The Space-Fractional Poisson Process

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Abstract

In this paper we introduce the space-fractional Poisson process whose state probabilities $p_k^\alpha(t)$, $t>0$, $\alpha\in(0,1]$, are governed by the equations $(\mathrm{d}/\mathrm{d}t)p_k(t) = -\lambda^\alpha(1-B)p_k^\alpha(t)$, where $(1-B)^\alpha$ is the fractional difference operator found in the study of time series analysis. We explicitly obtain the distributions $p_k^\alpha(t)$, the probability generating functions $G_\alpha(u,t)$, which are also expressed as distributions of the minimum of i.i.d. uniform random variables. The comparison with the time-fractional Poisson process is investigated and finally, we arrive at the more general space-time fractional Poisson process of which we give the explicit distribution.

Keywords: Space-fractional Poisson process; Backward shift operator; Discrete stable distributions; Stable subordinator; Space-time fractional Poisson process.

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1 Introduction

Fractional Poisson processes studied so far have been obtained either by considering renewal processes with intertimes between events represented by Mittag–Leffler distributions [Mainardi et al., 2004, Beghin and Orsingher, 2009] or by replacing the time derivative in the equations governing the state probabilities with the fractional derivative in the sense of Caputo.
In this paper we introduce a space-fractional Poisson process by means of the fractional difference operator

$$\Delta^\alpha = (1-B)^\alpha, \qquad \alpha\in(0,1], \tag{1.1}$$

which often appears in the study of long memory time series [Tsay, 2005].
The operator (1.1) implies a dependence of the state probabilities $p_k^\alpha(t)$ from all probabilities $p_j^\alpha(t)$, $j<k$. For $\alpha=1$ we recover the classical homogeneous Poisson process and the state probabilities $p_k(t)$ depend only on $p_{k-1}(t)$.
For the space-fractional Poisson process we obtain the following distribution:

$$p_k^\alpha(t) = \Pr\{N^\alpha(t)=k\} = \frac{(-1)^k}{k!}\sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-k)}, \qquad k\ge 0. \tag{1.2}$$

The distribution of the space-fractional Poisson process can be compared with that of the time-fractional Poisson process $N_\nu(t)$, $t > 0$, $\nu \in (0,1]$:

$$\Pr\{N_\nu(t) = k\} = \frac{(\lambda t^\nu)^k}{k!} \sum_{r=0}^{\infty} \frac{(r+k)!}{r!} \frac{(-\lambda t^\nu)^r}{\Gamma(\nu(k+r)+1)}, \qquad k \geq 0. \tag{1.3}$$

For $\alpha = \nu = 1$, from (1.2) and (1.3), we immediately arrive at the classical distribution of the homogeneous Poisson process.
The space-fractional Poisson process has probability generating function

$$G_\alpha(u,t) = \mathbb{E}u^{N^\alpha(t)} = e^{-\lambda^\alpha (1-u)^\alpha t}, \qquad |u| \leq 1, \tag{1.4}$$

and can be compared with its time-fractional counterpart

$$_\nu G(u,t) = \mathbb{E}u^{N_\nu(t)} = E_\nu\left(-\lambda(1-u)t^\nu\right), \qquad |u| \leq 1, \tag{1.5}$$

where

$$E_\nu(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\nu r+1)}, \qquad \nu > 0, \tag{1.6}$$

is the one-parameter Mittag–Leffler function.
We show below that the probability generating function of the space-time fractional Poisson process reads

$$_\nu G_\alpha(u,t) = E_\nu\left(-\lambda^\alpha (1-u)^\alpha t^\nu\right), \qquad |u| \leq 1, \tag{1.7}$$

and its distribution has the form

$$p_k^{\alpha,\nu}(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r+1)} \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-k)}, \qquad k \geq 0,\ \alpha \in (0,1],\ \nu \in (0,1]. \tag{1.8}$$

We also show that the space-fractional Poisson process $N^\alpha(t)$ can be regarded as a homogeneous Poisson process $N(t)$, subordinated to a positively skewed stable process $S^\alpha(t)$ with Laplace transform

$$\mathbb{E}e^{-zS^\alpha(t)} = e^{-tz^\alpha}, \qquad z > 0,\ t > 0. \tag{1.9}$$

In other words, we have the following equality in distribution

$$N^\alpha(t) \overset{\mathrm{d}}{=} N\left(S^\alpha(t)\right). \tag{1.10}$$

The representation (1.10) is similar to the following representation of the time-fractional Poisson process:

$$N_\nu(t) \overset{\mathrm{d}}{=} N\left(T_{2\nu}(t)\right), \tag{1.11}$$

where $T_{2\nu}(t)$, $t > 0$, is a process whose one-dimensional distribution is obtained by folding the solution to the time-fractional diffusion equation [Beghin and Orsingher, 2009].
Finally we can note that the probability generating function (1.7), for all $u \in (0,1)$, can be represented as

$$_\nu G_\alpha(u,t) = \Pr\left\{ \min_{0 \leq k \leq N_\nu(t)} X_k^{1/\alpha} \geq 1-u \right\}, \tag{1.12}$$

where the X_ks are i.i.d. uniformly distributed random variables.

2 Construction of the space-fractional Poisson process

In this section we describe the construction of an alternative fractional generalisation of the classical homogeneous Poisson process. First, let us recall some basic properties. Let us call

$$P_k(t) = \Pr\{N(t) = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad t > 0,\ \lambda > 0,\ k \ge 0, \tag{2.1}$$

the state probabilities of the classical homogeneous Poisson process $N(t)$, $t > 0$, of parameter $\lambda > 0$. It is well-known that the probabilities $p_k(t)$, $k \ge 0$, solve the Cauchy problems

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \\ p_k(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0. \end{cases} \end{cases} \tag{2.2}$$

Starting from (2.2), some time-fractional generalisations of the homogeneous Poisson process have been introduced in the literature (see e.g. Laskin [2003], Mainardi et al. [2004], Beghin and Orsingher [2009]). These works are based on the substitution of the integer-order derivative operator appearing in (2.2) with a fractional-order derivative operator, such as the Riemann–Liouville fractional derivative (as in Laskin [2003]) or the Caputo fractional derivative (as in Beghin and Orsingher [2009]). In this paper instead, we generalise the integer-order space-difference operator as follows. First, we rewrite equation (2.2) as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = -\lambda(1-B) p_k(t), \\ p_k(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0, \end{cases} \end{cases} \tag{2.3}$$

where B is the so-called *backward shift operator* and is such that $B(p_k(t)) = p_{k-1}(t)$ and $B^r(p_k(t)) = B^{r-1}(B(p_k(t))) = p_{k-r}(t)$. The fractional difference operator $\Delta^\alpha = (1-B)^\alpha$ has been widely used in time series analysis for constructing processes displaying long memory, such as the autoregressive fractionally integrated moving average process (ARFIMA). For more information on long memory processes and fractional differentiation, the reader can consult Tsay [2005], page 89.
Formula (2.3) can now be easily generalised by writing

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k^\alpha(t) = -\lambda^\alpha (1-B)^\alpha p_k^\alpha(t), \quad \alpha \in (0,1], \\ p_k^\alpha(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0, \end{cases} \end{cases} \tag{2.4}$$

where $p_k^\alpha(t)$, $k \ge 0$, $t > 0$, represents the state probabilities of a space-fractional homogeneous Poisson process $N^\alpha(t)$, $t > 0$, i.e.

$$p_k^\alpha(t) = \Pr\{N^\alpha(t) = k\}, \qquad k \ge 0. \tag{2.5}$$

In turn, we have that (2.4) can also be written as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k^\alpha(t) = -\lambda^\alpha \sum_{r=0}^{k} \frac{\Gamma(\alpha+1)}{r!\Gamma(\alpha+1-r)} (-1)^r p_{k-r}^\alpha(t), \quad \alpha \in (0,1], \\ p_k^\alpha(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0. \end{cases} \end{cases} \tag{2.6}$$

Note that, in (2.6) we considered that $p_j^\alpha(t) = 0$, $j \in \mathbb{Z}^-$. Equation (2.6) can also be written as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k^\alpha(t) = -\lambda^\alpha p_k^\alpha(t) + \alpha\lambda^\alpha p_{k-1}^\alpha(t) - \frac{\alpha(\alpha-1)}{2!} p_{k-2}^\alpha(t) + \cdots + (-1)^{k+1} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} p_0^\alpha(t), \\ p_k^\alpha(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0. \end{cases} \end{cases} \tag{2.7}$$

By applying the reflection property of the gamma function $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for $z = r - \alpha$, we have also that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} p_k^\alpha(t) = -\lambda^\alpha p_k^\alpha(t) + \alpha \lambda^\alpha p_{k-1}^\alpha(t) + \frac{\lambda^\alpha \sin(\pi\alpha)}{\pi} \sum_{r=2}^k B(\alpha+1, r-\alpha) p_{k-r}^\alpha(t), \\ p_k^\alpha(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0, \end{cases} \end{cases} \tag{2.8}$$

where the sum is considered equal to zero when $k < 2$ and $B(x, y)$ is the beta function. From (2.7) and (2.8) we see that for $\alpha = 1$ we retrieve equation (2.6) of the homogeneous Poisson process.
An example of process whose state probabilities $\tilde{p}_k(t)$, depend on all $\tilde{p}_j(t)$, $j < k$, is the iterated Poisson process $\tilde{N}(t) = N_1(N_2(t))$, where $N_1(t)$ and $N_2(t)$ are independent homogeneous Poisson processes. The process $\tilde{N}(t)$ is analysed in Orsingher and Polito [2011].

Theorem 2.1. *Let $N^\alpha(t)$, $t > 0$, be a space-fractional homogeneous Poisson process of parameter $\lambda > 0$ and let $G_\alpha(u, t) = \mathbb{E} u^{N^\alpha(t)}$, $|u| \le 1$, $\alpha \in (0, 1]$, be its probability generating function. The Cauchy problem satisfied by $G_\alpha(u, t)$ is*

$$\begin{cases} \frac{\partial}{\partial t} G_\alpha(u, t) = -\lambda^\alpha G_\alpha(u, t)(1-u)^\alpha, & |u| \le 1, \\ G_\alpha(u, 0) = 1, \end{cases} \tag{2.9}$$

with solution

$$G_\alpha(u, t) = e^{-\lambda^\alpha t (1-u)^\alpha}, \qquad |u| \le 1, \tag{2.10}$$

that is, the probability generating function of a discrete stable distribution (see Devroye [1993], page 349).

Proof. Starting from (2.6), we have that

$$\begin{aligned} \frac{\partial}{\partial t} G_\alpha(u, t) &= -\lambda^\alpha \Gamma(\alpha+1) \sum_{r=0}^\infty \sum_{k=r}^\infty u^k \frac{(-1)^r}{r!\Gamma(\alpha+1-r)} p_{k-r}^\alpha(t) \\ &= -\lambda^\alpha \Gamma(\alpha+1) \sum_{r=0}^\infty \sum_{k=0}^\infty \frac{u^{k+r}(-1)^r}{r!\Gamma(\alpha+1-r)} p_k^\alpha(t) \\ &= -\lambda^\alpha \Gamma(\alpha+1) G_\alpha(u, t) \sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\alpha+1-r)} \\ &= -\lambda^\alpha G_\alpha(u, t)(1-u)^\alpha, \end{aligned} \tag{2.11}$$

thus obtaining formula (2.9). It immediately follows that

$$G_\alpha(u, t) = e^{-\lambda^\alpha t (1-u)^\alpha}, \qquad |u| \le 1. \tag{2.12}$$

□

Remark 2.1. *Note that, for $\alpha = 1$, formula* (2.10) *reduces to the probability generating function of the classical homogeneous Poisson process. Furthermore, from* (2.10) *we have that $\mathbb{E}[N^\alpha(t)]^j = \infty$, $j \in \mathbb{N}$, $\alpha \in (0, 1)$.*

Remark 2.2. *Let X_k, $k = 1, \dots,$ be i.i.d. Uniform$[0, 1]$ random variables, then*

$$G_\alpha(u, t) = e^{-\lambda^\alpha t(1-u)^\alpha} = Pr\left\{ \min_{0 \le k \le N(t)} X_k^{1/\alpha} \ge 1 - u \right\}, \qquad u \in (0, 1), \tag{2.13}$$

where $N(t)$, $t > 0$, is a classical homogeneous Poisson process of rate λ^α with the assumption that $\min(X_k^{1/\alpha}) = 1$ *when $N(t) = 0$.*

Theorem 2.2. *The discrete stable state probabilities of a space-fractional homogeneous Poisson process $N^\alpha(t)$, $t > 0$, can be written as*

$$\begin{aligned} p_k^\alpha(t) &= Pr\{N^\alpha(t) = k\} \\ &= \frac{(-1)^k}{k!} \sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \\ &= \frac{(-1)^k}{k!} \, {}_1\psi_1 \left[\begin{array}{c} (1, \alpha) \\ (1-k, \alpha) \end{array} \middle| -\lambda^\alpha t \right], \qquad t > 0,\ k \geq 0, \end{aligned} \tag{2.14}$$

where ${}_h\psi_j(z)$ is the generalised Wright function (see Kilbas et al. [2006], page 56, formula (1.11.14)).

Proof. By expanding the probability generating function (2.10) we have that

$$\begin{aligned} G_\alpha(u,t) &= e^{-\lambda^\alpha t(1-u)^\alpha} \\ &= \sum_{r=0}^\infty \frac{[-\lambda^\alpha t(1-u)^\alpha]^r}{r!} \\ &= \sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!} \sum_{m=0}^\infty \frac{(-u)^m \Gamma(\alpha r + 1)}{m! \Gamma(\alpha r + 1 - m)} \\ &= \sum_{m=0}^\infty u^m \frac{(-1)^m}{m!} \sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - m)}. \end{aligned} \tag{2.15}$$

From this, formula (2.14) immediately follows. □

Remark 2.3. *Note that the discrete stable distribution* (2.14) *(which for $\alpha = 1$ reduces to the Poisson distribution) can also be written as*

$$\begin{aligned} p_k^\alpha(t) &= \frac{(-1)^k}{k!} \sum_{r=0}^\infty \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \\ &= \frac{(-1)^k}{k!} \int_0^\infty e^{-w} \sum_{r=0}^\infty \frac{(-\lambda^\alpha t w^\alpha)^r}{r! \Gamma(\alpha r + 1 - k)}. \end{aligned} \tag{2.16}$$

Theorem 2.3. *Let $S^\gamma(t)$, $t > 0$, $\gamma \in (0,1)$, be a γ-stable subordinator, that is a positively skewed stable process such that*

$$\mathbb{E} e^{-zS^\gamma(t)} = e^{-tz^\gamma}, \qquad z > 0,\ t > 0, \tag{2.17}$$

and with transition function $q_\gamma(s,t)$. For a space-fractional Poisson process $N^\alpha(t)$, $t > 0$, $\alpha \in (0,1]$, with rate $\lambda > 0$, the following representation holds in distribution:

$$N^\alpha(S^\gamma(t)) \stackrel{d}{=} N^{\alpha\gamma}(t). \tag{2.18}$$

Proof. In order to prove the representation (2.18) it is sufficient to observe that

$$\int_0^\infty G_\alpha(u,s) q_\gamma(s,t) \mathrm{d}s = \int_0^\infty e^{-\lambda^\alpha s(1-u)^\alpha} q_\gamma(s,t) \mathrm{d}s = e^{-\lambda^{\alpha\gamma} t (1-u)^{\alpha\gamma}}. \tag{2.19}$$

□

Remark 2.4. *Note that, when $\alpha = 1$, formula* (2.18) *reduces to*

$$N(S^\gamma(t)) \stackrel{d}{=} N^\gamma(t), \tag{2.20}$$

and this reveals a second possible way of constructing a space-fractional Poisson process.

Consider now the first-passage time at k of the space-fractional Poisson process

$$\tau_k^\alpha(t) = \inf\{t : N^\alpha(t) = k\}, \qquad k \geq 0. \tag{2.21}$$

Since $\Pr\{\tau_k^\alpha < t\} = \Pr\{N^\alpha(t) \geq k\}$, we have that

$$\Pr\{\tau_k^\alpha < t\} = \sum_{m=k}^{\infty} \frac{(-1)^m}{m!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - m)}. \tag{2.22}$$

Hence

$$\Pr\{\tau_k^\alpha \in \mathrm{d}s\} = \sum_{m=k}^{\infty} \frac{(-1)^m}{m!} \sum_{r=1}^{\infty} \frac{(-\lambda^\alpha)^r t^{r-1}}{(r-1)!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - m)}. \tag{2.23}$$

Note that for $\alpha = 1$ we obtain the classical Erlang process. First we have

$$\begin{aligned} \Pr\{\tau_k^1 < t\} &= \sum_{m=k}^{\infty} \frac{(-1)^m}{m!} \sum_{r=m}^{\infty} \frac{(-\lambda t)^r}{(r-m)!} \\ &= e^{-\lambda t} \sum_{m=k}^{\infty} \frac{(\lambda t)^m}{m!}, \end{aligned} \tag{2.24}$$

and therefore

$$\begin{aligned} \Pr\{\tau_k^1 \in \mathrm{d}s\} &= -\lambda e^{-\lambda t} \sum_{m=k}^{\infty} \frac{(\lambda t)^m}{m!} + \lambda e^{-\lambda t} \sum_{m=k}^{\infty} \frac{(\lambda t)^{m-1}}{(m-1)!} \\ &= -\lambda e^{-\lambda t} \sum_{m=k}^{\infty} \frac{(\lambda t)^m}{m!} + \lambda e^{-\lambda t} \sum_{m=k-1}^{\infty} \frac{(\lambda t)^m}{m!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}. \end{aligned} \tag{2.25}$$

Remark 2.5. *Note that, with a little effort, fractionality can be introduced also in time. In this case, for example, the fractional differential equation governing the probability generating function is*

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} {}_\nu G_\alpha(u,t) = -\lambda^\alpha {}_\nu G_\alpha(u,t)(1-u)^\alpha, & |u| \leq 1,\ \nu \in (0,1],\ \alpha \in (0,1], \\ {}_\nu G_\alpha(u,0) = 1, \end{cases} \tag{2.26}$$

where $\partial^\nu / \partial t^\nu$ is the Caputo fractional derivative operator (see Kilbas et al. [2006]). By means of Laplace transforms, some simple manipulations lead to

$${}_\nu G_\alpha(u,t) = E_\nu(-\lambda^\alpha t^\nu (1-u)^\alpha), \qquad |u| \leq 1, \tag{2.27}$$

where $E_\nu(x)$ is the Mittag–Leffler function [Kilbas et al., 2006]. In turn, by expanding the above probability generating function we have that

$$p_k^{\alpha,\nu}(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)}, \qquad k \geq 0,\ \alpha \in (0,1],\ \nu \in (0,1]. \tag{2.28}$$

When $\alpha = 1$ these probabilities easily reduce to those of a fractional Poisson process (see Beghin and Orsingher [2009]):

$$p_k^{1,\nu}(t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda t^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(r+1)}{\Gamma(r-k+1)} \tag{2.29}$$

$$
\begin{aligned}
&= \frac{(-1)^k}{k!} \sum_{r=k}^{\infty} \frac{(-\lambda t^\nu)^r}{\Gamma(\nu r+1)} \frac{r!}{(r-k)!} \\
&= \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{(\lambda t^\nu)^r}{\Gamma(\nu r+1)}.
\end{aligned}
$$

Moreover, let X_k*,* $k = 1, \dots,$ *be i.i.d. Uniform*$[0,1]$ *random variables and* $N_\nu(t)$*,* $t > 0$*, be a homogeneous time-fractional Poisson process of rate* λ^α *with the assumption that* $\min(X_k^{1/\alpha}) = 1$ *when* $N_\nu(t) = 0$*. The probability generating function* ${}_\nu G_\alpha(u,t)$*, for* $u \in (0,1)$*, can be written as*

$$
\begin{aligned}
{}_\nu G_\alpha(u,t) &= E_\nu(-\lambda^\alpha t^\nu (1-u)^\alpha) \qquad (2.30) \\
&= \sum_{r=0}^{\infty} (-1)^r \frac{(\lambda^\alpha t^\nu)^r (1-u)^{\alpha r}}{\Gamma(\nu r+1)} \\
&= \sum_{r=0}^{\infty} (-1)^r \frac{(\lambda^\alpha t^\nu)^r}{\Gamma(\nu r+1)} \sum_{k=0}^{r} (-1)^k \binom{r}{k} [1-(1-u)^\alpha]^k \\
&= \sum_{k=0}^{\infty} [1-(1-u)^\alpha]^k \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{(\lambda^\alpha t^\nu)^r}{\Gamma(\nu r+1)} \\
&= \sum_{k=0}^{\infty} \left[Pr\left(X_k^{1/\alpha} \geq 1-u \right) \right]^k Pr\left\{ N_\nu(t) = k \right\} \\
&= Pr\left\{ \min_{0 \leq k \leq N_\nu(t)} X_k^{1/\alpha} \geq 1-u \right\}.
\end{aligned}
$$

Formula (2.30) *shows that the contribution of the space-fractionality affects only the uniform random variables* $X_k^{1/\alpha}$ *while the time-fractionality only the driving process* $N_\nu(t)$*.*

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