}(d)\right.$ condition) Let $0<r_{0}:=\operatorname{rank} A_{0}<p, r_{0}^{\max }:=p-r_{0}$ and define $\alpha_{0}, \beta_{0}$ by the rank factorization $A_{0}=-\alpha_{0} \beta_{0}^{\prime}$. Moreover, for $j=1,2, \ldots$ define $\alpha_{j}, \beta_{j}$ by the rank factorization

$$
\begin{equation*}
P_{a_{j \perp}} A_{j, 1} P_{b_{j \perp}}=-\alpha_{j} \beta_{j}^{\prime}, \quad a_{j}:=\left(\alpha_{0}, \ldots, \alpha_{j-1}\right), \quad b_{j}:=\left(\beta_{0}, \ldots, \beta_{j-1}\right), \tag{6.4}
\end{equation*}
$$

where $P_{x}$ denotes the orthogonal projection onto the space spanned by $x$ and

$$
A_{h+1, n}:=\left\{\begin{array}{cl}
A_{n} & \text { for } h=0  \tag{6.5}\\
A_{h, n+1}+A_{h, 1} \sum_{i=0}^{h-1} \bar{\beta}_{i} \bar{\alpha}_{i}^{\prime} A_{i+1, n} & \text { for } h=1,2, \ldots
\end{array}, \quad n=0,1, \ldots .\right.
$$

Finally, let

$$
\begin{equation*}
r_{j}:=\operatorname{rank}\left(P_{a_{j \perp}} A_{j, 1} P_{b_{j \perp}}\right), \quad r_{j}^{\max }:=p-\sum_{i=0}^{j-1} r_{i} . \tag{6.6}
\end{equation*}
$$

Then, a necessary and sufficient condition for $A(z)$ to have an inverse with pole of order $d=1,2, \ldots$ at $z=z_{\omega}-\operatorname{called} \operatorname{Pole}(d)$ condition - is that

$$
\begin{cases}r_{j}<r_{j}^{\max } & \text { (reduced rank condition) for } j=1, \ldots, d-1 \\ r_{d}=r_{d}^{\max } & \text { (full rank condition) for } j=d\end{cases}
$$

Observe that because rank $P_{a_{j \perp}} A_{j, 1} P_{b_{j \perp}}=\operatorname{rank} a_{j \perp}^{\prime} A_{j, 1} b_{j \perp}$, one has $r_{j}=\operatorname{rank} a_{j \perp}^{\prime} A_{j, 1} b_{j \perp} ;$ hence $d=1$ if and only if

$$
r_{1}=r_{1}^{\max }, \quad \text { where } \quad r_{1}=\operatorname{rank} \alpha_{0 \perp}^{\prime} A_{1} \beta_{0 \perp} \quad \text { and } \quad r_{1}^{\max }=p-r_{0} .
$$

This corresponds to the condition in Howlett (1982, Theorem 3) and to the $I(1)$ condition in Johansen (1991, Theorem 4.1). Similarly, one has $d=2$ if and only if $r_{1}<r_{1}^{\max }$,

$$
r_{2}=r_{2}^{\max }, \quad \text { where } \quad r_{2}=\operatorname{rank} a_{2 \perp}^{\prime}\left(A_{2}+A_{1} \bar{\beta}_{0} \bar{\alpha}_{0}^{\prime} A_{1}\right) b_{2 \perp} \quad \text { and } \quad r_{2}^{\max }=p-r_{0}-r_{1}
$$

which corresponds to the $I(2)$ condition in Johansen (1992, Theorem 3).
Theorem 6.2 is thus a generalization of the Johansen's $I(1)$ and $I(2)$ conditions and shows that, in order to have a pole of order $d$ in the inverse, one needs $d+1$ rank conditions on $A(z)$ : the first $j=0, \ldots, d-1$ are reduced rank conditions, $r_{j}<r_{j}^{\max }$, that establish that the order of the pole is greater than $j$; the last one is a full rank condition, $r_{d}=r_{d}^{\max }$, that establishes that the order of the pole is exactly equal to $d$. These requirements make up the POLE ( $d$ ) condition.

Theorem 6.3 (Local Smith factorization) For $j=0, \ldots, d$ and $n=1,2, \ldots$, define the $p \times r_{j}$ matrix functions $\gamma_{j}(z)$ from

$$
\begin{equation*}
\gamma_{j, 0}^{\prime}:=\beta_{j}^{\prime}, \quad \gamma_{j, n}^{\prime}:=-\bar{\alpha}_{j}^{\prime} A_{j+1, n}, \quad \gamma_{j}(z)^{\prime}:=\sum_{n=0}^{\infty} \gamma_{j, n}^{\prime}\left(z-z_{\omega}\right)^{n}, \tag{6.7}
\end{equation*}
$$

and define the $p \times p$ matrix functions $\Gamma(z)$ and $\Lambda(z)$ from
$\Gamma(z):=\left(\begin{array}{c}\gamma_{0}(z)^{\prime} \\ \vdots \\ \gamma_{d}(z)^{\prime}\end{array}\right), \quad \Lambda(z):=\left(\begin{array}{ccc}\left(z-z_{\omega}\right)^{0} I_{r_{0}} & & \\ & \ddots & \\ & & \left(z-z_{\omega}\right)^{d} I_{r_{d}}\end{array}\right), \quad \Phi(z)=A(z) \Gamma(z)^{-1} \Lambda(z)^{-1}$.
Then $\Gamma(z), \Phi(z)$ are analytic and invertible on $D\left(z_{\omega}, \rho_{\circ}\right)$ for some $\rho_{\circ}>0$ and one can choose the factors $E(z), M(z), H(z)$ in (3.4) as

$$
E(z)=\Phi(z), \quad M(z)=\Lambda(z), \quad H(z)=\Gamma(z)
$$

That is, $\Lambda(z)$ and $\Phi(z), \Gamma(z)$ are respectively the local Smith form and extended canonical system of root functions of $A(z)$ at $z_{\omega}$.

Theorem 6.3 shows that the LRF fully characterizes the elements of the local Smith factorization of $A(z)$ at $z_{\omega}$. In fact, the values of $j$ with $r_{j}>0$ in the LRF provide the distinct partial multiplicities of $A(z)$ at 1 and $r_{j}$ gives the number of partial multiplicities that are equal to a given $j$; this characterizes the local Smith form $\Lambda(z)$. Moreover, it also provides constructions of extended canonical system of root functions.
Remark that the $j$-th block of rows in $\Gamma(z) C(z)=\left(z-z_{\omega}\right)^{d} \Lambda(z)^{-1} \Phi(z)^{-1}$ can be written as

$$
\begin{equation*}
\gamma_{j}(z)^{\prime} C(z)=\left(z-z_{\omega}\right)^{d-j} \xi_{j}(z)^{\prime}, \quad j=0, \ldots, d \tag{6.9}
\end{equation*}
$$

where $\gamma_{j}\left(z_{\omega}\right)^{\prime}=\beta_{j}^{\prime}$ and $\xi_{j}\left(z_{\omega}\right)^{\prime}$ have full row rank; here $\xi_{j}(z)^{\prime}$ denotes the corresponding block of rows in $\Phi(z)^{-1}$. This shows that $\gamma_{j}(z)^{\prime}$ are $r_{j}$ root functions of order $d-j$ of $C(z)$.
The next result presents the Triangular representation.
Proposition 6.4 (Triangular representation) Let $X_{t}$ in (6.1) satisfy the POLE $(d)$ condition on $A(z)$ and define

$$
\Gamma_{\circ}(L):=\binom{\varphi(L)^{\prime}}{\hdashline \bar{\beta}_{d}^{\prime}-}, \quad \varphi(L)^{\prime}:=\left(\begin{array}{c}
\gamma_{0}^{(d-1)}(L)^{\prime}  \tag{6.10}\\
\gamma_{1}^{(d-2)}(L)^{\prime} \\
\vdots \\
\gamma_{d-1}^{(0)}(L)^{\prime}
\end{array}\right)=\left(\begin{array}{c}
\beta_{0}^{\prime}+\sum_{k=1}^{d-1}\left(-z_{\omega}\right)^{k} \gamma_{0, k}^{\prime} \Delta_{\omega}^{k} \\
\beta_{1}^{\prime}+\sum_{k=1}^{d-2}\left(-z_{\omega}\right)^{k} \gamma_{1, k}^{\prime} \Delta_{\omega}^{k} \\
\vdots \\
\beta_{d-1}^{\prime}
\end{array}\right),
$$

where $\gamma_{j}^{(d-j-1)}(z)^{\prime}=\sum_{k=0}^{d-j-1} \gamma_{j, k}^{\prime}\left(z-z_{\omega}\right)^{k}$ is the truncation of order $d-j-1$ of the root functions $\gamma_{j}(z)^{\prime}$ in (6.7). Then $X_{t}$ is $I(d)$ and it admits the Triangular Representation

$$
\Lambda(L) \Gamma_{\circ}(L) X_{t} \sim I(0)
$$

where no linear combination exists of the l.h.s. that is integrated of lower order.
Observe that $\varphi(z)^{\prime}$ is not unique and not of minimal polynomial order, as discussed in the next section. The following example applies the above concepts in the $I(2)$ case.

Example 6.5 (I(2) example continued) Consider Example 2.10. Applying truncation to the rows of $\left(\beta_{0}+\Delta \beta_{2} \delta^{\prime}\right)^{\prime}$, see Proposition 4.5, one finds that the columns in $\beta_{0}^{\prime}$ are root functions of $C(z)$ at 0 of order at least $\min (2,1)=1$. Consider now one row in $\left(\beta_{0}+\Delta \beta_{2} \delta^{\prime}+\Delta^{2} A\right)^{\prime}$ for some matrix $A$; this root function is of order 2 by Proposition 3.6 , and its truncation to degree 1, i.e. to the corresponding column of $\left(\beta_{0}+\Delta \beta_{2} \delta^{\prime}\right)^{\prime}$ is still of order 2 by Proposition 4.7. Finally consider one row in $\left(\beta_{0}+\Delta A\right)^{\prime}$, which gives a root function of order at least 1 ; its truncation to a polynomial of degree 0 gives the corresponding row of $\beta_{0}^{\prime}$, which has order at least 1 by Proposition 4.5. In fact the rows of $\beta_{0}^{\prime}$ give root functions of order equal to 1 or to 2 , when the corresponding entries in $\delta$ in $\left(\beta_{0}+\Delta \beta_{2} \delta^{\prime}\right)^{\prime}$ are equal to 0 .

## 7. MINIMAL BASES

This section shows that when a rational basis exists for the cointegrating space (see Theorem 6.1) the results in Forney (1975) can be applied to obtain polynomial bases of minimal degree.

Suppose that some $r \times p$ basis $G(z), r<p$, for a given vector space $\mathcal{W}$ of rational vectors is given. In the following, the $j$-row of $G(z)$ is indicated as $g_{j}(z)^{\prime}$, and it is the $j$-th element of the basis. Various modifications of the original basis $G^{(0)}(z)$ are indicated as $G^{(h)}(z)$ for $h=1,2,3$.

Definition 7.1 (Degree of $G(z)$ ) If $G(z)$ is polynomial, the degree of its $j$-th row $\operatorname{deg} g_{j}(z)^{\prime}$, is defined as the maximum degree of its elements, and the degree of $G(z)$ is defined as $\operatorname{deg} G(z):=\sum_{j=1}^{r} \operatorname{deg} g_{j}(z)^{\prime}$, i.e. the sum of the degrees of its rows.

In the following, the expansion of $G(z)$ around any point $z_{0}$ is indicated as $G(z)=$ $\sum_{h=0}^{n} G_{h}\left(z-z_{o}\right)^{h}$, where $G_{h}$ are $r \times p$ matrices and $n:=\max _{j=1, \ldots, r} \operatorname{deg} g_{j}(z)^{\prime}$. The reduction algorithm proposed by Forney (1975) page 497-498 consists of the following 3 steps; indicate the initial basis as $G^{(0)}(z)$.

1. If $G^{(0)}(z)$ is not polynomial, multiply each row by its least common denominator to obtain a polynomial basis $G^{(1)}(z)$.
2. Reduce the given polynomial basis $G^{(1)}(z)$ to a basis $G^{(2)}(z)$ for the module of $1 \times p$ row vector polynomials that are numerators of vectors in $\mathcal{W}$.
3. Reduce the resulting basis $G^{(2)}(z)$ to a basis $G^{(3)}(z)$ with a full-row-rank high order coefficient matrix, i.e., a "row proper" basis.
This procedure gives a final basis $G(z)=G^{(3)}(z)$ which has lowest degree, see Forney (1975) Section 3. Each of these steps is described in more detail in the following, showing how each step consists of a change of basis.

### 7.1. Step 1

If $G^{(0)}(z)$ is polynomial, the algorithm sets $G^{(1)}(z)=G^{(0)}(z)$; otherwise $G^{(0)}(z)$ is rational, and its $j$-th row $g_{j}(z)^{\prime}$ has representation

$$
\begin{equation*}
g_{j}(z)^{\prime}=\frac{b_{j}(z)^{\prime}}{q_{j}(z)} \tag{7.1}
\end{equation*}
$$

where $b_{j}(z)^{\prime}$ is a polynomial row vector and $q_{j}(z)$ is a scalar polynomials, and the elements in $b_{j}(z)$ and $q_{j}(z)$ do not have common factors. ${ }^{8}$ Representation (7.1) can always be achieved by choosing $q_{j}(z)$ as the least common denominator of the elements in $b_{j}(z)$, see Remark 5.2.
The first step consist in computing

$$
G^{(1)}(z):=\left(\begin{array}{c}
b_{1}(z)^{\prime}  \tag{7.2}\\
\vdots \\
b_{r}(z)^{\prime}
\end{array}\right)=\left(\begin{array}{ccc}
q_{1}(z) & & \\
& \ddots & \\
& & q_{r}(z)
\end{array}\right)\left(\begin{array}{c}
g_{1}(z)^{\prime} \\
\vdots \\
g_{r}(z)^{\prime}
\end{array}\right)=: Q(z) G^{(0)}(z)
$$

which amounts in selection $b_{j}(z)^{\prime}$ as the rows of $G^{(1)}(z)$. One sees that $G^{(1)}(z)$ is obtained by a change of basis from $G^{(0)}(z)$ with the pre-multiplication of the square polynomial matrix $Q(z)$.

### 7.2. Step 2

Consider $G^{(1)}(z)$, which is a polynomial basis of $\mathcal{W}$. The second step consists in finding the points in $\mathbb{C}$ for which this basis has reduced rank, and to use a rank factorization to

[^7]eliminate common polynomials from the basis. One can start by calculating the greatest common divisor $p(z)$ of all $r \times r$ minors of $G^{(1)}(z)$. If $p(z)=1$ this step is complete, and the algorithm sets $G^{(2)}(z)=G^{(1)}(z)$; otherwise one needs to compute the zeros of $p(z)$, $z_{1}, \ldots, z_{k}$ say.
Consider a generic zero $z_{h}$, and call $S(z)$ the current basis; in the first instance $S(z)=$ $G^{(1)}(z)$. Observe that $S\left(z_{h}\right)$ has reduced rank, i.e. it has a rank factorization of the type $S\left(z_{h}\right)=a_{h} b_{h}^{\prime}$, where $a_{h}$ is $r \times s_{h}$ and of full column rank with $s_{h}<r .{ }^{9}$ For simplicity assume (or reorder) elements in the basis $S(z)$ so as to have $\operatorname{deg} s_{j}(z)^{\prime} \geq \operatorname{deg} s_{j+1}(z)^{\prime}$, where $s_{j}(z)^{\prime}$ is the $j$-th row of $S(z)$; this ensures that the row vector with highest polynomial degree is listed first, and so forth.
Next, expand $S(z)$ around $z=z_{h}$; one finds
\[

$$
\begin{equation*}
S(z)=a_{h} b_{h}^{\prime}+\sum_{j=1}^{n} S_{j}\left(z-z_{h}\right)^{j} . \tag{7.3}
\end{equation*}
$$

\]

This implies that

$$
a_{h \perp}^{\prime} S(z)=\left(\begin{array}{ccc}
\left(z-z_{h}\right)^{n_{1}} & &  \tag{7.4}\\
& \ddots & \\
& & \left(z-z_{h}\right)^{n_{r-s_{h}}}
\end{array}\right) \widetilde{S}(z)=: H_{h}(z) \widetilde{S}(z)
$$

for some $n_{1} \geq \cdots \geq n_{r-s_{h}} \geq 1$. In fact, because of orthogonality one has $a_{h \perp}^{\prime} a_{h}=0$; moreover, $a_{h \perp}^{\prime}$ may annihilate also other terms $S_{j}$ in the expansion (7.3), but this is not indicated explicitly here for simplicity.
The resulting basis is constructed as follows

$$
\begin{equation*}
R(z)=\underbrace{Q_{h}\binom{H_{h}(z)^{-1} a_{h \perp}^{\prime}}{\bar{a}_{h}^{\prime}}}_{K_{h}(z)} S(z)=Q_{h}\binom{\widetilde{S}(z)}{\bar{a}_{h}^{\prime} S(z)} \tag{7.5}
\end{equation*}
$$

where $Q_{h}$ is a permutation matrix that ensures that the rows of the basis $R(z)$ with higher degree come first. Note that $\widetilde{S}(z)$ has lower degree than $a_{h \perp}^{\prime} S(z)$, see (7.4), and $\bar{a}_{h}^{\prime} S(z)$ does not imply any changes in the degree; this implies that $\operatorname{deg} R(z)<\operatorname{deg} S(z)$.
This process is repeated setting $S(z)$ equal to the current new basis $R(z)$ and selecting a new zero $z_{h}$, until $R(z)$ has full row rank for all $z \in \mathbb{C}$. The final basis has representation

$$
\begin{equation*}
G^{(2)}(z)=K(z) G^{(1)}(z), \quad K(z):=\prod_{h=1}^{q} K_{h}(z) \tag{7.6}
\end{equation*}
$$

where $h=1, \ldots, k$ enumerate all steps of type (7.5). One sees that $G^{(2)}(z)$ is obtained by a change of basis from $G^{(1)}(z)$ with the pre-multiplication of the square rational matrix $K(z)$, i.e. that also this step produces a change of basis, which eliminates all common polynomials with zeros in the complex plane $\mathbb{C}$.

[^8]
### 7.3. Step 3

The last step eliminates common polynomials with zeros at infinity. Consider the polynomial basis $G^{(2)}(z)=\sum_{j=0}^{n} G_{j}^{(2)} z^{j}$ where $n:=\max _{j=1, \ldots, r} \operatorname{deg} g_{j}(z)^{\prime}$, where $g_{j}(z)^{\prime}$ is the $j$-th row of $G^{(2)}(z)$. Consider $G_{n}^{(2)}$; if this matrix is of reduced rank, $s<r$ say, then one can reduce the degree of $G^{(2)}(z)$ further, otherwise the algorithm stops setting $G^{(3)}(z)=G^{(2)}(z)$.
Let $G_{n}^{(2)}=a b^{\prime}$ be the rank factorization of the coefficient of degree $n$; one can compute the minimal basis $G^{(3)}(z)$ as $G^{(3)}(z)=U G^{(2)}(z)=\sum_{j=0}^{n} G_{j}^{(3)} z^{j}$ where

$$
\begin{equation*}
U:=\binom{\bar{a}^{\prime}}{a_{\perp}^{\prime}} \quad \text { and } \quad G_{n}^{(3)}=U G_{n}^{(2)}=\binom{b^{\prime}}{0} \tag{7.7}
\end{equation*}
$$

The degree of $G^{(3)}(z)$ is lower than the one of $G^{(2)}(z)$ by $r-s$, and because $b^{\prime}$ has full rank, the degree of the basis cannot be reduced further.
One sees that $G^{(3)}(z)$ is obtained by a change of basis from $G^{(2)}(z)$ with the premultiplication of the square matrix $U$. Overall, the procedure in Forney (1975) calculates

$$
G^{(3)}(z)=\underbrace{U K(z) Q(z)}_{M(z)} G^{(0)}(z)
$$

where $M(z)=U H(z) Q(z)$ is a square matrix of rational functions that performs a change of basis from the initial basis $G^{(0)}(z)$ to a minimal one $G^{(3)}(z)$, where $K(z)$ is defined in Step 2 see (7.6) and $Q(z)$ is defined in Step 1 see (7.2).

## 8. FROM CANONICAL SYSTEM OF ROOT FUNCTIONS TO MINIMAL BASES IN THE $I(2)$ CASE

One can note that $\varphi(z)^{\prime}$ in the triangular representation in Proposition 6.4 is not necessarily of minimal order. The algorithm of Forney in Section 7 can be applied to each block of rows in $\varphi(z)^{\prime}$ in (6.10) to reduce the basis to minimal order. This section illustrates this procedure in the $I(2)$ example, and shows that this delivers the separation of the cases of

1. non-polynomial cointegrating relations reducing the order of integration from 2 to 0 ;
2. polynomial cointegrating relations reducing the order of integration from 2 to 0 .

Remark that the process of obtaining minimal bases does not lead to a unique basis, which leaves open the possibility to further restrict the basis using economic rationale, as discussed in Johansen (1995) for $I(1)$ and in Mosconi and Paruolo (2017) for $I(d)$ processes, $d>1$. Sometimes, obtaining a minimal basis may be at variance with economic interpretation; in this case one can choose how to best combine requirements of minimality with economic interpretability.

### 8.1. Step 1 in $I(2)$

Consider the triangular representation of an $I(2)$ system, see (6.10):

$$
\begin{equation*}
\Gamma_{\circ}(z):=\binom{\varphi(z)^{\prime}}{\hdashline \bar{\beta}_{2}^{\prime}-}, \quad \varphi(z)^{\prime}:=\binom{\gamma_{0}^{(1)}(z)^{\prime}}{\frac{\gamma_{1}^{(0)}(z)^{\prime}}{(0)}}=\left(\underline{\beta}_{0}^{\prime}+\underline{\gamma}_{0,1}^{\prime}\left(z-z_{\omega}\right)\right), \tag{8.1}
\end{equation*}
$$

to which the procedure of Forney (1975) can be applied only to $\gamma_{0}^{(1)}(z)^{\prime}$ because $\gamma_{1}^{(0)}(z)^{\prime}=$ $\beta_{1}^{\prime}$ is already of minimal degree, equal to 0 .

Set $G^{(0)}(z)=\gamma_{0}^{(1)}(z)^{\prime}=\beta_{0}^{\prime}+\gamma_{0,1}^{\prime}\left(z-z_{\omega}\right)$, where $\gamma_{0,1}^{\prime}=-\bar{\alpha}_{0}^{\prime} A_{1}$, see (6.5) and (6.7), and note that $G^{(0)}(z)$ is already polynomial. Hence, Step 1 does not apply, and $G^{(1)}(z)=$ $G^{(0)}(z)$.

### 8.2. Step 2 in $I(2)$

Next consider Step 2, and equation (7.3). One wishes to find some zero $z_{h}$ and some corresponding $a_{h \perp}^{\prime}$ so as to satisfy (7.3). Denoting $v^{\prime}=a_{h \perp}^{\prime}$ and $u=z_{h}-z_{\omega}$, a generic zero $z_{h}$ of $S(z)=G^{(1)}(z)$ satisfies $v^{\prime} S\left(z_{h}\right)=0$, i.e.

$$
\begin{equation*}
v^{\prime}\left(\beta_{0}^{\prime}-\bar{\alpha}_{0}^{\prime} A_{1} u\right)=0 \tag{8.2}
\end{equation*}
$$

where $u$ is a scalar. Note that $u=0$ is not a possible zero of $S(z)$, because $S\left(z_{\omega}\right)=\beta_{0}^{\prime}$ is of full row rank, so that $u \neq 0$.
Post-multiplying (8.2) by the square non-singular matrix ( $\bar{\beta}_{0}, \bar{\beta}_{1}, \bar{\beta}_{2}$ ) one finds

$$
\begin{align*}
v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1} \bar{\beta}_{0} & =\lambda v^{\prime}, \quad \lambda:=u^{-1} \neq 0,  \tag{8.3}\\
v^{\prime} \alpha_{0}^{\prime} A_{1} \bar{\beta}_{1} & =0  \tag{8.4}\\
v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1} \bar{\beta}_{2} & =0 \tag{8.5}
\end{align*}
$$

where $\lambda=u^{-1} \neq 0$ in (8.3) because $u \in \mathbb{C}$; note also that $u \neq 0$ has been simplified in (8.4) and (8.5). This proves the following proposition.

Proposition 8.1 (Step 2 condition in $I(2)$ ) A necessary and sufficient condition for Step 2 to be non-empty is that (8.3), (8.4) and (8.5) hold simultaneously, i.e. that ( $\lambda, v_{j}^{\prime}$ ) is a non-zero eigenvalue - left eigenvector pair ${ }^{10}$ of $\bar{\alpha}_{0}^{\prime} A_{1} \bar{\beta}_{0}$, where $v_{j}^{\prime}$ is the $j$-th row in $v^{\prime}$, and that the left eigenvectors $v^{\prime}$ are orthogonal to $\bar{\alpha}_{0}^{\prime} A_{1}\left(\bar{\beta}_{1}, \bar{\beta}_{2}\right)$.

Observe that from (8.2), with $u=z_{h}-z_{\omega}$, one finds

$$
\begin{align*}
v^{\prime} S(z) & =v^{\prime} \beta_{0}^{\prime}-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)=v^{\prime} \beta_{0}^{\prime}-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{h}+u\right), \\
& =v^{\prime}\left(\beta_{0}^{\prime}-\bar{\alpha}_{0}^{\prime} A_{1} u\right)-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{h}\right)=-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{h}\right) \tag{8.6}
\end{align*}
$$

so that $\widetilde{S}(z)=-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}$ and $H_{h}(z)=z-z_{h}$ in (7.4).
Eq. (7.5) asks to replace $r_{0}-s_{h}$ rows in $S(z)$, which all have degree equal to 1 , with the $r_{0}-s_{h}$ rows in $\widetilde{S}(z)=-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}$, which all have degree equal to 0 . Note that from (8.3) one finds that $v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1} P_{\beta_{0}}=\lambda v^{\prime} \beta_{0}^{\prime}$, i.e. $v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}=\lambda v^{\prime} \beta_{0}^{\prime}$ because $P_{\beta_{0}}=I-P_{\beta_{1}}-P_{\beta_{2}}$ and because of (8.4) and (8.5). Hence these cointegrating relations are in the span of $\beta_{0}^{\prime}$, and one finds

$$
R(z)=\left(\begin{array}{cc}
0 & I_{s_{h}}  \tag{8.7}\\
I_{r_{0}-s_{h}} & 0
\end{array}\right)\binom{-v^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}}{v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)}=\binom{v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)}{-\lambda v^{\prime} \beta_{0}^{\prime}} .
$$

One can hence replace $\gamma_{0}^{(1)}(z)^{\prime}$ with any multiple of $R(z), \widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$ say. This implies that the extended canonical system of root functions $\Gamma_{0}(z)$ in the triangular representation can be replaced by

$$
\widetilde{\Gamma}(z):=\binom{\widetilde{\varphi}(z)^{\prime}}{\hdashline \bar{\beta}_{2}^{\prime}-}, \quad \widetilde{\varphi}(z)^{\prime}=\binom{\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}}{-\gamma_{1}^{(0)}(z)^{\prime}}=\left(\begin{array}{c}
v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)  \tag{8.8}\\
v^{\prime} \bar{\beta}_{0}^{\prime} \\
\hdashline \bar{\beta}_{1}^{\prime}
\end{array}\right),
$$

where the factor $-\lambda \neq 0$ has been eliminated from the second block.

[^9]Remark 8.2 (Separation between $\mathrm{CI}(2,2)$ and polynomial cointegration - part 1) Note that the degree of $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$ is lower than the one of $\gamma_{0}^{(1)}(z)^{\prime}$, while maintaining the property that $\widetilde{\gamma}_{0}^{(1)}(L)^{\prime} x_{t} \sim I(0)$. This implies that the cointegrating relations $v^{\prime} \beta_{0}^{\prime}$ are $C I(2,2)$ in the sense of the definition of Engle and Granger (1987) quoted in the Introduction (Section 1 ), while the remaining cointegrating vectors $v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)$ are polynomial cointegrating vectors, which also reduce the processes to $I(0)$.

Remark 8.3 (Complex vectors at frequency 0) Consider the leading case of $z_{\omega}$ real, as for $\omega=0$ with $z_{\omega}=1$. All matrices in the expansions (6.2) and (6.5) are real, and so are the coefficient matrices in $\gamma_{0}^{(1)}(z)^{\prime}$. Observe that the non-zero eigenvalue - left eigenvector pair $\left(\lambda, v_{j}^{\prime}\right)$ of $\bar{\alpha}_{0}^{\prime} A_{1} \bar{\beta}_{0}$, may be complex, because the real matrix $\bar{\alpha}_{0}^{\prime} A_{1} \bar{\beta}_{0}$ need not be symmetric. This would imply that the cointegrating relations $v^{\prime} \beta_{0}^{\prime}$ may have complex coefficients, which would be difficult to interpret from an economic point of view; this may make Step 2 unattractive in practice and/or at odds with alternative ways to restrict the basis of the cointegrating space.

### 8.3. Step 3 in $I(2)$

In Step 3, one considers $\widetilde{\varphi}(z)^{\prime}$ in (8.8), which contains $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$ and $\gamma_{1}^{(0)}(z)^{\prime}=\beta_{1}^{\prime}$. The highest polynomial degree is 1 and it is present only in the component $v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}(z-$ $z_{\omega}$ ) in $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$, because its other component $v^{\prime} \beta_{0}^{\prime}$ has degree 0 .

The non-zero entry $v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}$ in the first block of $s_{h}$ rows of the coefficient of degree 1 , has rank decomposition $v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}=c b^{\prime}$, with $c$ and $b$ of full column rank $s$. In case $s<s_{h}$, then $c_{\perp}^{\prime}$ is non-zero and of dimension $s_{h}-s$ and one can replace $v_{\perp}^{\prime} \beta_{0}^{\prime}-v_{\perp}^{\prime} \bar{\alpha}_{0}^{\prime} A_{1}\left(z-z_{\omega}\right)$ in $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$ with

$$
\begin{equation*}
\binom{c_{\perp}^{\prime} v_{\perp}^{\prime} \beta_{0}^{\prime}}{\bar{c}^{\prime} v_{\perp}^{\prime} \beta_{0}^{\prime}-b^{\prime}\left(z-z_{\omega}\right)} \tag{8.9}
\end{equation*}
$$

Eq. (8.9) corresponds to (7.7) in Step 3 with $a=\left(c^{\prime}, 0\right)^{\prime}$. Eq. (8.9) leads to replace $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$ with $\gamma_{0}^{\dagger(1)}(z)^{\prime}$ chosen proportional to (8.9), and the extended canonical system of root functions $\widetilde{\Gamma}(z)$ in the triangular representation (8.1) can be replaced by the minimal basis

$$
\Gamma^{\dagger}(z):=\binom{\varphi^{\dagger}(z)^{\prime}}{-\bar{\beta}_{2}^{\prime}-}, \quad \varphi^{\dagger}(z)^{\prime}=\binom{\gamma_{0}^{\dagger(1)}(z)^{\prime}}{\hdashline \gamma_{1}^{(0)}(z)^{\prime}}=\left(\begin{array}{c}
\bar{c}^{\prime} v_{\perp}^{\prime} \beta_{0}^{\prime}-b^{\prime}\left(z-z_{\omega}\right)  \tag{8.10}\\
c_{\perp}^{\prime} v_{\perp}^{\prime} \beta_{0}^{\prime} \\
v^{\prime} \beta_{0}^{\prime} \\
\beta_{1}^{\prime}
\end{array}\right)
$$

where $\gamma_{0}^{\dagger(1)}(z)^{\prime}$ has minimal degree.
Remark 8.4 (Separation between $\mathrm{CI}(2,2)$ and polynomial cointegration - part 2) Note that the degree of $\gamma_{0}^{\dagger(1)}(z)^{\prime}$ is lower than the one of $\widetilde{\gamma}_{0}^{(1)}(z)^{\prime}$, while maintaining the property that $\gamma_{0}^{\dagger(1)}(L)^{\prime} x_{t} \sim I(0)$. This implies that the cointegrating relations $c_{\perp}^{\prime} v_{\perp}^{\prime} \beta_{0}^{\prime}$ are also $C I(2,2)$. This provides the complete separation between $\mathrm{CI}(2,2)$ cointegrating relations and polynomial cointegrating relations.

## 9. MINIMALITY AND ECONOMIC IDENTIFICATION IN THE $I(2)$ CASE

This section discusses parametric identification of cointegrating vectors in the $I(2)$ case, and its relations with the choice of minimal bases described in the previous section. Cointegration parameters are here grouped in the matrices $\beta^{\prime}, \gamma^{\prime}$ and $\phi^{\prime}$, respectively of dimensions $r_{0} \times p, r_{1} \times p$ and $r_{0} \times r_{2}$. Here $\beta^{\prime}$ is a matrix of parameters with the same dimensions of $\beta_{0}^{\prime}, \gamma^{\prime}$ has the same dimensions of $\beta^{\prime}$ and $\phi^{\prime}$ is such that $\beta_{1}^{\prime} \phi$ is square and full rank and the rows of $\phi^{\prime}$ belong to $\operatorname{span}\left(\beta_{0}, \beta_{1}\right)^{\prime} .{ }^{11}$
Let $X_{t}$ be $I(2)$ with MA representation $\Delta^{2} X_{t}=F(L) \varepsilon_{t}$ where $F(L) \varepsilon_{t}$ is $I(0)$. The triangular representation of Stock and Watson (1993) can be written as, see Mosconi and Paruolo (2017) eq. (3.8) (3.9), ${ }^{12}$

$$
\left(\begin{array}{c}
\beta^{\prime}+\gamma^{\prime} \Delta \\
\phi^{\prime} \Delta \\
b_{2}^{\prime} \Delta^{2}
\end{array}\right) X_{t}=H(L) \varepsilon_{t}
$$

where $H(L) \varepsilon_{t}$ is $\mathrm{I}(0)$ with $H(1)$ of full rank, i.e. non-cointegrating, $b_{2}$ is any set of vectors such that $b_{2}^{\prime} \beta_{2}$ is square and invertible, and $\phi^{\prime}$ is a linear combination of $\beta^{\prime}$ and $\beta_{1}^{\prime}$ such that that $\phi^{\prime} \beta_{1}$ is square and invertible. Finally $\gamma^{\prime}$ is any matrix such that $\gamma^{\prime} \beta_{2}=\bar{\alpha}^{\prime} A_{1} \beta_{2}$.
Theorem 1 in Mosconi and Paruolo (2017) shows that such an $X_{t}$ also satisfies the equivalent triangular representation

$$
\left(\begin{array}{c}
\beta^{\circ \prime}+\gamma^{\circ \prime} \Delta  \tag{9.1}\\
\phi^{\circ \prime} \Delta \\
b_{2}^{\prime} \Delta^{2}
\end{array}\right) X_{t}=H^{\circ}(L) \varepsilon_{t}
$$

where $H^{\circ}(L) \varepsilon_{t}$ is $\mathrm{I}(0)$ with $H^{\circ}(1)$ of full rank, i.e. non-cointegrating,

$$
\begin{equation*}
\beta^{\circ \prime}:=Q_{00} \beta^{\prime}, \quad \phi^{\circ \prime}:=Q_{\phi \phi} \phi^{\prime}+Q_{\phi \beta} \beta^{\prime}, \quad \gamma^{\circ \prime}=Q_{00} \gamma^{\prime}+Q_{0 \phi} \phi^{\prime}+Q_{0 \beta} \beta^{\prime} \tag{9.2}
\end{equation*}
$$

where $Q_{00}$ and $Q_{\phi \phi}$ are square and invertible and $Q_{i j}$ are blocks, $i, j=0, \phi, \beta$ of the matrix

$$
Q:=\left(\begin{array}{ccc}
Q_{00} & Q_{0 \gamma} & Q_{0 \beta}  \tag{9.3}\\
r_{0} \times r_{0} & & \\
0 & Q_{\gamma \gamma} & Q_{\gamma \beta} \\
0 & 0 & Q_{00}
\end{array}\right)
$$

They show that (9.2) describes the set of observationally equivalent parameter points; this corresponds to the $I(2)$ identification problem of the long-run relations that also appears in the Error Correction form.
Theorem 2 in Mosconi and Paruolo (2017) considers linear restrictions on the parameters in $\beta^{\prime}, \gamma^{\prime}, \phi^{\prime}$, and provides rank and order identification conditions for these restrictions to identify $\beta^{\prime}, \gamma^{\prime}, \phi^{\prime}$, i.e. to reduce the set of transformation matrices $Q$ to the identity $I$. These conditions are referred to in the following as economic linear restrictions.
These conditions can be used to calculate how many restrictions need to be imposed in addition to the ones already present in the minimal basis (8.10) to obtain identification. Note that (8.10) has form

$$
\gamma_{0}^{\dagger(1)}(z)^{\prime}=\beta^{\prime}+\left(\begin{array}{l}
b^{\prime}  \tag{9.4}\\
0 \\
0
\end{array}\right)(1-z)
$$

[^10]where $\beta^{\prime}=a \beta_{0}^{\prime}$ has no restrictions and $\gamma^{\prime}$, the coefficient to $\Delta$ has zero-restrictions in the second and third blocks. One could achieve minimality of the basis with additional restrictions which make economic sense.
If the minimal basis in (8.10) or in (9.4) was not interpretable, economic linear restrictions may provide an alternative way to achieve identification imposing restrictions directly on $\gamma_{0}(z)^{\prime}$. In this way economic linear restrictions and minimality restrictions can be used as complements for the identification of bases of cointegrating spaces.

## 10. CONCLUSIONS

This paper discusses the notion of cointegrating space for general $I(d)$ processes. It makes a number of observations and contributions to the literature. Specifically, it observes that the notion of cointegrating space was formally introduced in the literature by Johansen (1991) for the case of $\mathrm{I}(1)$ VAR system. The definition of the cointegrating space is simplest in the $I(1)$ case, because there is no need to consider vector polynomials in the lag operator.
Engle and Yoo (1991) introduced the notion of polynomial cointegrating vectors in parallel with the related one of multicointegration in Granger and Lee (1989). It appears however, that the literature did not attempt to define the notion of cointegrating space in the general polynomial case yet. This paper fills this gap. In this context, this paper recognises that cointegrating vectors are in general root functions, which have been analysed at length in the mathematical and engineering literature, see e.g. Gohberg et al. (1993). This allows to characterise a number of properties of cointegrating vectors.

Observing that root functions can be truncated to polynomials, the notion of rational cointegrating space is argued to be the proper notion of cointegrating space in the $I(d)$ case $d>1$. It is then observed that a basis of this rational cointegrating space can be chosen to be polynomial.
It is next shown that the extended local rank factorization of Franchi and Paruolo (2016) can be used to deliver a canonical system of root functions, and hence to produce a basis for the rational cointegrating space. This result is constructive, as it gives an explicit way to derive such as basis from the VAR polynomial. This basis is not necessarily of minimal polynomial degree, however.
The 3-step procedure of Forney (1975) to reduce this basis to minimality is first re-stated in terms of rank factorizations. Next it is applied to the basis for the rational cointegrating space obtained by extended local rank factorization to obtain a minimal basis. It is shown how this minimal basis separates the multi-cointegrating and the $\mathrm{CI}(2,2)$ cointegrating vectors within the basis in the $I(2)$ case.
The minimal basis is still not unique. Similarly to the identification results for the long run relations in the $\mathrm{I}(1)$ case discussed in Johansen (1995), this paper links to the results on identification in $I(d)$ systems of cointegrating vectors in Mosconi and Paruolo (2017). It is shown how these results can be applied to restrict the minimal basis to obtain economically interpretable coefficients.

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## APPENDIX A: $(\mathcal{Q}, F(Z))$ IS CLOSED UNDER ADDITION AND MULTIPLICATION BY A SCALAR

It is simple to verify that $(\mathcal{Q}, F(z))$ is closed under vector addition and under multiplication by a scalar. In fact, consider two vectors $a(z)^{\prime}$ and $b(z)^{\prime}$ in $\mathcal{Q}, a(z)^{\prime}=\left(f_{1}(z), \ldots, f_{p}(z)\right)$ and $b(z)^{\prime}=\left(g_{1}(z), \ldots, g_{p}(z)\right)$, with $f_{j}(z)=c_{j}(z) / d_{j}(z)$ and $g_{j}(z)=h_{j}(z) / k_{j}(z)$, and $d_{j}(z), k_{j}(z) \neq 0$. Then the $j$-th entry in the $1 \times p$ row vector $a(z)^{\prime}+c(z)^{\prime}$ is given by

$$
f_{j}(z)+g_{j}(z)=\frac{c_{j}(z)}{d_{j}(z)}+\frac{h_{j}(z)}{k_{j}(z)}=\frac{c_{j}(z) k_{j}(z)+d_{j}(z) h_{j}(z)}{d_{j}(z) k_{j}(z)}=\frac{m_{j}(z)}{n_{j}(z)}, \text { say. }
$$

Note that $m_{j}(z)$ and $n_{j}(z)$ are still polynomials in $F[z]$, and that $n_{j}(z)=d_{j}(z) k_{j}(z) \neq 0$ because both $d_{j}(z), k_{j}(z) \neq 0$. Hence $a(z)^{\prime}+c(z)^{\prime} \in \mathcal{Q}$, i.e. $\mathcal{Q}$ is closed under vector addition.

Moreover, for $v(z) \in F(z)$ with representation $v(z)=h(z) / k(z)$, with $k(z) \neq 0$ and polynomials
$h(z), k(z) \in F[z]$, the scalar multiplication $v(z) a(z)^{\prime}$ gives a $1 \times p$ row vector with $j$-th entry equal to

$$
v(z) f_{j}(z)=\frac{c_{j}(z) h(z)}{d_{j}(z) k(z)}=\frac{u_{j}(z)}{w_{j}(z)}, \text { say. }
$$

where again $u_{j}(z)$ and $w_{j}(z)$ are polynomials in $F[z]$, and $w_{j}(z)=d_{j}(z) k(z) \neq 0$ because both $d_{j}(z), k(z) \neq$ 0 . This shows that $v(z) a(z)^{\prime} \in \mathcal{Q}$, i.e. that $\mathcal{Q}$ is closed under multiplication by a scalar in $F(z)$.


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    ${ }^{1}$ JEL classification: C32.
    ${ }^{1}$ See Engle and Granger (1987) pages 253-254. Here $N$ in their notation is replaced by $p$ and $\alpha$ with $\beta$ for consistency with the rest of the paper.

[^1]:    ${ }^{2}$ Here, for any set of vectors $a_{1}, \ldots, a_{n}, \operatorname{span}\left(a_{1}, \ldots, a_{n}\right)$ is defined as the set $\left\{v: v=\sum_{j=1}^{n} a_{i} u_{i}, u_{i} \in\right.$ $F\}$, where $F$ is the relevant field.

[^2]:    ${ }^{3}$ This result is usually stated as $x_{t}=u_{t}-a_{0}$ where $a_{0}:=x_{0}-v_{0}$ is a generic constant, see e.g. Hannan and Deistler (1988) eq. (1.2.15).

[^3]:    ${ }^{4}$ Note that this case is covered by Proposition 4.5 below.

[^4]:    ${ }^{5}$ Theorem 6.3 provides two constructions of the local Smith factorization.

[^5]:    ${ }^{6}$ Note that $F(z)$ contains the multiplicative inverse; in fact every non-zero $c(z) \in F(z)$ with representation $a(z) / d(z)$, with $d(z) \neq 0$ has multiplicative inverse $d(z) / a(z)$, with $a(z) \neq 0$ (because $c(z)$ is non-zero).

[^6]:    ${ }^{7}$ See Hungerford (1980) page 233.

[^7]:    ${ }^{8} q_{j}(z)$ can be chosen to be monic.

[^8]:    ${ }^{9}$ Forney (1975) replaces one row vector of the basis at a time. Here these steps are aggregated into the substitution of several row vectors at a time.

[^9]:    ${ }^{10}$ Note that all left eigenvectors $v_{j}^{\prime}$ are associated with the same eigenvalue $\lambda$.

[^10]:    ${ }^{11}$ For details and motivation of these choices, see Mosconi and Paruolo (2017).
    ${ }^{12}$ Mosconi and Paruolo (2017) address the identification problem for any $I(d)$ system, $d=2,3, \ldots$.

