

Random-time processes governed by differential equations of fractional distributed order

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Abstract

We analyze here different types of fractional differential equations, under the assumption that their fractional order $\nu \in (0, 1]$ is random with probability density $n(\nu)$. We start by considering the fractional extension of the recursive equation governing the homogeneous Poisson process $N(t), t > 0$. We prove that, for a particular (discrete) choice of $n(\nu)$, it leads to a process with random time, defined as $N(\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)), t > 0$. The distribution of the random time argument $\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)$ can be expressed, for any fixed t , in terms of convolutions of stable-laws. The new process $N(\tilde{\mathcal{T}}_{\nu_1, \nu_2})$ is itself a renewal and can be shown to be a Cox process. Moreover we prove that the survival probability of $N(\tilde{\mathcal{T}}_{\nu_1, \nu_2})$, as well as its probability generating function, are solution to the so-called fractional relaxation equation of distributed order (see [16]).

In view of the previous results it is natural to consider diffusion-type fractional equations of distributed order. We present here an approach to their solutions in terms of composition of the Brownian motion $B(t), t > 0$ with the random time $\tilde{\mathcal{T}}_{\nu_1, \nu_2}$. We thus provide an alternative to the constructions presented in Mainardi and Pagnini [19] and in Chechkin et al. [6], at least in the double-order case.

Key words: Fractional differential equations of distributed order; Stable laws; Generalized Mittag-Leffler functions; Processes with random time; Renewal process; Cox process.

1 Introduction

In the last decade an increasing attention has been drawn to fractional extensions of the Poisson process: see, among the others, [25], [10], [13], [28], [15], [17], [18], [5]. In particular, the analysis carried out by Beghin and Orsingher [2] starts from the generalization of the equation governing the Poisson process, where the time-derivative is substituted by the fractional derivative (in the Caputo sense) of order $\nu \in (0, 1]$:

$$\frac{d^\nu p_k}{dt^\nu} = -\lambda(p_k - p_{k-1}), \quad k \geq 0, \quad (1.1)$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases} \quad (1.2)$$

and $p_{-1}(t) = 0$. The solution to this equation has been expressed as the density of the random-time process called Fractional Poisson process (FPP) and defined as

$$\mathcal{N}_\nu(t) = N(\mathcal{T}_{2\nu}(t)), \quad t > 0. \quad (1.3)$$

Here N denotes the standard homogeneous Poisson process with rate parameter $\lambda > 0$, while $\mathcal{T}_{2\nu}(t), t > 0$ is a random process (independent from N) with density given by the folded solution to the fractional diffusion equation

$$\frac{\partial^{2\nu} v}{\partial t^{2\nu}} = c^2 \frac{\partial^2 v}{\partial y^2}, \quad t > 0, y \in \mathbb{R}. \quad (1.4)$$

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Later, in [3], the distribution of the FPP \mathcal{N}_ν has been expressed as

$$p_k^\nu(t) = (\lambda t^\nu)^k E_{\nu, \nu k+1}^{k+1}(-\lambda t^\nu), \quad k \geq 0, t > 0, \quad (1.5)$$

in terms of the so-called Generalized Mittag-Leffler (GML) function, which is defined as

$$E_{\alpha, \beta}^\gamma(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j z^j}{j! \Gamma(\alpha j + \beta)}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(\gamma) > 0, \quad (1.6)$$

where $(\gamma)_j = \gamma(\gamma+1)\dots(\gamma+j-1)$ (for $j = 1, 2, \dots$, and $\gamma \neq 0$) is the Pochhammer symbol and $(\gamma)_0 = 1$ (see [11], p.45). Moreover a higher-order generalization of the previous results has been obtained in [3] by introducing “higher-order fractional derivatives” in (1.1) and analyzing the following equation

$$\frac{d^{m\nu} p_k}{dt^{m\nu}} + \binom{n}{1} \lambda \frac{d^{(n-1)\nu} p_k}{dt^{(n-1)\nu}} + \dots + \binom{n}{n-1} \lambda^{n-1} \frac{d^\nu p_k}{dt^\nu} = -\lambda^n (p_k - p_{k-1}), \quad k \geq 0, \quad (1.7)$$

where $\nu \in (0, 1)$, subject to the initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}, \quad \text{for } 0 < \nu < 1 \quad (1.8)$$

$$\left. \frac{d^j}{dt^j} p_k(t) \right|_{t=0} = 0 \quad j = 1, \dots, n-1, \quad k \geq 0, \quad \text{for } \frac{1}{n} < \nu < 1$$

and $p_{-1}(t) = 0$. The solution to (1.7) was given by the following finite sum of GML functions:

$$\tilde{p}_k^n(t) = \sum_{j=1}^n \binom{n}{j} (\lambda t^\nu)^{n(k+1)-j} E_{\nu, n(k+1)-\nu j+1}^{kn+n}(-\lambda t^\nu). \quad (1.9)$$

The corresponding process was proved to be a renewal, linked to $\mathcal{N}_\nu(t)$, $t > 0$, by the following relationship

$$\tilde{p}_k^n(t) = p_{nk}^\nu(t) + p_{nk+1}^\nu(t) + \dots + p_{nk+n-1}^\nu(t), \quad t > 0.$$

Thus it can be interpreted as a FPP which “records” only the k -th order events and disregards the other ones (for an application to the theory of random motions at finite velocity, see [4]).

We will introduce here the assumption that the fractional order ν of the derivative appearing in equation (1.1) is itself random, with distribution $n(\nu)$, $\nu \in (0, 1]$: i.e.

$$\int_0^1 \frac{d^\nu p_k}{dt^\nu} n(\nu) d\nu = -\lambda (p_k - p_{k-1}), \quad k \geq 0, \nu \in (0, 1]. \quad (1.10)$$

More precisely, we will concentrate on the case of a double-order discrete distribution of ν , i.e.

$$n(\nu) = n_1 \delta(\nu - \nu_1) + n_2 \delta(\nu - \nu_2), \quad 0 < \nu_1 < \nu_2 \leq 1, \quad (1.11)$$

for $n_1, n_2 \geq 0$ and such that $n_1 + n_2 = 1$. The assumption (1.11) has been already considered in the context of fractional relaxation (see [16]), as well as for fractional kinetic equations and, in the last case, it leads to the so-called diffusion with retardation (see [19]). As we will see in the next section this assumption on ν produces a form of the solution which is much more complicated than (1.5) and (1.9), since it involves infinite sums of GML functions. Nevertheless the renewal property is still valid and a subordinating relationship similar to (1.3) holds for the corresponding process, which can be defined as

$$N(\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)), \quad t > 0. \quad (1.12)$$

In this case the random time $\tilde{\mathcal{T}}_{\nu_1, \nu_2}$ is represented by a process whose transition density can be expressed either as an infinite sum of Wright functions or by convolutions of stable laws.

In section 3 we will investigate the relationship between the previous results and the diffusion equation of fractional distributed order

$$\int_0^1 \frac{\partial^\nu v}{\partial t^\nu} n(\nu) d\nu = \frac{\partial^2 v}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \quad v(x, 0) = \delta(x), \quad (1.13)$$

for $0 < \nu \leq 1$. Equations like (1.13) have been already studied in [6] and [19] in connection with the kinetic description of anomalous diffusions. It has been proved by Chechkin et al. [6] that the solution $v(x, t)$, $x \in \mathbb{R}, t > 0$, is a probability density function and that the corresponding process is subordinated to the Brownian motion via the following relationship

$$v(x, t) = \int_0^{+\infty} \frac{e^{-x^2/4\pi u}}{\sqrt{4\pi u}} G(u, t) du. \quad (1.14)$$

In (1.14) the function G is defined by its Laplace transform

$$\mathcal{L}\{G(u, t); \eta\} = \frac{\int_0^1 n(\nu) \eta^\nu d\nu}{\eta} e^{-u \int_0^1 n(\nu) \eta^\nu d\nu}. \quad (1.15)$$

In the special case of double-order fractional derivative in (1.13) these authors focus on the behavior of the second moment of $v(x, t)$, which suggests that the process can be interpreted as a “diffusion with retardation”, in this case. Moreover, under assumption (1.11), equation (1.13) can be seen as a particular case (for $\gamma = 2$) of the equation (2.35) below, which is analyzed in [26]. In this paper only the Fourier transform of the solution is given in explicit form, in terms of infinite sums of generalized Mittag-Leffler (GML) functions. Finally the solution to (1.13) has been analytically expressed in terms of generalized Wright functions by [19].

We prove here that the solution to equation (1.13) with the assumption (1.11), i.e.

$$n_1 \frac{\partial^{\nu_1} v}{\partial t^{\nu_1}} + n_2 \frac{\partial^{\nu_2} v}{\partial t^{\nu_2}} = \frac{\partial^2 v}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \quad v(x, 0) = \delta(x), \quad n_1, n_2 > 0, \quad (1.16)$$

for $0 < \nu_1 < \nu_2 \leq 1$, coincides with the density of the random-time process

$$B(\widetilde{\mathcal{T}}_{\nu_1, \nu_2}(t)), \quad t > 0, \quad (1.17)$$

where B is the standard Brownian motion and the time argument $\widetilde{\mathcal{T}}_{\nu_1, \nu_2}$ is the same as in (1.12), thus writing in explicit form the density G in (1.14) as an infinite sum of Wright functions or by convolutions of stable laws.

Finally we note that the density of the random time $\widetilde{\mathcal{T}}_{\nu_1, \nu_2}$ appearing in the processes (1.12) and (1.17) coincides with the solution to the equation (1.13), when a different hypothesis on the density $n(\nu)$ is assumed, i.e.

$$n(\nu) = n_1^2 \delta(\nu - 2\nu_1) + n_2^2 \delta(\nu - 2\nu_2) + 2n_1 n_2 \delta(\nu - (\nu_1 + \nu_2)), \quad 0 < \nu_1 < \nu_2 \leq 1, \quad (1.18)$$

for $n_1, n_2 \geq 0$ and such that $n_1 + n_2 = 1$. Therefore the equation governing the process $\widetilde{\mathcal{T}}_{\nu_1, \nu_2}(t)$, $t > 0$ turns out to be

$$\left(n_1 \frac{\partial^{\nu_1} v}{\partial t^{\nu_1}} + n_2 \frac{\partial^{\nu_2} v}{\partial t^{\nu_2}} \right)^2 = \frac{\partial^2 v}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \quad n_1, n_2 > 0, \quad (1.19)$$

for $0 < \nu_1 < \nu_2 \leq 1$, with the usual initial conditions and $v_t(x, 0) = 0$, in addition.

Equations (1.16) and (1.19) are proved to govern deeply different processes: while the former is linked, for any value of ν_1, ν_2 to a diffusion with retardation (see also [6] and [7]), the same is not true for the second equation, which, depending on the value of the random indexes, produces a subdiffusion or a superdiffusion.

2 The recursive equation of distributed order

2.1 The double-order fractional case

We begin by considering the following fractional recursive differential equation

$$\int_0^1 \frac{d^\nu p_k}{dt^\nu} n(\nu) d\nu = -\lambda(p_k - p_{k-1}), \quad k \geq 0, \quad (2.1)$$

where, by assumption,

$$n(\nu) \geq 0, \quad \int_0^1 n(\nu) d\nu = 1, \quad \nu \in (0, 1] \quad (2.2)$$

subject to the initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}, \quad (2.3)$$

with $p_{-1}(t) = 0$. We apply in (2.1) the definition of fractional derivative in the sense of Caputo, that is, for $m \in \mathbb{N}$,

$$\frac{d^\nu}{dt^\nu} u(t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{1}{(t-s)^{1+\nu-m}} \frac{d^m}{ds^m} u(s) ds, & \text{for } m-1 < \nu < m \\ \frac{d^m}{dt^m} u(t), & \text{for } \nu = m \end{cases}, \quad (2.4)$$

(see, for example, [11], p.92). As a special case, for $n(\nu) = \delta(\nu - \bar{\nu})$, and a particular value of $\bar{\nu} \in (0, 1)$, equation (2.1) reduces to (1.1), which governs the so-called FPP $\mathcal{N}_\nu(t)$, $t > 0$ (see, for details, [15], [17] and [18]).

In order to get an analytic expression for the solution to (2.1), we adopt here the following particular form for the density of the fractional order ν :

$$n(\nu) = n_1 \delta(\nu - \nu_1) + n_2 \delta(\nu - \nu_2), \quad 0 < \nu_1 < \nu_2 \leq 1, \quad (2.5)$$

for $n_1, n_2 \geq 0$ and such that $n_1 + n_2 = 1$ (conditions (2.2) are trivially fulfilled). The density (2.5) has been already used by [19] and [6], in the analysis of the so-called double-order time-fractional diffusion equation, and corresponds to the case of a subdiffusion with retardation (see next section for details). Moreover, it was applied in [16] in the context of fractional relaxation with distributed order.

Under assumption (2.5), equation (2.1) becomes

$$n_1 \frac{d^{\nu_1} p_k}{dt^{\nu_1}} + n_2 \frac{d^{\nu_2} p_k}{dt^{\nu_2}} = -\lambda(p_k - p_{k-1}), \quad k \geq 0. \quad (2.6)$$

By taking the Laplace transform of (2.6) we get the following first result.

Theorem 2.1 *The Laplace transform of the solution to equation (2.6), under conditions (2.3), is given by*

$$\mathcal{L}\{\bar{p}_k^\nu; \eta\} = \frac{\lambda^k n_1 \eta^{\nu_1-1} + \lambda^k n_2 \eta^{\nu_2-1}}{(\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2})^{k+1}}, \quad (2.7)$$

for any $k \geq 0$.

Proof Formula (2.7) can be easily obtained by applying to (2.6) the expression for the Laplace transform of the Caputo derivative, i.e.

$$\begin{aligned} \mathcal{L}\left\{\frac{d^\nu u}{dt^\nu}; \eta\right\} &= \int_0^\infty e^{-\eta t} \frac{d^\nu}{dt^\nu} u(t) dt \\ &= \eta^\nu \mathcal{L}\{u(t); \eta\} - \sum_{r=0}^{m-1} \eta^{\nu-r-1} \left. \frac{d^r}{dt^r} u(t) \right|_{t=0}, \end{aligned} \quad (2.8)$$

where $m = \lfloor \nu \rfloor + 1$, which yields, for $k \geq 1$,

$$n_1 \eta^{\nu_1} \mathcal{L} \{ \bar{p}_k^\nu; \eta \} + n_2 \eta^{\nu_2} \mathcal{L} \{ \bar{p}_k^\nu; \eta \} = -\lambda \left[\mathcal{L} \{ \bar{p}_k^\nu; \eta \} - \mathcal{L} \{ \bar{p}_{k-1}^\nu; \eta \} \right]. \quad (2.9)$$

By recursively using (2.9) we get

$$\mathcal{L} \{ \bar{p}_k^\nu; \eta \} = \left(\frac{\lambda}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}} \right)^k \mathcal{L} \{ \bar{p}_0^\nu; \eta \}, \quad k \geq 1. \quad (2.10)$$

For $k = 0$, we get, instead:

$$\mathcal{L} \{ \bar{p}_0^\nu; \eta \} = \frac{n_1 \eta^{\nu_1-1} + n_2 \eta^{\nu_2-1}}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}}, \quad (2.11)$$

which, together with (2.10), gives (2.7). \square

We can not use a direct method in order to invert analytically the Laplace transform (2.7). Indeed an explicit inversion formula is available only for $k = 0$, while for $k > 0$ the presence of the power $k + 1$ makes the analytic inversion too complicated. For $k = 0$, we can apply the well-known expression of the Laplace transform of the GML function defined in (1.6) (see [11], p.47), i.e.

$$\mathcal{L} \{ t^{\gamma-1} E_{\beta, \gamma}^\delta(\omega t^\beta); \eta \} = \frac{\eta^{\beta\delta-\gamma}}{(\eta^\beta - \omega)^\delta}, \quad (2.12)$$

(where $\text{Re}(\beta) > 0$, $\text{Re}(\gamma) > 0$, $\text{Re}(\delta) > 0$ and $s > |\omega|^{\frac{1}{\text{Re}(\beta)}}$) and the resulting formulae (26) and (27) of [26].

Therefore we get

$$\begin{aligned} & \bar{p}_0^\nu(t) \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^r E_{\nu_2, (\nu_2-\nu_1)r+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right) - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^{r+1} E_{\nu_2, (\nu_2-\nu_1)(r+1)+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right), \end{aligned} \quad (2.13)$$

under condition $|n_1 \eta^{\nu_1} / (n_2 \eta^{\nu_2} + \lambda)| < 1$ (which is fulfilled, for $\nu_2 > \nu_1$, $\lambda > 0$).

For $k > 0$, we adopt an approach similar to those used in [1], [2], [21], [22], [23] and [24] (for different types of fractional differential equations), which leads to an expression of the solution in terms of convolutions of known distributions. In particular we will resort to the class of completely asymmetric stable laws (of index less than one). More precisely, let us denote by $\bar{p}_\alpha(\cdot; z)$, for $j = 1, 2$, the density of a stable random variable X_α of index $\alpha \in (0, 1)$ and parameters equal to $\beta = 1$, $\mu = 0$ and $\sigma = \left(|z| \cos \frac{\pi\alpha}{2} \right)^{1/\alpha}$ (see [27] for the definitions and the properties of this class of stable laws). As well-known, X_α is endowed by the following Laplace transform

$$\mathcal{L} \{ \bar{p}_\alpha(\cdot; y); \eta \} = e^{-\frac{\eta^\alpha}{\cos(\pi\alpha/2)} y}, \quad (2.14)$$

which will be particularly useful in inverting (2.7). We need moreover the following result proved in [21]: the solution to the following fractional diffusion equation

$$\begin{cases} \frac{\partial^{2\alpha} v}{\partial t^{2\alpha}} = c^2 \frac{\partial^2 v}{\partial y^2}, & t > 0, y \in \mathbb{R}, c \in \mathbb{R} \\ v(y, 0) = \delta(y), & \text{for } 0 < \alpha < 1 \\ v_t(y, 0) = 0, & \text{for } 1/2 < \alpha < 1 \end{cases}. \quad (2.15)$$

can be expressed as

$$v_{2\alpha}(y, t) = \frac{1}{2c\Gamma(1-\alpha)} \int_0^t \frac{\bar{p}_\alpha(s; |y|)}{(t-s)^\alpha} ds = \frac{1}{2c} I^\alpha \{ \bar{p}_\alpha(\cdot; |y|) \}(t), \quad t > 0, y \in \mathbb{R}, \quad (2.16)$$

where $I^\alpha \{ \cdot \}$ denotes the Riemann-Liouville fractional integral of order α . By $\bar{v}_{2\alpha}(y, t)$ we will denote the folded solution to (2.15), i.e.

$$\bar{v}_{2\alpha}(y, t) = \begin{cases} 2v_{2\alpha}(y, t), & y > 0 \\ 0, & y < 0 \end{cases}. \quad (2.17)$$

Theorem 2.2 *The solution to equation (2.6), under conditions (2.3), is given, for any $k \geq 0$, and $t > 0$, by*

$$\begin{aligned}\bar{p}_k^\gamma(t) &= \int_0^{+\infty} p_k(y) q_{v_1, v_2}(y, t) dy \\ &= \frac{1}{k!} \int_0^{+\infty} y^k e^{-y} \left[\int_0^t \bar{p}_{v_2}(t-s; y) \bar{v}_{2v_1}(y, s) ds + \int_0^t \bar{p}_{v_1}(t-s; y) \bar{v}_{2v_2}(y, s) ds \right] dy,\end{aligned}\quad (2.18)$$

where $p_k, k \geq 0$, represents the distribution of the standard homogeneous Poisson process $N(t), t > 0$ (with intensity 1) and $\bar{p}_{v_j}(\cdot; z)$ denotes the density of the stable random variable X_{v_j} of index $v_j \in (0, 1)$, for $j = 1, 2$, with parameters equal $\beta = 1, \mu = 0$ and $\sigma = \left(\frac{n_j}{\lambda}|y| \cos \frac{\pi v_j}{2}\right)^{1/v_j}$.

Proof We observe that (2.7) can be rewritten as follows

$$\begin{aligned}\mathcal{L}\{\bar{p}_k^\gamma; \eta\} &= \left(\frac{n_1}{\lambda}\eta^{v_1-1} + \frac{n_2}{\lambda}\eta^{v_2-1}\right) \mathcal{L}\left\{p_k; \frac{n_1}{\lambda}\eta^{v_1} + \frac{n_2}{\lambda}\eta^{v_2}\right\} \\ &= \left(\frac{n_1}{\lambda}\eta^{v_1-1} + \frac{n_2}{\lambda}\eta^{v_2-1}\right) \int_0^{+\infty} p_k(y) e^{-(\frac{n_1}{\lambda}\eta^{v_1} + \frac{n_2}{\lambda}\eta^{v_2})y} dy,\end{aligned}\quad (2.19)$$

since, for the distribution of N , the Laplace transform reads

$$\mathcal{L}\{p_k; \eta\} = \frac{1}{(1+\eta)^{k+1}}, \quad k \geq 0.$$

The exponential in (2.19) coincides with the Laplace transform of the following convolution of the stable laws \bar{p}_{v_1} and \bar{p}_{v_2} :

$$g_{v_1, v_2}(w; y) = \int_0^w \bar{p}_{v_1}(w-x; y) \bar{p}_{v_2}(x; y) dx. \quad (2.20)$$

Therefore, by considering that

$$\eta^{v-1} = \frac{1}{\Gamma(1-v)} \int_0^{+\infty} e^{-\eta t} t^{-v} dt,$$

we obtain

$$\begin{aligned}\bar{p}_k^\gamma(t) &= \frac{n_1}{\lambda \Gamma(1-v_1)} \int_0^t (t-w)^{-v_1} \mathcal{L}^{-1} \left\{ \int_0^{+\infty} p_k(y) e^{-(\frac{n_1}{\lambda}\eta^{v_1} + \frac{n_2}{\lambda}\eta^{v_2})y} dy; w \right\} dw + \\ &\quad + \frac{n_2}{\lambda \Gamma(1-v_2)} \int_0^t (t-w)^{-v_2} \mathcal{L}^{-1} \left\{ \int_0^{+\infty} p_k(y) e^{-(\frac{n_1}{\lambda}\eta^{v_1} + \frac{n_2}{\lambda}\eta^{v_2})y} dy; w \right\} dw \\ &= \frac{n_1}{\lambda \Gamma(1-v_1)} \int_0^t (t-w)^{-v_1} \left(\int_0^{+\infty} p_k(y) g_{v_1, v_2}(w; y) dy \right) dw + \\ &\quad + \frac{n_2}{\lambda \Gamma(1-v_2)} \int_0^t (t-w)^{-v_2} \left(\int_0^{+\infty} p_k(y) g_{v_1, v_2}(w; y) dy \right) dw.\end{aligned}\quad (2.21)$$

By inserting (2.20) into (2.21) and changing the integration's order, we get

$$\begin{aligned}\bar{p}_k^\gamma(t) &= \frac{n_1}{\lambda \Gamma(1-v_1)} \int_0^{+\infty} p_k(y) \left(\int_0^t \bar{p}_{v_2}(x; y) dx \int_x^t (t-w)^{-v_1} \bar{p}_{v_1}(w-x; y) dw \right) dy \\ &\quad + \frac{n_2}{\lambda \Gamma(1-v_2)} \int_0^{+\infty} p_k(y) \left(\int_0^t \bar{p}_{v_1}(x; y) dx \int_x^t (t-w)^{-v_2} \bar{p}_{v_2}(w-x; y) dw \right) dy \\ &= \frac{n_1}{\lambda} \int_0^{+\infty} p_k(y) \left(\int_0^t \bar{p}_{v_2}(x; y) I^{v_1} \{ \bar{p}_{v_1}(\cdot; y) \} (t-x) dx \right) dy \\ &\quad + \frac{n_2}{\lambda} \int_0^{+\infty} p_k(y) \left(\int_0^t \bar{p}_{v_1}(x; y) I^{v_2} \{ \bar{p}_{v_2}(\cdot; y) \} (t-x) dx \right) dy.\end{aligned}$$

By considering (2.16) and (2.17), for $c = \lambda/n_j$, for $j = 1, 2$, formula (2.18) immediately follows. \square

Remark 2.1 The previous result shows that the solution to (2.6) can be expressed as the probability distribution of a standard Poisson process $N(t)$, $t > 0$, composed with a random time argument with transition density $q_{v_1, v_2}(y, t)$, that will be denoted as $\tilde{\mathcal{T}}_{v_1, v_2}$ (independent from N): thus we can write

$$\tilde{p}_k^v(t) = \Pr\{N(\tilde{\mathcal{T}}_{v_1, v_2}(t)) = k\}, \quad k \geq 0, t > 0. \quad (2.22)$$

It is proved in [2] that the solution to the fractional equation (1.1) is the density of the composition of $N(t)$, $t > 0$ with a random time argument $\mathcal{T}_v(t)$, whose density is given by $\tilde{v}_{2v}(y, t)$. The properties of this process have been extensively analyzed in [3]: it turns out to be a Cox process, with directing measure equal to $\Lambda((0, t]) \equiv \mathcal{T}_v(t)$. We will prove below that an analogous result is valid for the process $\tilde{N}_{v_1, v_2}(t) \equiv N(\tilde{\mathcal{T}}_{v_1, v_2}(t))$ introduced here. Moreover we will check that it is also a renewal process.

We derive now a series expression for the transition density $q_{v_1, v_2}(y, t)$ of the random time-argument $\tilde{\mathcal{T}}_{v_1, v_2}(t)$, $t > 0$, which is alternative to the integral one given in Theorem 2.2.

Theorem 2.3 *The density $q_{v_1, v_2}(y, t)$ of the random time-argument $\tilde{\mathcal{T}}_{v_1, v_2}(t)$, $t > 0$ can be expressed as follows*

$$\begin{aligned} & q_{v_1, v_2}(y, t) \\ &= \frac{n_1}{\lambda t^{v_1}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{n_2 |y|}{\lambda t^{v_2}} \right)^r \mathcal{W}_{-v_1, 1-v_2 r-v_1} \left(-\frac{n_1 |y|}{\lambda t^{v_1}} \right) + \frac{n_2}{\lambda t^{v_2}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{n_1 |y|}{\lambda t^{v_1}} \right)^r \mathcal{W}_{-v_2, 1-v_1 r-v_2} \left(-\frac{n_2 |y|}{\lambda t^{v_2}} \right), \end{aligned} \quad (2.23)$$

where

$$\mathcal{W}_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha > -1, \beta \in \mathbb{C}, x \in \mathbb{R}$$

is the Wright function.

Proof We recall that the solution to the diffusion equation (2.15) can be expressed as

$$v_{2\alpha}(y, t) = \frac{1}{2ct^\alpha} \mathcal{W}_{-\alpha, 1-\alpha} \left(-\frac{|y|}{ct^\alpha} \right), \quad t > 0, y \in \mathbb{R}, \quad (2.24)$$

(see [14], for details). Then we get from (2.18) that

$$\begin{aligned} & q_{v_1, v_2}(y, t) \\ &= \frac{n_1}{\lambda} \int_0^t \bar{p}_{v_2}(t-s; |y|) \frac{1}{s^{v_1}} \mathcal{W}_{-v_1, 1-v_1} \left(-\frac{n_1 |y|}{\lambda s^{v_1}} \right) ds + \frac{n_2}{\lambda} \int_0^t \bar{p}_{v_1}(t-s; |y|) \frac{1}{s^{v_2}} \mathcal{W}_{-v_2, 1-v_2} \left(-\frac{n_2 |y|}{\lambda s^{v_2}} \right) ds. \end{aligned} \quad (2.25)$$

We now consider the series representation of the stable law of order $\alpha \in (0, 1)$ given in [8] (formula (6.10), p.583) and already used (with some corrections), in the fractional context, in [22]:

$$\bar{p}_\alpha(x; \gamma, 1) = \frac{\alpha}{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\alpha(r+1))}{r!} x^{-\alpha(r+1)-1} \sin \left[\frac{\pi}{2} (\gamma + \alpha)(r+1) \right]. \quad (2.26)$$

In (2.26) the canonical Feller representation for the stable laws (with null position parameter μ) has been used, i.e.

$$\bar{p}_\alpha(x; \gamma, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \exp \left\{ -\zeta |\theta|^\alpha e^{-\frac{i\pi}{2} \frac{\theta}{|\theta|}} \right\} d\theta, \quad \alpha \neq 1;$$

hence we must convert the parameters appearing there into those used here, as follows:

$$\begin{aligned} \alpha &= v_j \\ \gamma &= \frac{2}{\pi} \arctan \left(\tan \frac{\pi v_j}{2} \right) = v_j = \alpha \\ \zeta &= \frac{\sigma^{v_j}}{\cos \frac{\pi v_j}{2}} = \frac{n_j |y| \cos \frac{\pi v_j}{2}}{\lambda \cos \frac{\pi v_j}{2}} = \frac{n_j}{\lambda} |y|. \end{aligned}$$

By taking into account the self-similarity property, the stable densities appearing in (2.25) become

$$\begin{aligned}
\bar{p}_{\nu_j}(x; \nu_j, \frac{n_j}{\lambda}|y|) &= \frac{\lambda^{1/\nu_j}}{(n_j|y|)^{1/\nu_j}} \bar{p}_{\nu_j}\left(\frac{x\lambda^{1/\nu_j}}{(n_j|y|)^{1/\nu_j}}; \nu_j, 1\right) \\
&= \frac{\lambda^{1/\nu_j}}{\pi(n_j|y|)^{1/\nu_j}} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_j r + 1)}{r!} \left(\frac{x\lambda^{1/\nu_j}}{(n_j|y|)^{1/\nu_j}}\right)^{-\nu_j r - 1} \sin(\pi \nu_j r) \\
&= \frac{1}{\pi x} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_j r + 1)}{r!} \left(\frac{n_j|y|}{\lambda^{1/\nu_j} x^{\nu_j}}\right)^r \sin(\pi \nu_j r),
\end{aligned} \tag{2.27}$$

so that (2.25) reads

$$\begin{aligned}
&q_{\nu_1, \nu_2}(y, t) \\
&= \frac{n_1}{\lambda \pi} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_2 r + 1)}{r!} \left(\frac{n_2}{\lambda}|y|\right)^r \sin(\pi \nu_2 r) \int_0^t \frac{1}{(t-s)^{\nu_2 r + 1} s^{\nu_1}} \mathcal{W}_{-\nu_1, 1-\nu_1}\left(-\frac{n_1|y|}{\lambda s^{\nu_1}}\right) ds \\
&\quad + \frac{n_2}{\lambda \pi} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_1 r + 1)}{r!} \left(\frac{n_1}{\lambda}|y|\right)^r \sin(\pi \nu_1 r) \int_0^t \frac{1}{(t-s)^{\nu_1 r + 1} s^{\nu_2}} \mathcal{W}_{-\nu_2, 1-\nu_2}\left(-\frac{n_2|y|}{\lambda s^{\nu_2}}\right) ds \\
&= \frac{n_1}{\lambda \pi} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_2 r + 1)}{r!} \left(\frac{n_2}{\lambda}|y|\right)^r \sin(\pi \nu_2 r) \sum_{l=0}^{\infty} \frac{(-n_1|y|/\lambda)^l}{l! \Gamma(-\nu_1 l + 1 - \nu_1)} \int_0^t \frac{1}{(t-s)^{\nu_2 r + 1} s^{\nu_1(l+1)}} ds + \\
&\quad + \frac{n_2}{\lambda \pi} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_1 r + 1)}{r!} \left(\frac{n_1}{\lambda}|y|\right)^r \sin(\pi \nu_1 r) \sum_{l=0}^{\infty} \frac{(-n_2|y|/\lambda)^l}{l! \Gamma(-\nu_2 l + 1 - \nu_2)} \int_0^t \frac{1}{(t-s)^{\nu_1 r + 1} s^{\nu_2(l+1)}} ds \\
&= \frac{n_1}{\lambda \pi t^{\nu_1}} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_2 r + 1)}{r!} \left(\frac{n_2|y|}{\lambda t^{\nu_2}}\right)^r \sin(\pi \nu_2 r) \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma(-\nu_2 r)}{\Gamma(1 - \nu_1 l - \nu_1 - \nu_2 r)} \left(-\frac{n_1|y|}{\lambda t^{\nu_1}}\right)^l + \\
&\quad + \frac{n_2}{\lambda \pi t^{\nu_2}} \sum_{r=0}^{\infty} (-1)^{r-1} \frac{\Gamma(\nu_1 r + 1)}{r!} \left(\frac{n_1|y|}{\lambda t^{\nu_1}}\right)^r \sin(\pi \nu_1 r) \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma(-\nu_1 r)}{\Gamma(1 - \nu_2 l - \nu_2 - \nu_1 r)} \left(-\frac{n_2|y|}{\lambda t^{\nu_2}}\right)^l \\
&= \frac{n_1}{\lambda t^{\nu_1}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{n_2|y|}{\lambda t^{\nu_2}}\right)^r \sum_{l=0}^{\infty} \frac{1}{l! \Gamma(1 - \nu_1 l - \nu_1 - \nu_2 r)} \left(-\frac{n_1|y|}{\lambda t^{\nu_1}}\right)^l + \\
&\quad + \frac{n_2}{\lambda t^{\nu_2}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{n_1|y|}{\lambda t^{\nu_1}}\right)^r \sum_{l=0}^{\infty} \frac{1}{l! \Gamma(1 - \nu_2 l - \nu_2 - \nu_1 r)} \left(-\frac{n_2|y|}{\lambda t^{\nu_2}}\right)^l,
\end{aligned} \tag{2.28}$$

where, in the last step, we have used the reflection formula of the Gamma function. \square

Remark 2.2 Consider the special case $n_1 = 0, n_2 = 1$: the distribution of the random order ν reduces, in this case, to

$$n(\nu) = \delta(\nu - \nu_2), \quad 0 < \nu_2 \leq 1, \tag{2.29}$$

so that equation (2.6) becomes the fractional equation (1.1), with $\nu = \nu_2$. The Laplace transform (2.7) simplifies to

$$\mathcal{L}\{\tilde{p}_k^\nu; \eta\} = \frac{\lambda^k \eta^{\nu_2 - 1}}{(\lambda + \eta^{\nu_2})^{k+1}}, \tag{2.30}$$

thus giving the well-known distribution

$$p_k^\nu(t) = \lambda^k t^{\nu_2} E_{\nu_2, \nu_2 k + 1}^{k+1}(-\lambda t^{\nu_2}),$$

for any $k \geq 0$ (see [3] for details). The result of Theorem 2.2 can be specialized as follows, for $n_1 = 0, n_2 = 1$:

$$p_k^\nu(t) = \int_0^{+\infty} p_k(y) q_{\nu_1, \nu_2}(y, t) dy = \frac{1}{k!} \int_0^{+\infty} y^k e^{-y} \bar{v}_{2\nu_2}(y, t) dy. \tag{2.31}$$

In (2.31) we have taken into account that the density of the stable random variable X_{v_1} with $\mu = 0$ and $\sigma = \left(\frac{n_1}{\lambda} y \cos \frac{\pi v_1}{2}\right)^{1/v_1}$ degenerates to the Dirac's delta function (i.e. $\bar{p}_{v_1}(w - x; y) = \delta(x - w)$), so that the density (2.20) becomes $g_{v_2}(w; y) = \bar{p}_{v_2}(w; y)$ and (2.21) easily yields (2.31). The latter coincides with the result proved in [2] and already recalled in Remark 2.1.

As far as Theorem 2.3 is concerned, by putting $n_1 = 0$, $n_2 = 1$, the density of the random time-argument can be expressed as follows:

$$q_{v_2}(y, t) = \frac{1}{\lambda t^{v_2}} \mathcal{W}_{-v_2, 1-v_2} \left(-\frac{|y|}{\lambda t^{v_2}} \right) = \bar{v}_{2v_2}(y, t), \quad (2.32)$$

since in (2.23) only the term $r = 0$ of the sum survives. To sum up, the FPP analyzed in [2] is equal in distribution to the random time process $N(\mathcal{T}_v(t))$, whose density can be expressed as a simple Wright function; on the other hand, in the distributed order case, the situation is more complicated. The density of the process $N(\tilde{\mathcal{T}}_{v_1, v_2}(t))$ is written in terms of infinite sums of Wright functions. Moreover, in the single-order case the density $\bar{v}_{2v_2}(y, t)$ coincides with the folded solution to the fractional diffusion equation (2.15); in the double-order case the relationship between the density $q_{v_1, v_2}(y, t)$ and the fractional diffusion equation of distributed order is more complicated, as we will prove in the next section.

Let us now focus on the probability generating function of the process \tilde{N}_{v_1, v_2} , which can be expressed in terms of GML functions (1.6), as the following result shows.

Theorem 2.4 *The probability generating function $\tilde{G}_{v_1, v_2}(u, t)$ of the process \tilde{N}_{v_1, v_2} is equal to*

$$\begin{aligned} \tilde{G}_{v_1, v_2}(u, t) &\equiv \sum_{k=0}^{\infty} u^k \bar{p}_k^v(t) \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2-v_1}}{n_2} \right)^r E_{v_2, (v_2-v_1)r+1}^{r+1} \left(-\frac{\lambda(1-u)t^{v_2}}{n_2} \right) + \\ &\quad - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2-v_1}}{n_2} \right)^{r+1} E_{v_2, (v_2-v_1)(r+1)+1}^{r+1} \left(-\frac{\lambda(1-u)t^{v_2}}{n_2} \right). \end{aligned} \quad (2.33)$$

Proof The Laplace transform of \tilde{G}_{v_1, v_2} can be written, by taking into account formula (2.7), as

$$\begin{aligned} \mathcal{L}\{\tilde{G}_{v_1, v_2}(u, \cdot); \eta\} &= \sum_{k=0}^{\infty} \frac{u^k \lambda^k n_1 \eta^{v_1-1}}{(\lambda + n_1 \eta^{v_1} + n_2 \eta^{v_2})^{k+1}} + \sum_{k=0}^{\infty} \frac{u^k \lambda^k n_2 \eta^{v_2-1}}{(\lambda + n_1 \eta^{v_1} + n_2 \eta^{v_2})^{k+1}} \\ &= \frac{n_1 \eta^{v_1-1}}{\lambda(1-u) + n_1 \eta^{v_1} + n_2 \eta^{v_2}} + \frac{n_2 \eta^{v_2-1}}{\lambda(1-u) + n_1 \eta^{v_1} + n_2 \eta^{v_2}} \\ &= \frac{n_1}{n_2} \frac{\frac{\lambda}{n_2}(1-u) + \frac{n_1}{n_2} \eta^{v_1} + \eta^{v_2}}{\eta^{v_1-1}} + \frac{\frac{\lambda}{n_2}(1-u) + \frac{n_1}{n_2} \eta^{v_1} + \eta^{v_2}}{\eta^{v_2-1}}. \end{aligned} \quad (2.34)$$

By applying formula (26) and (24) of [26] to the first and second terms of (2.34) respectively and recalling that $v_2 > v_1$, the Laplace transform can be inverted as follows:

$$\begin{aligned} \tilde{G}_{v_1, v_2}(u, t) &= \frac{n_1}{n_2} t^{v_2-v_1} \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2-v_1}}{n_2} \right)^r E_{v_2, (v_2-v_1)(r+1)+1}^{r+1} \left(-\frac{\lambda(1-u)t^{v_2}}{n_2} \right) + \\ &\quad + \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2-v_1}}{n_2} \right)^r E_{v_2, (v_2-v_1)r+1}^{r+1} \left(-\frac{\lambda(1-u)t^{v_2}}{n_2} \right), \end{aligned}$$

which is equal to (2.33). \square

Remark 2.3 We observe that the infinite sum of GML functions in (2.33) coincides with the Fourier transform of the solution of the diffusion equation (fractional in time and space), analyzed in [26], i.e.

$$\frac{\partial^{\nu_2}}{\partial t^{\nu_2}} f(x, t) + \frac{n_1}{n_2} \frac{\partial^{\nu_1}}{\partial t^{\nu_1}} f(x, t) = c^2 \frac{\partial^\gamma}{\partial x^\gamma} f(x, t), \quad x \in \mathbb{R}, t > 0, \quad (2.35)$$

in the special case where $\gamma = 0$. In this case, for $c^2 = \lambda(1 - u)/n_2$, equation (2.35) reduces to the time-fractional equation

$$\frac{\partial^{\nu_2}}{\partial t^{\nu_2}} G(u, t) + \frac{n_1}{n_2} \frac{\partial^{\nu_1}}{\partial t^{\nu_1}} G(u, t) = \frac{\lambda(1 - u)}{n_2} G(u, t), \quad (2.36)$$

(with initial condition $G(u, 0) = 1$). Indeed the probability generating function $\widetilde{G}_{\nu_1, \nu_2}$ must solve equation (2.36), as the following steps easily show:

$$\begin{aligned} & n_2 \frac{\partial^{\nu_2}}{\partial t^{\nu_2}} \widetilde{G}_{\nu_1, \nu_2}(u, t) + n_1 \frac{\partial^{\nu_1}}{\partial t^{\nu_1}} \widetilde{G}_{\nu_1, \nu_2}(u, t) \\ &= \sum_{k=0}^{\infty} u^k \left[n_2 \frac{\partial^{\nu_2}}{\partial t^{\nu_2}} \widetilde{p}_k^\gamma(t) + n_1 \frac{\partial^{\nu_1}}{\partial t^{\nu_1}} \widetilde{p}_k^\gamma(t) \right] \\ &= -\lambda \sum_{k=0}^{\infty} u^k \widetilde{p}_k^\gamma(t) + \lambda u \sum_{k=1}^{\infty} u^{k-1} \widetilde{p}_{k-1}^\gamma(t) \\ &= -\lambda(1 - u) \widetilde{G}_{\nu_1, \nu_2}(u, t). \end{aligned}$$

Remark 2.4 By means of the probability generating function, we can check that the distribution $\widetilde{p}_k^\gamma(t)$, sums up to one, for $k = 0, 1, \dots$. For $u = 1$, formula (2.33) yields

$$\begin{aligned} \widetilde{G}_{\nu_1, \nu_2}(u, t) \Big|_{u=1} &= \sum_{k=0}^{\infty} \widetilde{p}_k^\gamma(t) \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^r \frac{1}{\Gamma((\nu_2 - \nu_1)r + 1)} - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^{r+1} \frac{1}{\Gamma((\nu_2 - \nu_1)(r + 1) + 1)} = 1, \end{aligned} \quad (2.37)$$

since only the term $j = 0$ in the expression (1.6) of the GML function survives.

Moreover, for $u = 0$, formula (2.33) gives the probability $\widetilde{p}_0^\gamma(t)$ (already obtained in (2.13)):

$$\begin{aligned} & \widetilde{G}_{\nu_1, \nu_2}(u, t) \Big|_{u=0} = \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^r E_{\nu_2, (\nu_2 - \nu_1)r + 1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right) - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^{r+1} E_{\nu_2, (\nu_2 - \nu_1)(r + 1) + 1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right) \\ &= \widetilde{p}_0^\gamma(t). \end{aligned} \quad (2.38)$$

We note moreover that, in the special case $n_1 = 0, n_2 = 1$, the probability generating function reduces to

$$G_{\nu_2}(u, t) = E_{\nu_2, 1}(-\lambda(1 - u)t^{\nu_2})$$

which coincides with the one obtained for the fractional Poisson process in [2], as expected.

We make use of Theorem 2.4 also in the evaluation of the exponential moments of the process $\widetilde{N}_{\nu_1, \nu_2}$ and in its resulting characterization as a Cox process.

Theorem 2.5 *The factorial moments of the process $\widetilde{N}_{\nu_1, \nu_2}$, with distribution $\widetilde{p}_k^\gamma(t)$ and probability generating function $\widetilde{G}_{\nu_1, \nu_2}(u, t)$ given in (2.33), are equal to*

$$\begin{aligned} & \mathbb{E} \left[\widetilde{N}_{\nu_1, \nu_2}(t) (\widetilde{N}_{\nu_1, \nu_2}(t) - 1) \dots (\widetilde{N}_{\nu_1, \nu_2}(t) - k + 1) \right] \\ &= \frac{n_1 \lambda^k t^{\nu_2 k + \nu_2 - \nu_1}}{n_2^{k+1}} k! E_{\nu_2 - \nu_1, \nu_2 k + \nu_2 - \nu_1 + 1}^{k+1} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right) + \frac{\lambda^k t^{\nu_2 k}}{n_2^k} k! E_{\nu_2 - \nu_1, \nu_2 k + 1}^{k+1} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right). \end{aligned} \quad (2.39)$$

Moreover \widetilde{N}_{v_1, v_2} is a Cox process with directing measure $\Lambda((0, t]) \equiv \widetilde{\mathcal{T}}_{v_1, v_2}(t)$, endowed with density $q_{v_1, v_2}(y, t)$.

Proof We take the k -th derivatives of \widetilde{G}_{v_1, v_2} with respect to u :

$$\begin{aligned} & \frac{\partial^k}{\partial u^k} \widetilde{G}_{v_1, v_2}(u, t) \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^r \frac{1}{r!} \sum_{j=0}^{\infty} \frac{(r+j)!}{j! \Gamma(v_2 j + (v_2 - v_1)r + 1)} \frac{\left(-\frac{\lambda t^{v_2}}{n_2} \right)^j (-1)^k (1-u)^{j-k} j!}{(j-k)!} + \\ & \quad - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^{r+1} \frac{1}{r!} \sum_{j=0}^{\infty} \frac{(r+j)!}{j! \Gamma(v_2 j + (v_2 - v_1)(r+1) + 1)} \frac{\left(-\frac{\lambda t^{v_2}}{n_2} \right)^j (-1)^k (1-u)^{j-k} j!}{(j-k)!}, \end{aligned}$$

which, for $u = 1$, becomes

$$\begin{aligned} \left. \frac{\partial^k}{\partial u^k} \widetilde{G}_{v_1, v_2}(u, t) \right|_{u=1} &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^r \frac{1}{r!} \frac{(r+k)! \left(\frac{\lambda t^{v_2}}{n_2} \right)^k}{\Gamma(v_2 k + (v_2 - v_1)r + 1)} \\ & \quad - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^{r+1} \frac{1}{r!} \frac{(r+k)! \left(\frac{\lambda t^{v_2}}{n_2} \right)^k}{\Gamma(v_2 k + (v_2 - v_1)(r+1) + 1)}. \end{aligned} \quad (2.40)$$

Formula (2.40) can be written as (2.39), by multiplying and dividing for $k!$.

In order to prove that $\widetilde{N}_{v_1, v_2}(t)$, $t > 0$ is a Cox process with directing measure equal to $\Lambda((0, t]) \equiv \widetilde{\mathcal{T}}_{v_1, v_2}(t)$, we adopt the characterization of Cox processes by its factorial moments. Indeed it is proved in [12] that for a Cox process they must coincide with the ordinary moments of its directing measure. Our goal is to show that this equivalence holds for \widetilde{N}_{v_1, v_2} and for the density $q_{v_1, v_2}(y, t)$ of its time argument, i.e. that

$$\mathbb{E} \left[\widetilde{\mathcal{T}}_{v_1, v_2}(t) \right]^k = \int_0^{+\infty} y^k q_{v_1, v_2}(y, t) dy \quad (2.41)$$

coincides with (2.39). We start by taking the Laplace transform of (2.23), which reads

$$\begin{aligned} \mathcal{L} \{ q_{v_1, v_2}(y, \cdot); \eta \} &= \frac{n_1}{\lambda} \sum_{r=0}^{\infty} \frac{(-n_2 |y|/\lambda)^r}{r!} \sum_{l=0}^{\infty} \frac{(-n_1 |y|/\lambda)^l}{l! \Gamma(-v_1 l + 1 - v_2 r - v_1)} \frac{\Gamma(1 - v_1 l - v_2 r - v_1)}{\eta^{1 - v_1 l - v_2 r - v_1}} + \\ & \quad + \frac{n_2}{\lambda} \sum_{r=0}^{\infty} \frac{(-n_1 |y|/\lambda)^r}{r!} \sum_{l=0}^{\infty} \frac{(-n_2 |y|/\lambda)^l}{l! \Gamma(-v_2 l + 1 - v_1 r - v_2)} \frac{\Gamma(1 - v_2 l - v_1 r - v_2)}{\eta^{1 - v_2 l - v_1 r - v_2}} \\ &= \frac{n_1 e^{-n_1 \eta^{v_1} |y|/\lambda}}{\lambda \eta^{1 - v_1}} \sum_{r=0}^{\infty} \frac{(-n_2 \eta^{v_2} |y|/\lambda)^r}{r!} + \frac{n_2 e^{-n_2 \eta^{v_2} |y|/\lambda}}{\lambda \eta^{1 - v_2}} \sum_{r=0}^{\infty} \frac{(-n_1 \eta^{v_1} |y|/\lambda)^r}{r!} \\ &= \frac{n_1 e^{-(n_1 \eta^{v_1} + n_2 \eta^{v_2}) |y|/\lambda}}{\lambda \eta^{1 - v_1}} + \frac{n_2 e^{-(n_2 \eta^{v_2} + n_1 \eta^{v_1}) |y|/\lambda}}{\lambda \eta^{1 - v_2}}, \end{aligned} \quad (2.42)$$

so that the Laplace transform of (2.41) becomes

$$\begin{aligned} & \mathcal{L} \left\{ \mathbb{E} \left[\widetilde{\mathcal{T}}_{v_1, v_2}(\cdot) \right]^k; \eta \right\} \\ &= \frac{n_1}{\lambda \eta^{1 - v_1}} \int_0^{+\infty} y^k e^{-(n_1 \eta^{v_1} + n_2 \eta^{v_2}) y/\lambda} dy + \frac{n_2}{\lambda \eta^{1 - v_2}} \int_0^{+\infty} y^k e^{-(n_1 \eta^{v_1} + n_2 \eta^{v_2}) y/\lambda} dy \\ &= \frac{n_1 \lambda^k \eta^{v_1 - 1} k!}{(n_1 \eta^{v_1} + n_2 \eta^{v_2})^{k+1}} + \frac{n_2 \lambda^k \eta^{v_2 - 1} k!}{(n_1 \eta^{v_1} + n_2 \eta^{v_2})^{k+1}}. \end{aligned} \quad (2.43)$$

We now take the Laplace transform of (2.39), by applying (2.12):

$$\begin{aligned} & \mathcal{L} \left\{ \mathbb{E} \left[\widetilde{N}_{v_1, v_2}(\cdot) \dots (\widetilde{N}_{v_1, v_2}(\cdot) - k + 1) \right]; \eta \right\} \\ &= \frac{n_1 \lambda^k}{n_2^{k+1}} k! \frac{\eta^{-v_1 k - 1}}{\left(\eta^{v_2 - v_1} + \frac{n_1}{n_2} \right)^{k+1}} + \frac{\lambda^k}{n_2^k} k! \frac{\eta^{v_2 - v_1 - v_1 k - 1}}{\left(\eta^{v_2 - v_1} + \frac{n_1}{n_2} \right)^{k+1}}. \end{aligned}$$

It is simply verified that the last expression coincides with (2.43). \square

Remark 2.5 For $k = 1$ we get from (2.39) the expected value of $\tilde{\mathcal{N}}_{v_1, v_2}$:

$$\begin{aligned}
& \mathbb{E} \tilde{\mathcal{N}}_{v_1, v_2}(t) \\
&= \frac{n_1 \lambda t^{2v_2 - v_1}}{n_2^2} E_{v_2 - v_1, 2v_2 - v_1 + 1}^2 \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right) + \frac{\lambda t^{v_2}}{n_2} E_{v_2 - v_1, v_2 + 1}^2 \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right) \\
&= \frac{n_1 \lambda t^{2v_2 - v_1}}{n_2^2} \sum_{j=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j \frac{j+1}{\Gamma(v_2 j - v_1 j + 2v_2 - v_1 + 1)} + \\
&\quad + \frac{\lambda t^{v_2}}{n_2} \sum_{j=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j \frac{j+1}{\Gamma(v_2 j - v_1 j + v_2 + 1)} \\
&= -\frac{\lambda t^{v_2}}{n_2} \sum_{l=1}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^l \frac{l}{\Gamma((v_2 - v_1)l + v_2 + 1)} + \\
&\quad + \frac{\lambda t^{v_2}}{n_2} \sum_{j=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j \frac{j+1}{\Gamma((v_2 - v_1)j + v_2 + 1)} \\
&= \frac{\lambda t^{v_2}}{n_2} \sum_{j=0}^{\infty} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j \frac{1}{\Gamma((v_2 - v_1)j + v_2 + 1)} \\
&= \frac{\lambda t^{v_2}}{n_2} E_{v_2 - v_1, v_2 + 1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right).
\end{aligned} \tag{2.44}$$

Now consider again the particular case $n_1 = 0, n_2 = 1$; formula (2.39) reduces, in this case, to

$$\mathbb{E} [\mathcal{N}_{v_2}(t) (\mathcal{N}_{v_2}(t) - 1) \dots (\mathcal{N}_{v_2}(t) - k + 1)] = \frac{\lambda^k t^{v_2 k} k!}{\Gamma(v_2 k + 1)},$$

which coincides with the factorial moments of the FPP obtained in [2]. Analogously the expected value given in (2.44) reduces (for $n_1 = 0, n_2 = 1$) to

$$\mathbb{E} \mathcal{N}_{v_2}(t) = \frac{\lambda t^{v_2}}{\Gamma(v_2 + 1)}, \tag{2.45}$$

as expected. We observe that, in the distributed order case analyzed here, both the factorial moments and the expected value of $\tilde{\mathcal{N}}_{v_1, v_2}$ are expressed in terms of Mittag-Leffler functions; for $k = 1$ it is a two-parameter Mittag-Leffler function, while, for $k > 1$, we need a GML function with third parameter equal to $k+1$. This is analogously true, in view of Theorem 2.5, for the k -th order moments of the time argument $\tilde{\mathcal{T}}_{v_1, v_2}(t)$.

We concentrate now on the renewal property of $\tilde{\mathcal{N}}_{v_1, v_2}$: more precisely, we obtain the density of the waiting-time of the k -th event $\tilde{f}_k^v(t)$ (or, more exactly, its Laplace transform) and that of the interarrival times $\tilde{f}_1^v(t)$. The latter is expressed again by means of infinite sums of GML functions. The same is true for the survival probability $\tilde{\Psi}^v(t)$. We remark that $\tilde{f}_k^v(t)$ can be expressed as the k -th convolution of $\tilde{f}_1^v(t)$ and this implies that the process $\tilde{\mathcal{N}}_{v_1, v_2}$ is a renewal, since the waiting-time of the k -th event $T_k \equiv \inf \{t > 0 : \tilde{\mathcal{N}}_{v_1, v_2}(t) = k\}$ is given by the sum of k independent and identically distributed interarrival times $U_j, j = 1, \dots, k$.

Theorem 2.6 *The Laplace transform of the density $\tilde{f}_k^v(t) = \Pr \{T_k \in dt\}$ of the k -th event waiting-time T_k , is equal to*

$$\mathcal{L} \{ \tilde{f}_k^v; \eta \} = \left(\frac{\lambda}{\lambda + n_1 \eta^{v_1} + n_2 \eta^{v_2}} \right)^k, \quad k \geq 1. \tag{2.46}$$

The density of the interarrival time U_j is equal to \widetilde{f}_1^v , for any $j = 1, 2, \dots$ and can be written as

$$\widetilde{f}_1^v(t) = \frac{\lambda}{n_2} t^{\nu_2-1} \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^r E_{\nu_2, \nu_2+(\nu_2-\nu_1)r}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right). \quad (2.47)$$

Alternatively

$$\widetilde{f}_1^v(t) = \int_0^{+\infty} f_1(s) g_{\nu_1, \nu_2}(s, t) ds = \int_0^{+\infty} e^{-s} g_{\nu_1, \nu_2}(s, t) ds, \quad (2.48)$$

where f_1 denotes the interarrival-time density of the Poisson process N (i.e. $f_1(t) = e^{-t}$) and g_{ν_1, ν_2} is given in (2.20). Then $\widetilde{N}_{\nu_1, \nu_2}$ is a renewal process with renewal function given by

$$\widetilde{m}^v(t) = \frac{\lambda t^{\nu_2}}{n_2} E_{\nu_2-\nu_1, \nu_2+1} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right). \quad (2.49)$$

The survival probability $\widetilde{\Psi}^v(t) \equiv \Pr\{U_1 > t\}$ can be expressed as

$$\begin{aligned} \widetilde{\Psi}^v(t) &= 1 - \int_0^t \widetilde{f}_1^v(s) ds \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^r E_{\nu_2, (\nu_2-\nu_1)r+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right) - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^{r+1} E_{\nu_2, (\nu_2-\nu_1)(r+1)+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2} \right), \end{aligned} \quad (2.50)$$

which solves the relaxation equation of distributed order

$$n_1 \frac{\partial^{\nu_1}}{\partial t^{\nu_1}} \Psi(t) + n_2 \frac{\partial^{\nu_2}}{\partial t^{\nu_2}} \Psi(t) = -\lambda \Psi(t), \quad (2.51)$$

(with initial condition $\Psi(0) = 1$).

Proof Formula (2.46) easily follows from the following relationship

$$\begin{aligned} \mathcal{L}\{\widetilde{p}_k^v; \eta\} &= \int_0^{+\infty} e^{-\eta t} \Pr\{\widetilde{N}_{\nu_1, \nu_2}(t) = k\} dt \\ &= \int_0^{+\infty} e^{-\eta t} [\Pr\{T_k < t\} - \Pr\{T_{k+1} < t\}] dt \\ &= \frac{1}{\eta} [\mathcal{L}\{\widetilde{f}_k^v; \eta\} - \mathcal{L}\{\widetilde{f}_{k+1}^v; \eta\}], \end{aligned}$$

used together with (2.7). The Laplace transform of the density of the first interarrival time U_1 is equal to (2.46) for $k = 1$:

$$\mathcal{L}\{\widetilde{f}_1^v; \eta\} = \frac{\lambda}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}}, \quad (2.52)$$

and thus the density of the k -th event waiting time \widetilde{f}_k^v is expressed as the k -fold convolution of \widetilde{f}_1^v . This proves that $\widetilde{N}_{\nu_1, \nu_2}$ is a renewal process; its renewal function has been already calculated in (2.44). As a check we show that the well-known relationship between the Laplace transforms of $\widetilde{m}^v(t)$ and \widetilde{f}_1^v holds in this case:

$$\begin{aligned} \mathcal{L}\{\widetilde{m}^v; \eta\} &= \frac{\lambda}{\eta [n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}]} \\ &= \frac{\frac{\lambda}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}}}{\eta \left[1 - \frac{\lambda}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}} \right]} = \frac{\mathcal{L}\{\widetilde{f}_1^v; \eta\}}{\eta [1 - \mathcal{L}\{\widetilde{f}_1^v; \eta\}]}. \end{aligned}$$

The Laplace transform (2.52) can be inverted by applying formula (27) of [26], for $\alpha = \nu_2$, $a = n_1/n_2$, $\beta = \nu_1$ and $b = \lambda/n_2$, thus giving (2.47). We can rewrite moreover (2.52) as follows:

$$\begin{aligned} \mathcal{L}\{\widetilde{f}_1^v; \eta\} &= \frac{\lambda}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}} \\ &= \mathcal{L}\left\{f_1; \frac{n_1}{\lambda} \eta^{\nu_1} + \frac{n_2}{\lambda} \eta^{\nu_2}\right\}, \end{aligned} \quad (2.53)$$

where $f_1(t), t > 0$, is again the density of the interarrival times for the Poisson process; hence

$$\mathcal{L}\{\tilde{f}_1^\nu; \eta\} = \int_0^{+\infty} f_1(t) e^{-(\frac{n_1}{\lambda} \eta^{\nu_1} + \frac{n_2}{\lambda} \eta^{\nu_2})t} dt. \quad (2.54)$$

By inverting the Laplace transform (2.54), taking into account (2.21) and (2.20), we get (2.48).

We take the derivative of (2.50) and we show that $-\frac{d}{dt}\tilde{\Psi}^\nu(t) = \tilde{f}_1^\nu(t)$ given in (2.47): indeed

$$\begin{aligned} & -\frac{d}{dt}\tilde{\Psi}^\nu(t) \\ = & -(\nu_2 - \nu_1) \sum_{r=0}^{\infty} r \left(-\frac{n_1}{n_2}\right)^r t^{(\nu_2 - \nu_1)r-1} E_{\nu_2, (\nu_2 - \nu_1)r+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2}\right) + \\ & - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2}\right)^r \frac{1}{r!} \sum_{j=0}^{\infty} \frac{(r+j)! \left(-\frac{\lambda}{n_2}\right)^j j \nu_2 t^{\nu_2 j-1}}{j! \Gamma(\nu_2 j + (\nu_2 - \nu_1)r + 1)} + \\ & + (\nu_2 - \nu_1) \sum_{r=0}^{\infty} (r+1) \left(-\frac{n_1}{n_2}\right)^{r+1} t^{(\nu_2 - \nu_1)(r+1)-1} E_{\nu_2, (\nu_2 - \nu_1)(r+1)+1}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2}\right) + \\ & + \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2}\right)^{r+1} \frac{1}{r!} \sum_{j=0}^{\infty} \frac{(r+j)! \left(-\frac{\lambda}{n_2}\right)^j j \nu_2 t^{\nu_2 j-1}}{j! \Gamma(\nu_2 j + (\nu_2 - \nu_1)(r+1) + 1)} \\ = & -\frac{1}{t} \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2}\right)^r E_{\nu_2, (\nu_2 - \nu_1)r}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2}\right) + \frac{1}{t} \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2}\right)^{r+1} E_{\nu_2, (\nu_2 - \nu_1)(r+1)}^{r+1} \left(-\frac{\lambda t^{\nu_2}}{n_2}\right). \end{aligned}$$

The latter expression can be shown to coincide with (2.47).

By noting that (2.50) is equal to (2.33) for $u = 0$, it is immediately proved that $\tilde{\Psi}^\nu$ solves equation (2.36) for $u = 0$, i.e. equation (2.51). Alternatively, it is easy to check that the Laplace transform of (2.50) is given by

$$\mathcal{L}\{\tilde{\Psi}^\nu; \eta\} = \frac{n_1 \eta^{\nu_1-1} + n_2 \eta^{\nu_2-1}}{\lambda + n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2}},$$

which coincides with the solution to the Laplace transform of equation (2.51). \square

Remark 2.6 In the special case $n_1 = 0, n_2 = 1$, from (2.47) we retrieve the density of the interarrival times of the fractional Poisson process (see [2]):

$$f_1^\nu(t) = \lambda t^{\nu_2-1} E_{\nu_2, \nu_2}(-\lambda t^{\nu_2}). \quad (2.55)$$

Likewise the survival probability (2.50) reduces to

$$\Psi^\nu(t) = E_{\nu_2, 1}(-\lambda t^{\nu_2}).$$

It is interesting to analyze the asymptotic behavior of the waiting time densities and of the renewal function and to compare these expressions with the corresponding formulas obtained for the fractional Poisson process. To this purpose we need to prove the following integral representation for the GML function:

$$E_{\nu, \beta}^k(-ct^\nu) = \frac{t^{1-\beta}}{2\pi i} \int_0^{+\infty} r^{\nu k - \beta} e^{-rt} \frac{e^{i\pi\beta}(r^\nu + ce^{-i\pi\nu})^k - e^{-i\pi\beta}(r^\nu + ce^{i\pi\nu})^k}{[r^{2\nu} + 2r^\nu c \cos(\nu\pi) + c^2]^k} dr. \quad (2.56)$$

We start by checking that, for $k = 1$, formula (2.56) coincides with the form given for $E_{\nu, \beta}(-t^\nu)$ in [2], i.e.

$$E_{\nu, \beta}(-ct^\nu) = \frac{t^{1-\beta}}{2\pi i} \int_0^{+\infty} r^{\nu k - \beta} e^{-rt} \frac{r^\nu \sin(\beta\pi) + c \sin((\beta - \nu)\pi)}{r^{2\nu} + 2r^\nu c \cos(\nu\pi) + c^2} dr. \quad (2.57)$$

In order to prove formula (2.56) we multiply and divide the m -th term in the series expression of $E_{\nu,\beta}^k(-ct^\nu)$ for $\sin((\beta + \nu m)\pi)/\pi$ and apply again the reflection formula of the Gamma function, as follows

$$\begin{aligned}
& E_{\nu,\beta}^k(-ct^\nu) \tag{2.58} \\
&= \frac{1}{(k-1)!} \sum_{m=0}^{\infty} \frac{(m+k-1)!(-ct^\nu)^m}{m!\Gamma(\nu m + \beta)} \frac{\sin((\beta + \nu m)\pi)}{\pi} \Gamma(1 - \nu m - \beta) \Gamma(\nu m + \beta) \\
&= \frac{t^{1-\beta}}{\pi(k-1)!} \sum_{m=0}^{\infty} \frac{(m+k-1)!(-c)^m}{m!\Gamma(\nu m + \beta)} \sin((\beta + \nu m)\pi) \int_0^\infty e^{-rt} r^{-\nu m - \beta} dr \int_0^\infty e^{-y} y^{\nu m + \beta - 1} dy \\
&= \frac{t^{1-\beta}}{\pi(k-1)!} \int_0^\infty e^{-rt} dr \int_0^\infty e^{-ry} y^{\beta-1} \sum_{m=0}^{\infty} \frac{(m+k-1)!(-cy^\nu)^m}{m!\Gamma(\nu m + \beta)} \frac{e^{i\pi\nu m + i\pi\beta} - e^{-i\pi\nu m - i\pi\beta}}{2i} dy \\
&= \frac{t^{1-\beta}}{2\pi i(k-1)!} \int_0^\infty dy \int_0^\infty e^{-r(y+t)} y^{\beta-1} \left[e^{i\pi\beta} E_{\nu,\beta}^k(-cy^\nu e^{i\pi\nu}) - e^{-i\pi\beta} E_{\nu,\beta}^k(-cy^\nu e^{-i\pi\nu}) \right] dr \\
&= [\text{by (2.12)}] \\
&= \frac{t^{1-\beta}}{2\pi i} \int_0^\infty e^{-rt} \left[e^{i\pi\beta} \frac{r^{\nu k - \beta}}{(r^\nu + ce^{i\pi\nu})^k} - e^{-i\pi\beta} \frac{r^{\nu k - \beta}}{(r^\nu + ce^{-i\pi\nu})^k} \right] dr.
\end{aligned}$$

This coincides with (2.56). The asymptotic behavior of $E_{\nu,\beta}^k(-ct^\nu)$ can be obtained from (2.56) and reads, for $t \rightarrow \infty$:

$$\begin{aligned}
E_{\nu,\beta}^k(-ct^\nu) &= \frac{t^{-\nu k}}{2\pi i} \int_0^\infty e^{-z} z^{\nu k - \beta} \left[\frac{e^{i\pi\beta} (\frac{z^\nu}{t^\nu} + ce^{-i\pi\nu})^k - e^{-i\pi\beta} (\frac{z^\nu}{t^\nu} + ce^{i\pi\nu})^k}{(\frac{z^{2\nu}}{t^{2\nu}} + 2c\frac{z^\nu}{t^\nu} \cos \pi\nu + c^2)^k} \right] dz \\
&= \frac{\Gamma(1 - \beta + \nu k)}{\pi c^k t^{\nu k}} \sin(\pi(\beta - \nu k)) + o(t^{-\nu k}) \tag{2.59} \\
&= \frac{1}{c^k t^{\nu k} \Gamma(\beta - \nu k)} + o(t^{-\nu k}), \quad t \rightarrow \infty.
\end{aligned}$$

For $k = 1$, formula (2.59) reduces to the one holding for the Mittag-Leffler function, which can be deduced by (2.57), i.e.

$$E_{\nu,\beta}(-ct^\nu) = \frac{1}{ct^\nu \Gamma(\beta - \nu)} + o(t^{-\nu}), \quad t \rightarrow \infty. \tag{2.60}$$

For $t \rightarrow 0$, we get instead that

$$\begin{aligned}
\lim_{t \rightarrow 0^+} E_{\nu,\beta}^k(-ct^\nu) &= \lim_{t \rightarrow 0^+} \frac{1}{2\pi i} \int_0^\infty e^{-z} z^{\nu k - \beta} \left[\frac{e^{i\pi\beta} (z^\nu + ce^{-i\pi\nu} t^\nu)^k - e^{-i\pi\beta} (z^\nu + ce^{i\pi\nu} t^\nu)^k}{(z^{2\nu} + 2cz^\nu t^\nu \cos \pi\nu + c^2 t^{2\nu})^k} \right] dz \\
&= \frac{1}{\pi} \int_0^\infty e^{-z} z^{2\nu k - \beta} \frac{\sin(\pi\beta)}{z^{2\nu k}} dz = \frac{\Gamma(1 - \beta)}{\pi} \sin(\pi\beta) = \frac{1}{\Gamma(\beta)}. \tag{2.61}
\end{aligned}$$

The asymptotic behavior for small t can be deduced directly by the series expression of $E_{\nu,\beta}^k(-ct^\nu)$: indeed we get, for $0 \leq t \ll 1$,

$$E_{\nu,\beta}^k(-ct^\nu) \simeq \frac{1}{\Gamma(\beta)} - \frac{ct^\nu k}{\Gamma(\beta + \nu)},$$

which reduces, for $k = 1$, to the well-known expression (see [20], formula (3.13)).

The interarrival-time density (2.47) can be rewritten, by applying (2.58), as

$$\begin{aligned}
& \tilde{f}_1^{\vee}(t) \tag{2.62} \\
&= \frac{\lambda t^{\nu_2-1}}{n_2} \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^r \frac{t^{1-\nu_2-(\nu_2-\nu_1)r}}{2\pi i} \cdot \\
&\quad \cdot \int_0^{\infty} e^{-zt} z^{\nu_2(r+1)-\nu_2-(\nu_2-\nu_1)r} \left[\frac{e^{i\pi\nu_2+i\pi(\nu_2-\nu_1)r}}{(z^{\nu_2} + \frac{\lambda}{n_2} e^{i\pi\nu_2})^{r+1}} - \frac{e^{-i\pi\nu_2-i\pi(\nu_2-\nu_1)r}}{(z^{\nu_2} + \frac{\lambda}{n_2} e^{-i\pi\nu_2})^{r+1}} \right] dz \\
&= \frac{\lambda e^{i\pi\nu_2}}{2\pi i} \int_0^{\infty} \frac{e^{-zt} dz}{n_1 e^{i\pi(\nu_2-\nu_1)} z^{\nu_1} + n_2 z^{\nu_2} + \lambda e^{i\pi\nu_2}} - \frac{\lambda e^{-i\pi\nu_2}}{2\pi i} \int_0^{\infty} \frac{e^{-zt} dz}{n_1 e^{-i\pi(\nu_2-\nu_1)} z^{\nu_1} + n_2 z^{\nu_2} + \lambda e^{-i\pi\nu_2}} \\
&= \left[w = \frac{z}{t} \right] \\
&= \frac{\lambda}{2\pi i} \int_0^{\infty} e^{-w} dw \left[\frac{e^{i\pi\nu_2}}{n_1 e^{i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{1-\nu_1} + n_2 w^{\nu_2} t^{1-\nu_2} + \lambda e^{i\pi\nu_2} t} + \right. \\
&\quad \left. - \frac{e^{-i\pi\nu_2}}{n_1 e^{-i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{1-\nu_1} + n_2 w^{\nu_2} t^{1-\nu_2} + \lambda e^{-i\pi\nu_2} t} \right] \\
&\simeq \frac{\lambda \sin(\pi\nu_2) t^{\nu_2-1} \Gamma(1-\nu_2)}{\pi n_2} = \frac{\lambda t^{\nu_2-1}}{n_2 \Gamma(\nu_2)}, \quad t \rightarrow 0
\end{aligned}$$

which shows that, for $t \rightarrow 0$, the asymptotic behavior of \tilde{f}_1^{\vee} depends only on the larger fractional index ν_2 . The same conclusion can be drawn by looking at the series expansion of $\tilde{f}_1^{\vee}(t)$ given in (2.47).

For $t \rightarrow \infty$, from the sixth line of (2.62), we have instead that

$$\begin{aligned}
& \tilde{f}_1^{\vee}(t) \tag{2.63} \\
&= \frac{\lambda t^{-1}}{2\pi i} \int_0^{\infty} e^{-w} dw \cdot \\
&\quad \cdot \frac{n_1 w^{\nu_1} t^{-\nu_1} (e^{i\pi\nu_1} - e^{-i\pi\nu_1}) + n_2 w^{\nu_2} t^{-\nu_2} (e^{i\pi\nu_2} - e^{-i\pi\nu_2})}{(n_1 e^{i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{i\pi\nu_2}) (n_1 e^{-i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{-i\pi\nu_2})} \\
&\simeq \frac{n_1 \nu_1 t^{-1-\nu_1} \sin(\pi\nu_1) \Gamma(\nu_1)}{\lambda \pi} = \frac{n_1 \nu_1 t^{-1-\nu_1}}{\lambda \Gamma(1-\nu_1)}, \quad t \rightarrow \infty,
\end{aligned}$$

which depends only on the smaller fractional index ν_1 . Both asymptotic expressions (2.62) and (2.63) are exactly the same as for a fractional Poisson process of single order equal to ν_2 and ν_1 respectively (see [2], formulae (2.38) and (2.36)).

Analogously we can analyze the asymptotics of the survival probability, which turns out to be, for $t \rightarrow \infty$,

$$\begin{aligned}
& \tilde{\Psi}^{\vee}(t) \tag{2.64} \\
&= \frac{n_2 t^{-\nu_2} e^{i\pi}}{2\pi i} \int_0^{\infty} \frac{e^{-w} w^{\nu_2-1} dw}{n_1 e^{i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{i\pi\nu_2}} + \\
&\quad - \frac{n_2 t^{-\nu_2} e^{-i\pi}}{2\pi i} \int_0^{\infty} \frac{e^{-w} w^{\nu_2-1} dw}{n_1 e^{-i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{-i\pi\nu_2}} + \\
&\quad + \frac{n_1 t^{-\nu_1} e^{i\pi+i\pi(\nu_2-\nu_1)}}{2\pi i} \int_0^{\infty} \frac{e^{-w} w^{\nu_1-1} dw}{n_1 e^{i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{i\pi\nu_2}} + \\
&\quad - \frac{n_1 t^{-\nu_1} e^{-i\pi-i\pi(\nu_2-\nu_1)}}{2\pi i} \int_0^{\infty} \frac{e^{-w} w^{\nu_1-1} dw}{n_1 e^{-i\pi(\nu_2-\nu_1)} w^{\nu_1} t^{-\nu_1} + n_2 w^{\nu_2} t^{-\nu_2} + \lambda e^{-i\pi\nu_2}} \\
&\simeq \frac{n_1 t^{-\nu_1} \sin(\pi(1-\nu_1)) \Gamma(\nu_1)}{\lambda \pi} = \frac{n_1 t^{-\nu_1}}{\lambda \Gamma(1-\nu_1)}, \quad t \rightarrow \infty.
\end{aligned}$$

For $t \rightarrow 0$, by writing down the first terms of the series expansion in (2.50) (at least for $j = 0, 1, 2$

and $r = 0, 1, 2$) and doing some manipulations, we finally get

$$\widetilde{\Psi}^\nu(t) = 1 - \frac{\lambda}{n_2} \frac{t^{\nu_2}}{\Gamma(\nu_2 + 1)} + o(t^{\nu_2}), \quad 0 \leq t < 1. \quad (2.65)$$

As far as the renewal function is concerned, its asymptotic behavior can be represented as follows:

$$\widetilde{m}^\nu(t) \simeq \frac{\lambda t^{\nu_1}}{n_1 \Gamma(\nu_1 + 1)}, \quad t \rightarrow \infty \quad (2.66)$$

and

$$\widetilde{m}^\nu(t) \simeq \frac{\lambda t^{\nu_2}}{n_2 \Gamma(\nu_2 + 1)}, \quad t \rightarrow 0. \quad (2.67)$$

From (2.66) it is evident that the mean waiting time, which coincides with $\lim_{t \rightarrow \infty} t / \widetilde{m}^\nu(t)$, is infinite, since $\nu_1 < 1$.

Remark 2.7 We remark that, in Theorem 2.6, the survival probability $\widetilde{\Psi}^\nu$ expressed in (2.50) is proved to be a solution of the relaxation equation of distributed order (2.51) under the double-order hypothesis (2.5). This result can be compared to the analysis in [16], where only the Laplace transform of the solution is presented, together with its asymptotic behavior. Formulae (2.64) and (2.65) coincide with the result (4.13) obtained therein, but here we provide an explicit formula of the solution, in terms of infinite sums of GML functions.

2.2 Interpolation between fractional and integer-order equation

We analyze now the following equation:

$$n_1 \frac{d^\nu p_k}{dt^\nu} + n_2 \frac{dp_k}{dt} = -\lambda(p_k - p_{k-1}), \quad k \geq 0, \quad \nu \in (0, 1) \quad (2.68)$$

which is obtained from (2.6), as a special case for $\nu_2 = 1$, under the usual initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}, \quad (2.69)$$

and $p_{-1}(t) = 0$. Equation (2.68) represents an interpolation between the standard and the fractional equation governing the Poisson process. Hence the solution, which will be denoted in this case by \widetilde{p}_k^ν , must coincide, for $n_1 = 0, n_2 = 1$ with the distribution of the homogeneous Poisson process, i.e. $p_k, k \geq 0$. On the other hand, for $n_1 = 1, n_2 = 0$ we must retrieve the distribution of the fractional Poisson process, i.e. $p_k^\nu, k \geq 0$, given in (1.5).

The Laplace transform of the solution to equation (2.68) can be obtained directly by putting $\nu_1 = \nu$ and $\nu_2 = 1$ in the result of Theorem 2.1, so that we get

$$\mathcal{L}\{\widetilde{p}_k^\nu; \eta\} = \frac{\lambda^k n_1 \eta^{\nu_1-1} + \lambda^k n_2}{(\lambda + n_1 \eta^{\nu_1} + n_2 \eta)^{k+1}}, \quad k \geq 0. \quad (2.70)$$

for any $k \geq 0$. In order to invert this expression we adapt the result of Theorem 2.2 as follows.

Theorem 2.7 *The solution to equation (2.68), under conditions (2.69), are given, for any $k \geq 0$, and $t > 0$, by*

$$\begin{aligned} \widetilde{p}_k^\nu(t) &= \int_0^{+\infty} p_k(y) q_\nu(y, t) dy \\ &= \frac{1}{k!} \int_0^{+\infty} y^k e^{-y} \left[n_1 I^\nu(\overline{\overline{p}}_\nu(\cdot; y))(t) + n_2 \overline{\overline{p}}_\nu(t; y) \right] dy. \end{aligned} \quad (2.71)$$

Here $\overline{\overline{p}}_\nu(\cdot; y)$ denotes the stable law of the random variable X_ν of index $\nu \in (0, 1)$ and parameters equal to $\beta = 1, \mu = n_2|y|/\lambda$ and $\sigma = \left(\frac{n_1}{\lambda}|y|\cos\frac{\pi\nu}{2}\right)^{1/\nu}$.

Proof We observe that (2.70) can be rewritten as follows

$$\mathcal{L}\{\bar{p}_k^\nu; \eta\} = \left(\frac{n_1}{\lambda}\eta^{\nu-1} + \frac{n_2}{\lambda}\right)\mathcal{L}\left\{p_k; \frac{n_1}{\lambda}\eta^\nu + \frac{n_2}{\lambda}\eta\right\},$$

so that we get, by an argument analogous to Theorem 2.2,

$$\begin{aligned}\bar{p}_k^\nu(t) &= \frac{n_1}{\lambda\Gamma(1-\nu)} \int_0^t (t-w)^{-\nu} \left(\int_0^{+\infty} p_k(y)g_\nu(w; y)dy \right) dw + \\ &\quad + \frac{n_2}{\lambda} \int_0^{+\infty} p_k(y)g_\nu(t; y)dy\end{aligned}\tag{2.72}$$

The density in (2.72) can be expressed as follows

$$\begin{aligned}g_\nu(w; y) &= \mathcal{L}^{-1}\left\{e^{-(\frac{n_1}{\lambda}\eta^\nu + \frac{n_2}{\lambda}\eta)|y|}; w\right\} \\ &= \int_0^w \bar{p}_\nu(w-x; y)\delta\left(x - \frac{n_2|y|}{\lambda}\right)dx \\ &= \bar{\bar{p}}_\nu(w; y);\end{aligned}$$

hence

$$\begin{aligned}\bar{p}_k^\nu(t) &= \int_0^{+\infty} p_k(y) \left[\frac{n_1}{\lambda\Gamma(1-\nu)} \int_0^t (t-w)^{-\nu} \bar{\bar{p}}_\nu(w; y)dw + \frac{n_2}{\lambda} \bar{\bar{p}}_\nu(w; y) \right] dy \\ &= \int_0^{+\infty} p_k(y) \left[\frac{n_1}{\lambda} I^\nu(\bar{\bar{p}}_\nu(w; y)) + \frac{n_2}{\lambda} \bar{\bar{p}}_\nu(w; y) \right] dy,\end{aligned}$$

which coincides with (2.71). \square

Remark 2.8 For $n_2 = 0$ and $n_1 = 1$, we get that $\bar{\bar{p}}_\nu(t; y) = \bar{p}_\nu(t; y)$. Therefore formula (2.71) reduces to

$$\begin{aligned}\bar{p}_k^\nu(t) &= \frac{1}{k!} \int_0^{+\infty} y^k e^{-y} I^\nu(\bar{p}_\nu(\cdot; y))(t) dy \\ &= \frac{1}{k!} \int_0^{+\infty} y^k e^{-y} \bar{v}_{2\nu}(\cdot; y)(t) dy\end{aligned}\tag{2.73}$$

as for the single-order fractional equation (see (2.31) and (2.16)-(2.17)). On the other hand, for $n_1 = 0$ and $n_2 = 1$, it is $\bar{\bar{p}}_\nu(t; y) = \delta(y)$ and $\bar{p}_k^\nu(t) = p_k(t)$, since, in this case, equation (2.68) reduces to the equation governing the Poisson distribution.

Remark 2.9 A particular feature in this section is that for the process governed by (2.68) the probability generating function \bar{G}_ν , as well as the probability of zero events \bar{p}_0^ν and the interarrival time density $\bar{f}_1^\nu(t)$, can be expressed as infinite sums of the Kummer confluent hypergeometric function ${}_1F_1(\alpha, \beta; x)$. The latter is defined as

$${}_1F_1(\alpha; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} \frac{z^j}{j!}, \quad z, \alpha \in \mathbb{C}, \quad \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{2.74}$$

where $(\gamma)_r = \gamma(\gamma+1)\dots(\gamma+r-1)$ (for $r = 1, 2, \dots$, and $\gamma \neq 0$) and $(\gamma)_0 = 1$, or, in integral form, as

$${}_1F_1(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} e^{zt} dt, \quad 0 < \Re(\alpha) < \Re(\gamma), \tag{2.75}$$

(see [9], p.1085). Indeed it is well-known the following relationship between the GML function with first parameter equal to one and ${}_1F_1(\alpha, \gamma; x)$:

$$E_{1,\gamma}^\alpha(z) = \frac{1}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; z)$$

(see [11], p.62).

By specializing result (2.33), the probability generating function $\widetilde{G}_v(u, t)$ is equal to

$$\begin{aligned} \widetilde{G}_v(u, t) &= \sum_{r=0}^{\infty} \frac{\left(-\frac{n_1 t^{1-v}}{n_2}\right)^r}{\Gamma(r(1-v)+1)} {}_1F_1\left(r+1; r(1-v)+1; -\frac{\lambda(1-u)t}{n_2}\right) + \\ &- \sum_{r=0}^{\infty} \frac{\left(-\frac{n_1 t^{1-v}}{n_2}\right)^{r+1}}{\Gamma((r+1)(1-v)+1)} {}_1F_1\left(r+1; (r+1)(1-v)+1; -\frac{\lambda(1-u)t}{n_2}\right). \end{aligned}$$

Analogously, from (2.47) the interarrival time density reads, in this case,

$$\widetilde{f}_1^v(t) = \frac{\lambda}{n_2} \sum_{r=0}^{\infty} \frac{\left(-\frac{n_1 t^{1-v}}{n_2}\right)^r}{\Gamma(r-vr+1)} {}_1F_1\left(r+1; r-vr+1; -\frac{\lambda t}{n_2}\right). \quad (2.76)$$

Remark 2.10 The expected value of the renewal process $\widetilde{N}_v(t)$, $t > 0$ with distribution (2.71) is given by

$$\mathbb{E}\widetilde{N}_v(t) = \frac{\lambda t}{n_2} E_{1-v,2}\left(-\frac{n_1 t^{1-v_1}}{n_2}\right), \quad (2.77)$$

so that we get this asymptotic behavior

$$\mathbb{E}\widetilde{N}_v(t) \simeq \frac{\lambda t^v}{n_1 \Gamma(1+v)}, \quad t \rightarrow \infty.$$

This expression coincides with (2.45), for $n_1 = 1$: the mean value is not influenced by the presence of the first derivative. On the contrary, for $t \rightarrow 0$, we obtain from (2.77) that

$$\mathbb{E}\widetilde{N}_v(t) \simeq \frac{\lambda t}{n_2}, \quad t \rightarrow 0,$$

i.e. the usual expected value of the Poisson process. Therefore the first derivative dominates equation (2.68) asymptotically, as $t \rightarrow 0$.

3 Diffusion equations of distributed order

We study equation (1.13) in the double-order hypothesis (2.5), i.e.

$$n_1 \frac{\partial^{v_1} v}{\partial t^{v_1}} + n_2 \frac{\partial^{v_2} v}{\partial t^{v_2}} = \frac{\partial^2 v}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \quad v(x, 0) = \delta(x), \quad n_1, n_2 > 0, \quad (3.1)$$

for $0 < v_1 < v_2 \leq 1$. Equation (3.1) can be viewed also as the particular case (for $\gamma = 2$) of equation (2.35) analyzed in [26]. In that paper only the Fourier transform of the solution is given in explicit form, in terms of infinite sums of GML functions. Our aim is to give an explicit form of the solution, by using an approach similar to the previous section and providing an expression of the density of the random time in the subordinating relationship (1.14). This turns out to coincide with the density of the random time $\widetilde{T}_{v_1, v_2}(t)$, i.e. with q_{v_1, v_2} given in (2.18) or (2.23).

Theorem 3.1 *The solution to equation (3.1), is given by*

$$\begin{aligned} \widetilde{v}_{v_1, v_2}(x, t) &= \int_0^{+\infty} f(x, y) q_{v_1, v_2}(y, t) dy \\ &= \int_0^{+\infty} \frac{e^{-x^2/4y}}{\sqrt{4\pi y}} \left[\int_0^t \overline{p}_{v_2}(t-s; y) \overline{v}_{2v_1}(y, s) ds + \int_0^t \overline{p}_{v_1}(t-s; y) \overline{v}_{2v_2}(y, s) ds \right] dy, \end{aligned} \quad (3.2)$$

where f is the transition density of a Brownian motion $B(t)$, $t > 0$, \overline{v}_{2v_j} is given in (2.16)-(2.17) and $\overline{p}_{v_j}(\cdot; y)$ denotes the stable law of the random variable X_{v_j} of index $v_j \in (0, 1)$ with parameters $\beta = 1$,

$\mu = 0$ and $\sigma = \left(n_j|y| \cos \frac{\pi \nu_j}{2}\right)^{1/\nu_j}$, for $j = 1, 2$. Alternatively the density q_{ν_1, ν_2} in (3.2) can be written as in (2.23).

Proof We take the Fourier transform of (3.1), so that we get

$$\begin{cases} n_1 \frac{\partial^{v_1} \tilde{V}}{\partial r^{v_1}} + n_2 \frac{\partial^{v_2} \tilde{V}}{\partial r^{v_2}} = -\theta^2 \tilde{V} \\ \tilde{V}(\theta, 0) = 1 \end{cases} \quad (3.3)$$

where

$$\tilde{V}_{\nu_1, \nu_2}(\theta, t) = \mathcal{F} \{ \tilde{v}_{\nu_1, \nu_2}; \theta \} = \int_{-\infty}^{+\infty} e^{i\theta x} \tilde{v}_{\nu_1, \nu_2}(x, t) dx.$$

Taking now the Laplace transform of (3.3) we get

$$\mathcal{L} \{ \tilde{V}_{\nu_1, \nu_2}; \theta, \eta \} = \frac{n_1 \eta^{\nu_1-1} + n_2 \eta^{\nu_2-1}}{n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2} + \theta^2}. \quad (3.4)$$

We can invert the Laplace transform, by noting that it coincides with (2.11) for $\lambda = \theta^2$, as follows:

$$\begin{aligned} & \tilde{V}_{\nu_1, \nu_2}(\theta, t) \\ &= \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^r E_{\nu_2, (\nu_2-\nu_1)r+1}^{r+1} \left(-\frac{\theta^2 t^{\nu_2}}{n_2} \right) - \sum_{r=0}^{\infty} \left(-\frac{n_1 t^{\nu_2-\nu_1}}{n_2} \right)^{r+1} E_{\nu_2, (\nu_2-\nu_1)(r+1)+1}^{r+1} \left(-\frac{\theta^2 t^{\nu_2}}{n_2} \right), \end{aligned} \quad (3.5)$$

thus obtaining a first form for the solution to (3.3). Since inverting the Fourier transform (3.5) seems not possible in closed form, we rewrite (3.4) as follows:

$$\mathcal{L} \{ \tilde{V}_{\nu_1, \nu_2}; \theta, \eta \} = (n_1 \eta^{\nu_1-1} + n_2 \eta^{\nu_2-1}) \frac{1}{2} \int_0^{+\infty} e^{-n_1 w \eta^{\nu_1} + n_2 w \eta^{\nu_2} - \theta^2 w} dw.$$

We note that the term $e^{-(n_1 \eta^{\nu_1} + n_2 \eta^{\nu_2})w}$ can be seen again as the convolution of two stable laws \bar{p}_{ν_j} of index $\nu_j \in (0, 1)$ and parameters equal to $\beta = 1$, $\mu = 0$ and $\sigma = \left(\frac{1}{2} n_j |w| \cos \frac{\pi \nu_j}{2}\right)^{1/\nu_j}$ for $j = 1, 2$ (see (2.14)). Therefore we get, alternatively to (3.5)

$$\begin{aligned} & \tilde{V}_{\nu_1, \nu_2}(\theta, t) \\ &= \frac{n_1}{\Gamma(1-\nu_1)} \int_0^t (t-z)^{-\nu_1} dz \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^z \bar{p}_{\nu_1}(z-x; w) \bar{p}_{\nu_2}(x; w) dx \right] dw + \\ & \quad + \frac{n_2}{\Gamma(1-\nu_2)} \int_0^t (t-z)^{-\nu_2} dz \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^z \bar{p}_{\nu_1}(z-x; w) \bar{p}_{\nu_2}(x; w) dx \right] dw \\ &= n_1 \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^t I^{\nu_1} \{ \bar{p}_{\nu_1}(\cdot; w) \} (x) \bar{p}_{\nu_2}(t-x; w) dx \right] dw + \\ & \quad + n_2 \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^t I^{\nu_2} \{ \bar{p}_{\nu_2}(\cdot; w) \} (x) \bar{p}_{\nu_1}(t-x; w) dx \right] dw \\ &= \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^t \bar{v}_{2\nu_1}(x, w) \bar{p}_{\nu_2}(t-x; w) dx \right] dw + \\ & \quad + \int_0^{+\infty} e^{-\theta^2 w} \left[\int_0^t \bar{v}_{2\nu_2}(x, w) \bar{p}_{\nu_1}(t-x; w) dx \right] dw, \end{aligned} \quad (3.6)$$

where again $\bar{v}_{2\nu_j}$ is the solution to equation (2.15) with $c^2 = 1/n_j$, $j = 1, 2$. Finally, we recognize in (3.6) the Fourier transform of the Gaussian density, with variance $2|w|$, so that we can write the subordination relationship (3.2). \square

The previous theorem shows that the solution to the double-order equation (3.1) can be seen as the density of the random-time process

$$\tilde{\mathcal{B}}_{\nu_1, \nu_2}(t) \equiv B(\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)), \quad t > 0, \quad (3.7)$$

where B is a Brownian motion (with infinitesimal variance equal to 2) and $\widetilde{\mathcal{T}}_{v_1, v_2}$ is the random time, independent from B , with density q_{v_1, v_2} given in (3.2) or, alternatively, in (2.23), for $\lambda = 1$. By using the results obtained in Theorem 2.5, we can evaluate the moments of $\widetilde{\mathcal{B}}_{v_1, v_2}$, as follows:

Theorem 3.2 *The k -th order moments of the process $\widetilde{\mathcal{B}}_{v_1, v_2}$ are given by*

$$\begin{aligned} & \mathbb{E} \widetilde{\mathcal{B}}_{v_1, v_2}^k(t) \\ &= \begin{cases} 0 & \text{for } k = 2h + 1 \\ \frac{n_1 t^{v_2 h + v_2 - v_1}}{n_2^{h+1}} (2h)! E_{v_2 - v_1, v_2 h + v_2 - v_1 + 1}^{h+1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right) + \frac{t^{v_2 h}}{n_2^h} (2h)! E_{v_2 - v_1, v_2 h + 1}^{h+1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right) & \text{for } k = 2h \end{cases} \end{aligned}$$

Proof By the definition (3.7) we can write

$$\mathbb{E} \widetilde{\mathcal{B}}_{v_1, v_2}^k(t) = \mathbb{E} \left[B(\widetilde{\mathcal{T}}_{v_1, v_2}(t)) \right]^k = \int_{-\infty}^{+\infty} x^k \int_0^{+\infty} f(x, y) q_{v_1, v_2}(y, t) dy dx.$$

The odd order moments of $\widetilde{\mathcal{B}}_{v_1, v_2}$ are obviously null, while the moments of order $2h$, for $h \in \mathbb{N}$, can be evaluated as follows:

$$\begin{aligned} & \mathbb{E} \widetilde{\mathcal{B}}_{v_1, v_2}^{2h}(t) \\ &= \int_0^{+\infty} q_{v_1, v_2}(y, t) \int_{-\infty}^{+\infty} x^{2h} \frac{e^{-x^2/4y}}{\sqrt{4\pi y}} dx dy \\ &= \left[x = \sqrt{4yw} \right] \\ &= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} q_{v_1, v_2}(y, t) \int_0^{+\infty} (4yw)^h \frac{e^{-w}}{\sqrt{w}} dw dy \\ &= \frac{2^{2h}}{\sqrt{\pi}} \Gamma\left(h + \frac{1}{2}\right) \int_0^{+\infty} y^h q_{v_1, v_2}(y, t) dy \\ &= 2 \frac{(2h-1)!}{(h-1)!} \int_0^{+\infty} y^h q_{v_1, v_2}(y, t) dy \\ &= \frac{n_1 t^{v_2 h + v_2 - v_1}}{n_2^{h+1}} (2h)! E_{v_2 - v_1, v_2 h + v_2 - v_1 + 1}^{h+1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right) + \frac{t^{v_2 h}}{n_2^h} (2h)! E_{v_2 - v_1, v_2 h + 1}^{h+1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right), \end{aligned}$$

where, in the last step, we have applied formula (2.39) and the relationship

$$\mathbb{E} \left[\widetilde{\mathcal{T}}_{v_1, v_2}(t) \right]^k = \mathbb{E} \left[\widetilde{\mathcal{N}}_{v_1, v_2}(t) (\widetilde{\mathcal{N}}_{v_1, v_2}(t) - 1) \dots (\widetilde{\mathcal{N}}_{v_1, v_2}(t) - k + 1) \right], \quad k \in \mathbb{N} \quad (3.8)$$

proved in Theorem 2.5. \square

Remark 3.1 We can check the previous result by noting that, for $h = 1$, we get the second-order moment obtained in [6] (see formula (16), for $\tau = D = 1$):

$$\begin{aligned} & \mathbb{E} \widetilde{\mathcal{B}}_{v_1, v_2}^2(t) \quad (3.9) \\ &= \frac{2n_1 t^{2v_2 - v_1}}{n_2^2} \sum_{j=0}^{\infty} \frac{(j+1)! \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j}{j! \Gamma((v_2 - v_1)j + 2v_2 - v_1 + 1)} + \frac{2t^{v_2 - v_1}}{n_2} \sum_{j=0}^{\infty} \frac{(j+1)! \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j}{j! \Gamma((v_2 - v_1)j + v_2 + 1)} \\ &= -\frac{2t^{v_2}}{n_2} \sum_{l=0}^{\infty} \frac{l \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^l}{\Gamma((v_2 - v_1)l + v_2 + 1)} + \frac{2t^{v_2}}{n_2} \sum_{j=0}^{\infty} \frac{(j+1) \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right)^j}{\Gamma((v_2 - v_1)j + v_2 + 1)} \\ &= \frac{2t^{v_2}}{n_2} E_{v_2 - v_1, v_2 + 1} \left(-\frac{n_1 t^{v_2 - v_1}}{n_2} \right). \end{aligned}$$

Our attention is now addressed to the equation solved by the density q_{v_1, v_2} of the time argument $\widetilde{\mathcal{T}}_{v_1, v_2}$ (which is shared by the processes $\widetilde{\mathcal{N}}_{v_1, v_2}$ and $\widetilde{\mathcal{B}}_{v_1, v_2}$). In analogy with the single-order fractional case this equation must be of “second-order” (involving the two fractional indexes v_1, v_2), but is not evidently given by (3.1). We prove in the next theorem that a further time-fractional derivative must be included in the diffusion equation (3.1) in order to obtain q_{v_1, v_2} as solution.

Theorem 3.3 *The density $q_{v_1, v_2}(x, t)$ coincides with the folded solution*

$$\bar{v}(x, t) = \begin{cases} 2v(x, t), & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (3.10)$$

of the following equation

$$\left(n_1 \frac{\partial^{v_1} v}{\partial t^{v_1}} + n_2 \frac{\partial^{v_2} v}{\partial t^{v_2}} \right)^2 = \frac{\partial^2 v}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \quad n_1, n_2 > 0, \quad (3.11)$$

for $0 < v_1 < v_2 \leq 1$, with initial conditions

$$\begin{cases} v(x, 0) = \delta(x), \text{ for } 0 < v_1 < v_2 \leq 1 \\ \frac{\partial}{\partial t} v(x, t) \big|_{t=0} = 0 \text{ for } \frac{1}{2} < v_1 < v_2 \leq 1 \end{cases} \quad (3.12)$$

Proof Let us take the Fourier transform of (2.42), which reads

$$\begin{aligned} \mathcal{L}\{Q_{v_1, v_2}; \theta, \eta\} &\equiv \int_{-\infty}^{\infty} e^{i\theta y} \mathcal{L}\{q_{v_1, v_2}; \eta\} dy \\ &= (n_1 \eta^{v_1-1} + n_2 \eta^{v_2-1}) \int_{-\infty}^{\infty} e^{i\theta y} e^{-(n_1 \eta^{v_1} + n_2 \eta^{v_2})|y|} dy \\ &= (n_1 \eta^{v_1-1} + n_2 \eta^{v_2-1}) \left[\frac{1}{n_1 \eta^{v_1} + n_2 \eta^{v_2} - i\theta} + \frac{1}{n_1 \eta^{v_1} + n_2 \eta^{v_2} + i\theta} \right] \\ &= \frac{2(n_1 \eta^{v_1} + n_2 \eta^{v_2})^2}{\eta(n_1^2 \eta^{2v_1} + n_2^2 \eta^{2v_2} + 2n_1 n_2 \eta^{v_1+v_2} + \theta^2)}, \end{aligned} \quad (3.13)$$

by taking, for simplicity, $\lambda = 1$. We take now the Laplace-Fourier transform of equation (3.11), by considering formula (2.8) together with the initial conditions (3.12):

$$\begin{aligned} n_1^2 \eta^{2v_1} \mathcal{L}\{Q_{v_1, v_2}; \theta, \eta\} - n_1^2 \eta^{2v_1-1} + n_2^2 \eta^{2v_2} \mathcal{L}\{Q_{v_1, v_2}; \theta, \eta\} - n_2^2 \eta^{2v_2-1} + \\ + 2n_1 n_2 \eta^{v_1+v_2} \mathcal{L}\{Q_{v_1, v_2}; \theta, \eta\} - 2n_1 n_2 \eta^{v_1+v_2-1} = -\theta^2 \mathcal{L}\{Q_{v_1, v_2}; \theta, \eta\}, \end{aligned}$$

whose solution coincides with (3.13), by taking into account (3.10). \square

Remark 3.2 Equation (3.11) can be itself interpreted as a distributed order fractional equation, by assuming a different density for the random fractional index v , which, in this case, is defined on the interval $(0, 2]$: indeed we can formulate $n(v)$, as follows:

$$n(v) = n_1^2 \delta(v - 2v_1) + n_2^2 \delta(v - 2v_2) + 2n_1 n_2 \delta(v - (v_1 + v_2)), \quad 0 < v_1 < v_2 \leq 1, \quad (3.14)$$

for $n_1, n_2 \geq 0$ and such that $n_1 + n_2 = 1$. The last condition is enough to fulfill the normalization requirement $n_1^2 + n_2^2 + 2n_1 n_2 = 1$.

By considering (2.39) together with (3.8) we can obtain the second-order moment of the diffusion process $\widetilde{\mathcal{T}}_{v_1, v_2}$ governed by equation (3.11), i.e.

$$\begin{aligned}
\mathbb{E}[\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)]^2 &= \frac{2n_1 t^{3\nu_2 - \nu_1}}{n_2^3} E_{\nu_2 - \nu_1, 3\nu_2 - \nu_1 + 1}^3 \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right) + \frac{2t^{2\nu_2}}{n_2^2} E_{\nu_2 - \nu_1, 2\nu_2 + 1}^3 \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right) \quad (3.15) \\
&= \frac{n_1 t^{3\nu_2 - \nu_1}}{n_2^3} \sum_{j=0}^{\infty} \frac{(j+2)(j+1)}{\Gamma((\nu_2 - \nu_1)j + 3\nu_2 - \nu_1 + 1)} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^j + \\
&\quad + \frac{t^{2\nu_2}}{n_2^2} \sum_{j=0}^{\infty} \frac{(j+2)(j+1)}{\Gamma((\nu_2 - \nu_1)j + 2\nu_2 + 1)} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^j \\
&= [l = j + 1] \\
&= -\frac{t^{2\nu_2}}{n_2^2} \sum_{l=1}^{\infty} \frac{(l+1)l}{\Gamma((\nu_2 - \nu_1)l + 2\nu_2 + 1)} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^l + \frac{t^{2\nu_2}}{n_2^2} \sum_{l=0}^{\infty} \frac{(l+2)(l+1)}{\Gamma((\nu_2 - \nu_1)l + 2\nu_2 + 1)} \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right)^l \\
&= \frac{2t^{2\nu_2}}{n_2^2} E_{\nu_2 - \nu_1, 2\nu_2 + 1}^2 \left(-\frac{n_1 t^{\nu_2 - \nu_1}}{n_2} \right).
\end{aligned}$$

We compare (3.15) with the second-order moment of the diffusion process $\tilde{\mathcal{B}}_{\nu_1, \nu_2}$ governed by equation (3.1) (which is given in (3.9)), noting that, apart from the similar structure, the second-order moment of $\tilde{\mathcal{T}}_{\nu_1, \nu_2}$ is expressed in terms of a GML function, instead of a standard one. For $t \rightarrow \infty$ the different asymptotic behavior is described below, by applying (2.59) and (2.60):

$$\begin{aligned}
\mathbb{E}[\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)]^2 &\simeq \frac{2t^{2\nu_1}}{n_1^2 \Gamma(1 + 2\nu_1)} \quad (3.16) \\
\mathbb{E}\tilde{\mathcal{B}}_{\nu_1, \nu_2}^2(t) &\simeq \frac{2t^{\nu_1}}{n_1 \Gamma(1 + \nu_1)}
\end{aligned}$$

For $t \rightarrow 0$, since the behavior of the GML and the standard Mittag-Leffler function coincides, we must apply, in both cases, formula (2.61). Hence we get:

$$\begin{aligned}
\mathbb{E}[\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)]^2 &\simeq \frac{2t^{2\nu_2}}{n_2^2 \Gamma(1 + 2\nu_2)} \quad (3.17) \\
\mathbb{E}\tilde{\mathcal{B}}_{\nu_1, \nu_2}^2(t) &\simeq \frac{2t^{\nu_2}}{n_2 \Gamma(1 + \nu_2)}.
\end{aligned}$$

We can conclude from (3.16) and (3.17) that, in the case where $\nu_1, \nu_2 < 1/2$, the effect of diffusion with retardation, which is characteristic of $\tilde{\mathcal{B}}_{\nu_1, \nu_2}$ (see also [6]), is emphasized for the process $\tilde{\mathcal{T}}_{\nu_1, \nu_2}$, since the difference between $2\nu_2$ and $2\nu_1$ is greater than for ν_2 and ν_1 .

A different conclusion should be drawn in the case where either ν_2 or both ν_1, ν_2 are greater than $1/2$. For $\nu_1 < 1/2$ and $\nu_2 > 1/2$, we can observe that the asymptotic behavior of $\mathbb{E}[\tilde{\mathcal{T}}_{\nu_1, \nu_2}(t)]^2$ drastically changes, at least for $t \rightarrow 0$: indeed in this case it goes to zero faster than t . For $\nu_1, \nu_2 > 1/2$, in addition to this effect, we note that, also for $t \rightarrow \infty$, the rate convergence is greater than in the standard diffusion case (besides that of diffusion with retardation). Therefore the process governed by (3.11), for $\nu_1, \nu_2 > 1/2$, can be interpreted as a diffusion with acceleration.

4 Concluding remarks

For the reader's convenience, we sum up in the following tables the most important results obtained in this paper, by highlighting the connections between the processes and the corresponding equations involved.

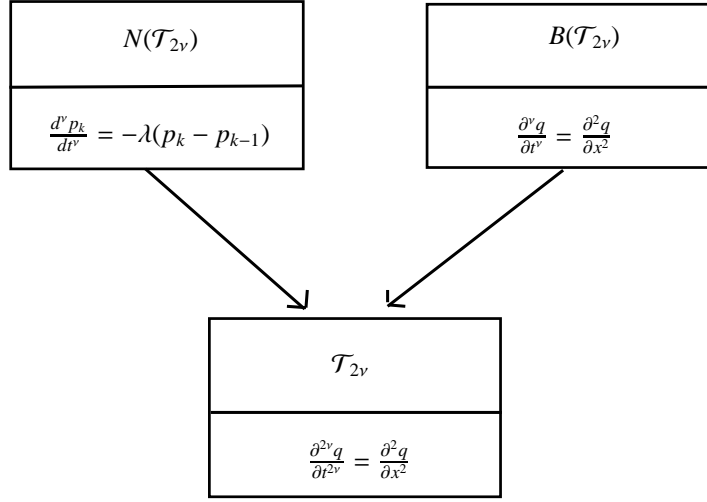


Figure 1: Single order fractional equations

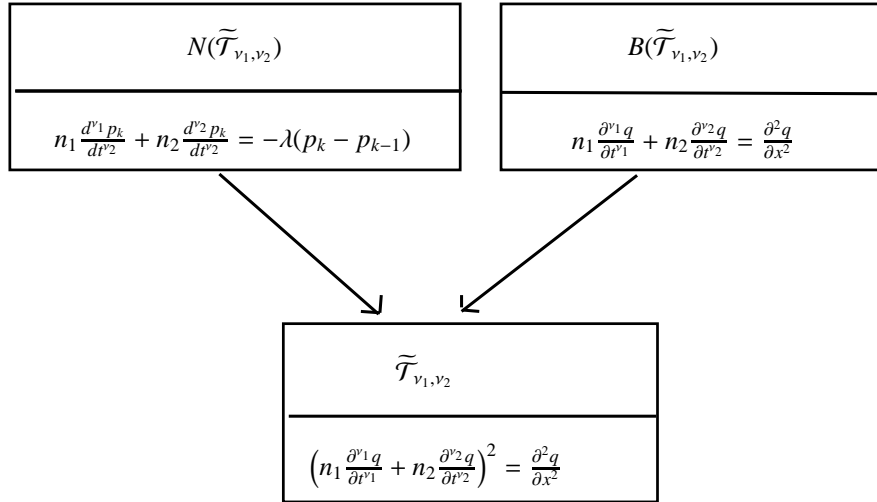


Figure 2: Double order fractional equations

$\widetilde{\Psi}_{\nu_1, \nu_2}(t)$
$n_1 \frac{\partial^{\nu_1} \Psi}{\partial t^{\nu_1}} + n_2 \frac{\partial^{\nu_2} \Psi}{\partial t^{\nu_2}} = -\lambda \Psi$

Figure 3: Fractional relaxation equation

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