HIGHER-ORDER LAPLACE EQUATIONS AND HYPER-CAUCHY DISTRIBUTIONS

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Abstract. In this paper we introduce new distributions which are solutions of higher-order Laplace equations. It is proved that their densities can be obtained by folding and symmetrizing Cauchy distributions. Another class of probability laws related to higher-order Laplace equations is obtained by composing pseudo-processes with positively-skewed Cauchy distributions which produce asymmetric Cauchy densities in the odd-order case. A special attention is devoted to the third-order Laplace equation where the connection between the Cauchy distribution and the Airy functions is obtained and analyzed.

1. Introduction

The Cauchy density

$$p(x,t) = \frac{1}{\pi}\frac{t}{(x^2+t^2)} \tag{1.1}$$

solves the Laplace equation (see Nane [8])

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

The n-dimensional counterpart of (1.1)

$$p(\mathbf{x},t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}}\frac{t}{\left(t^2+|\mathbf{x}|^2\right)^{\frac{n}{2}}}, \quad \mathbf{x} \in \mathbb{R}^{n-1},\ t > 0 \tag{1.3}$$

with characteristic function

$$\int_{\mathbb{R}^{n-1}} e^{i\langle \boldsymbol{\alpha}, \mathbf{x}\rangle} p(\mathbf{x},t)\, d\mathbf{x} = \exp\left(-t|\boldsymbol{\alpha}|\right) \tag{1.4}$$

solves the n-dimensional Laplace equation

$$\frac{\partial^2 p}{\partial t^2} + \sum_{j=1}^{n-1}\frac{\partial^2 p}{\partial x_j^2} = 0. \tag{1.5}$$

The inspiring idea of this paper is to investigate the class of distributions which satisfy the higher-order Laplace equations of the form

$$\frac{\partial^n u}{\partial t^n} + \frac{\partial^n u}{\partial x^n} = 0, \quad x \in \mathbb{R},\ t > 0 \tag{1.6}$$

In a previous paper of ours we have shown that the law

$$p_4(x,t) = \frac{t}{\pi\sqrt{2}}\frac{x^2+t^2}{x^4+t^4} \tag{1.7}$$

solves the fourth-order Laplace equation

$$\frac{\partial^4 u}{\partial t^4} + \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in \mathbb{R},\ t > 0. \tag{1.8}$$

In Section 2 we analyze distributions related to equations of the form

$$\frac{\partial^{2^n} u}{\partial t^{2^n}} + \frac{\partial^{2^n} u}{\partial x^{2^n}} = 0 \tag{1.9}$$

which can be expressed in many alternative forms. The decoupling of the 2^n-th order differential operator in (1.9)

$$\frac{\partial^{2^n}}{\partial t^{2^n}} + \frac{\partial^{2^n}}{\partial x^{2^n}} = \prod_{\substack{k=-(2^{n-1}-1)\\ k\ odd}}^{2^{n-1}-1} \left(\frac{\partial^2}{\partial t^2} + e^{i\frac{\pi k}{2^{n-1}}} \frac{\partial^2}{\partial x^2} \right)$$

suggests to represent distributions related to (1.9) as

$$p_{2^n}(x,t) = \frac{1}{\pi 2^{n-1}} \sum_{\substack{k=-(2^{n-1}-1)\\ k\ odd}}^{2^{n-1}-1} \frac{t\, e^{i\frac{\pi k}{2^n}}}{x^2 + (te^{i\frac{\pi k}{2^n}})^2}, \quad n \geq 2. \tag{1.10}$$

that is the superposition of Cauchy densities at imaginary times. Alternatively, we give a real-valued expression for (1.10) as

$$p_{2^n}(x,t) = \frac{t(x^2+t^2)}{2^{n-2}\pi(x^{2^n}+t^{2^n})} \sum_{\substack{k=1\\ k \text{ odd}}}^{2^{n-1}-1} \prod_{\substack{j=1,\, j \text{ odd}\\ j\neq k}}^{2^{n-1}-1} \left(x^4 + t^4 + 2x^2t^2 \cos\frac{j\pi}{2^{n-1}} \right). \tag{1.11}$$

The density (1.11) can also be represented as

$$p_{2^n}(x,t) = \frac{t(x^2+t^2)}{2^{n-2}\pi} \sum_{\substack{k=1\\ k \text{ odd}}}^{2^{n-1}-1} \frac{\cos\frac{k\pi}{2^n}}{x^4 + t^4 + 2x^2t^2\cos\frac{k\pi}{2^{n-1}}}, \quad n \geq 2. \tag{1.12}$$

Each component of the distribution (1.12) is produced by folding and symmetrizing the density of the r.v.

$$V(t) = C\left(t\cos\frac{k\pi}{2^n}\right) - t\sin\frac{k\pi}{2^n}, \quad t > 0,\ 1 \leq k \leq 2^{n-1}-1,\ k \text{ odd}$$

where $C(t)$, $t > 0$ is the Cauchy symmetric process. The distributions (1.12) differ from the Cauchy laws since they have a bimodal structure for all $n \geq 2$ as figures below show. For $n = 2$, the distribution (1.11) reduces to (1.7) if we assume that the inner product appearing in formula (1.11) is equal to one. Of course, the density (1.12) coincides with (1.7) for $n = 2$. For $n = 3$ we get from (1.11) and (1.12) that

$$\begin{aligned} p_{2^3}(x,t) =& \frac{t(x^2+t^2)}{\sqrt{2}\pi(x^8+t^8)} \left[(x^4+t^4-\sqrt{2}x^2t^2)\cos\frac{\pi}{8} + (x^4+t^4+\sqrt{2}x^2t^2)\sin\frac{\pi}{8} \right] \\ =& \frac{t(x^2+t^2)}{2\pi} \left[\frac{\sin\frac{\pi}{8}}{x^4+t^4-\sqrt{2}x^2t^2} + \frac{\cos\frac{\pi}{8}}{x^4+t^4+\sqrt{2}x^2t^2} \right]. \end{aligned} \tag{1.13}$$

In Orsingher and D'Ovidio [11] we have shown that the density (1.7) is the probability distribution of

$$Q(t) = F(T_t), \quad t > 0$$

where F is the Fresnel pseudo-process described in [11] and T_t, $t>0$ is the first passage time of a Brownian motion independent from F. We note that

$$\mathcal{Q}(t)=F(|B(t)|), \quad t>0$$

has density coinciding with the fundamental solution of the fourth-order heat equation

$$\frac{\partial u}{\partial t}=-\frac{\partial^4 u}{\partial x^4}.$$

We prove also that, for $k\in\mathbb{N}$, there are non-centered Cauchy distributions which solve the equations

$$\frac{\partial^{2k+1}u}{\partial t^{2k+1}}+\frac{\partial^{2k+1}u}{\partial x^{2k+1}}=0. \tag{1.14}$$

If $X_{2k+1}(t)$, $t>0$ is the pseudo-process whose density measure

$$\mu_{2k+1}(dx,t)=\mu\{X_{2k+1}(t)\in dx\}$$

solves the heat-type equations

$$\frac{\partial u}{\partial t}=-\frac{\partial^{2k+1}u}{\partial x^{2k+1}}, \quad k\in\mathbb{N} \tag{1.15}$$

and $S_{\frac{1}{2k+1}}(t)$, $t>0$ is a positively skewed stable process of order $\frac{1}{2k+1}$ we have that

$$Pr\{X_{2k+1}(S_{\frac{1}{2k+1}}(t))\in dx\}/dx=\frac{t\cos\frac{\pi}{2(2k+1)}}{\pi\left[\left(x+(-1)^{k+1}t\sin\frac{\pi}{2(2k+1)}\right)^2+t^2\cos^2\frac{\pi}{2(2k+1)}\right]}. \tag{1.16}$$

We show below that the densities (1.16) solve also the following second-order p.d.e.

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=2\sin\frac{\pi}{2(2k+1)}\frac{\partial^2 u}{\partial t\,\partial x}.$$

We have investigated in detail the case of third-order Laplace equation

$$\frac{\partial^3 u}{\partial t^3}+\frac{\partial^3 u}{\partial x^3}=0 \tag{1.17}$$

and have shown that

$$\begin{aligned} Pr\{X_3(S_{\frac{1}{3}}(t))\in dx\} =& dx\int_0^\infty \frac{1}{\sqrt[3]{3s}}Ai\left(\frac{x}{\sqrt[3]{3s}}\right)\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right)ds \\ =& dx\frac{\sqrt{3}}{2}t\frac{x-t}{x^3-t^3}=dx\frac{\sqrt{3}}{2}\frac{t}{x^2+xt+t^2} \\ =& dx\frac{t\cos\frac{\pi}{6}}{(x+t\sin\frac{\pi}{6})^2+t^2\cos^2\frac{\pi}{6}}. \end{aligned} \tag{1.18}$$

The pictures of the Cauchy distributions (1.16) show that the location parameter $t\sin\frac{\pi}{2(2k+1)}$ tends to zero as $k\to\infty$ while the scale parameter tends to one, $t\cos\frac{\pi}{2(2k+1)}\to t$. This means that the asymmetry of the Cauchy densities decreases as k increases and is maximal for $k=1$. The decrease of parameters of (1.16) (with k increasing) is due to the growing symmetrization of the fundamental solutions of equations (1.15).

By suitably combining the distribution (1.16) for $k=1$, we arrive at the density

$$p_6(x,t)=\frac{\sqrt{3}}{2^2\pi}t\frac{(x^2+t^2)\cos\frac{\pi}{6}+xt}{(x^2+t^2+xt\cos\frac{\pi}{6})^2+2x^2t^2\cos\frac{\pi}{3}} \tag{1.19}$$

which solves the equation

$$\frac{\partial^6 u}{\partial t^6}+\frac{\partial^6 u}{\partial x^6}=0. \tag{1.20}$$

The probability density (1.19) displays the unimodal structure of the Cauchy distribution.

2. Hyper Cauchy distributions

In this section we analyze the distribution related to Laplace-type equations of the form

$$\left(\frac{\partial^{2^n}}{\partial t^{2^n}}+\frac{\partial^{2^n}}{\partial x^{2^n}}\right)u=0, \quad n>1. \tag{2.1}$$

For $n\geq 2$ we obtain a new class of distributions having the form

$$p_{2^n}(x,t)=\frac{t(x^2+t^2)}{2^{n-2}\pi(x^{2^n}+t^{2^n})}g(x,t), \quad x\in\mathbb{R},\ t>0 \tag{2.2}$$

where $g(x,t)$ is a polynomial of order 2^n-2^2. For $n=2$, formula (2.2) yields the distribution

$$p_4(x,t)=\frac{t(x^2+t^2)}{\sqrt{2}\pi(x^4+t^4)}, \quad x\in\mathbb{R},\ t>0 \tag{2.3}$$

emerging in the analysis of Fresnel pseudo-processes (see Orsingher and D'Ovidio [11]).

The main result of this section is given in the next theorem.

Theorem 2.1. *The hyper Cauchy density*

$$p_{2^n}(x,t)=\frac{1}{\pi 2^{n-1}}\sum_{\substack{k=-(2^{n-1}-1)\\ k\ odd}}^{2^{n-1}-1}\frac{te^{i\frac{\pi k}{2^n}}}{x^2+(te^{i\frac{\pi k}{2^n}})^2} \tag{2.4}$$

solves the equation

$$\left(\frac{\partial^{2^n}}{\partial t^{2^n}}+\frac{\partial^{2^n}}{\partial x^{2^n}}\right)u=0, \quad x\in\mathbb{R},\ t>0, \quad n>1. \tag{2.5}$$

A real-valued expression of (2.4) *reads*

$$p_{2^n}(x,t)=\frac{t(x^2+t^2)}{\pi 2^{n-2}(x^{2^n}+t^{2^n})}\sum_{\substack{k=1\\ k\ odd}}^{2^{n-1}-1}\cos\frac{k\pi}{2^n}\prod_{\substack{k\neq j=1\\ j\ odd}}^{2^{n-1}-1}\left(x^4+t^4+2x^2t^2\cos\frac{j\pi}{2^{n-1}}\right) \tag{2.6}$$

or equivalently

$$p_{2^n}(x,t)=\frac{t(x^2+t^2)}{2^{n-2}\pi}\sum_{\substack{k=1\\ k\ odd}}^{2^{n-1}-1}\frac{\cos\frac{k\pi}{2^n}}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}}, \quad \textit{for}\ \ n>1. \tag{2.7}$$

Proof. In order to check that (2.4) satisfies equation (2.5) we resort to Fourier transforms

$$U(\beta,t)=\int_{-\infty}^{+\infty}e^{i\beta x}u(x,t)dx.$$

Equation (2.5) becomes

$$\frac{\partial^{2^n} U}{\partial t^{2^n}} + (-i\beta)^{2^n} U = \frac{\partial^{2^n} U}{\partial t^{2^n}} + \beta^{2^n} U = 0. \tag{2.8}$$

The solutions of the algebraic equation associated to (2.8) have the form

$$r_j = |\beta|\, e^{i\pi \frac{2j+1}{2^n}}, \quad 0 \le j \le 2^n - 1. \tag{2.9}$$

In order to construct bounded solutions to (2.8) we restrict ourselves to

$$U(\beta, t) = \frac{1}{2^{n-1}} \sum_{\substack{k=-(2^{n-1}-1) \\ k\ odd}}^{2^{n-1}-1} e^{-t|\beta| e^{i\frac{k\pi}{2^n}}} \tag{2.10}$$

where the normalizing constant in (2.10) is chosen equal to $1/2^{n-1}$ so that $U(\beta, 0) = 1$. The inverse of (2.10) is (2.4). We check directly that each term of (2.4) has Fourier transform solving equation (2.8). For all odd values of k, we have that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{i\beta x} \left(\frac{\partial^{2^n}}{\partial t^{2^n}} + \frac{\partial^{2^n}}{\partial x^{2^n}} \right) \left(\frac{t e^{i\frac{k\pi}{2^n}}}{x^2 + (t e^{i\frac{k\pi}{2^n}})^2} \right) dx \\
=& \frac{\partial^{2^n}}{\partial t^{2^n}} e^{-t|\beta| e^{i\frac{k\pi}{2^n}}} + (-i\beta)^{2^n} e^{-t|\beta| e^{i\frac{k\pi}{2^n}}} \\
=& \left(\beta^{2^n} e^{ik\pi} + i^{2^n} \beta^{2^n} \right) e^{-t|\beta| e^{i\frac{k\pi}{2^n}}} \\
=& \left((-1)^k \beta^{2^n} + \beta^{2^n} \right) e^{-t|\beta| e^{i\frac{k\pi}{2^n}}} = 0
\end{aligned}$$

because k is odd. In order to obtain (2.6) we observe that, in view of (2.4) we can write

$$p_{2^n}(x, t) = \frac{1}{\pi} \sum_{\substack{k=-(2^{n-1}-1) \\ k \text{ odd}}}^{2^{n-1}-1} \frac{c_k\, t^{|2k-1|} x^{2^n - |2k-1| - 1}}{\prod_{\substack{k=-(2^{n-1}-1) \\ k \text{ odd}}}^{2^{n-1}-1} (x^2 + (t e^{i\frac{k\pi}{2^n}})^2)}$$

where

$$\prod_{\substack{k=-(2^{n-1}-1) \\ k \text{ odd}}}^{2^{n-1}-1} (x^2 + (t e^{i\frac{k\pi}{2^n}})^2) = x^{2^n} + t^{2^n} \tag{2.11}$$

and c_k are constants evaluated below. Result (2.11) can be obtained directly by solving the equation $x^{2^n} + t^{2^n} = 0$ or by successively regrouping the terms of the right-hand side of (2.11). We have at first that

$$\begin{aligned}
\prod_{\substack{k=-(2^{n-1}-1) \\ k \text{ odd}}}^{2^{n-1}-1} (x^2 + (t e^{i\frac{k\pi}{2^n}})^2) &= \prod_{k=1,\, k \text{ odd}}^{2^{n-1}-1} \left(x^4 + t^4 + 2x^2 t^2 \cos \frac{k\pi}{2^{n-1}} \right) \\
&= \prod_{k=1,\, k \text{ odd}}^{2^{n-2}-1} \left(x^8 + t^8 + 2x^4 t^4 \cos \frac{k\pi}{2^{n-2}} \right) \\
&= \cdots \\
&= \left(x^{2^n} + t^{2^n} + 2x^2 t^2 \cos \frac{\pi}{2} \right)
\end{aligned}$$

$$=x^{2^n}+t^{2^n}.$$

In view of (2.11) we can rewrite (2.4) as

$$p_{2^n}(x,t)=\frac{t}{\pi 2^{n-1}(x^{2^n}+t^{2^n})}\sum_{\substack{k=-(2^{n-1}-1)\\ k\text{ odd}}}^{2^{n-1}-1}\prod_{\substack{j=-(2^{n-1}-1)\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^2+(te^{i\frac{2\pi j}{2^n}}))e^{i\frac{\pi k}{2^n}}$$

where

$$\begin{aligned}
&\sum_{\substack{k=-(2^{n-1}-1)\\ k\text{ odd}}}^{2^{n-1}-1}\prod_{\substack{j=-(2^{n-1}-1)\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^2+(te^{i\frac{2\pi j}{2^n}}))e^{i\frac{\pi k}{2^n}}\\
=&\sum_{\substack{k=-(2^{n-1}-1)\\ k\text{ odd}}}^{2^{n-1}-1}\prod_{\substack{j=1\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^4+t^4+2x^2t^2\cos\frac{\pi j}{2^{n-1}})(x^2+(te^{-i\frac{2\pi k}{2^n}}))e^{i\frac{\pi k}{2^n}}\\
=&\sum_{\substack{k=1\\ k\text{ odd}}}^{2^{n-1}-1}\prod_{\substack{j=1\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^4+t^4+2x^2t^2\cos\frac{\pi j}{2^{n-1}})(x^2e^{i\frac{k\pi}{2^n}}+t^2e^{-i\frac{k\pi}{2^n}}+x^2e^{-i\frac{k\pi}{2^n}}+t^2e^{i\frac{k\pi}{2^n}})\\
=&2(x^2+t^2)\sum_{\substack{k=1\\ k\text{ odd}}}^{2^{n-1}-1}\cos\frac{k\pi}{2^n}\prod_{\substack{j=1\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^4+t^4+2x^2t^2\cos\frac{\pi j}{2^{n-1}})
\end{aligned}$$

and thus

$$p_{2^n}(x,t)=\frac{t(x^2+t^2)}{\pi 2^{n-2}(x^{2^n}+t^{2^n})}\sum_{\substack{k=1\\ k\text{ odd}}}^{2^{n-1}-1}\cos\frac{k\pi}{2^n}\prod_{\substack{j=1\\ j\text{ odd},\,j\neq k}}^{2^{n-1}-1}(x^4+t^4+2x^2t^2\cos\frac{\pi j}{2^{n-1}}).$$

Furthermore, from the fact that

$$x^{2^n}+t^{2^n}=\prod_{k=1,\,k\text{ odd}}^{2^{n-2}-1}\left(x^8+t^8+2x^4t^4\cos\frac{k\pi}{2^{n-2}}\right)$$

we obtain that

$$p_{2^n}(x,t)=\frac{t(x^2+t^2)}{2^{n-2}\pi}\sum_{\substack{k=1\\ k\text{ odd}}}^{2^{n-1}-1}\frac{\cos\frac{k\pi}{2^n}}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}}.$$

□

Remark 2.2. In order to prove that the density (2.6) integrates to unity we present the following calculation

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{x^2+t^2}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}}dx=&2\int_0^{+\infty}\frac{x^2+t^2}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}}dx\\
=&\frac{2}{t}\int_0^{+\infty}\frac{y^2+1}{y^4+1+2y\cos\frac{k\pi}{2^{n-1}}}dy\\
=&\frac{2}{t}\int_0^{\frac{\pi}{2}}\frac{1}{\tan^4\theta+1+2\tan^2\theta\cos\frac{k\pi}{2^{n-1}}}\frac{d\theta}{\cos^4\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{t}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta\cos\frac{k\pi}{2^{n-1}}} \\
&= \frac{2}{t}\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-\frac{1-\cos\frac{k\pi}{2^{n-1}}}{2}\sin^2 2\theta} \\
&= \frac{2}{t}\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-\frac{1}{2}\left(1-\cos\frac{k\pi}{2^{n-1}}\right)\left(\frac{1-\cos 4\theta}{2}\right)} \\
&= \frac{1}{2t}\int_0^{2\pi} \frac{d\phi}{1-\frac{1-\cos\frac{k\pi}{2^{n-1}}}{4}+\frac{1}{4}\left(1-\cos\frac{k\pi}{2^{n-1}}\right)\cos\phi} \\
&= \frac{2}{t}\int_0^{2\pi} \frac{d\phi}{\left(3+\cos\frac{k\pi}{2^{n-1}}\right)+\left(1-\cos\frac{k\pi}{2^{n-1}}\right)\cos\phi} \\
&= \frac{2}{t}\frac{2\pi}{\sqrt{\left(3+\cos\frac{k\pi}{2^{n-1}}\right)^2-\left(1-\cos\frac{k\pi}{2^{n-1}}\right)^2}} \\
&= \frac{\pi\sqrt{2}}{t}\frac{1}{\sqrt{1+\cos\frac{k\pi}{2^{n-1}}}} \\
&= \frac{\pi}{t}\frac{1}{\cos\frac{k\pi}{2^n}}. \qquad (2.12)
\end{aligned}
$$

From (2.7), in view of (2.12), we can conclude that

$$\int_{-\infty}^{+\infty} p_{2^n}(x,t)\,dx = 1$$

Remark 2.3. From (2.4), for $n = 2$ we obtain that

$$p_4(x,t) = \frac{1}{2\pi}\left[\frac{te^{i\frac{\pi}{4}}}{x^2+(te^{i\frac{\pi}{4}})^2}+\frac{te^{-i\frac{\pi}{4}}}{x^2+(te^{-i\frac{\pi}{4}})^2}\right]$$

with Fourier transform

$$\int_{-\infty}^{+\infty} e^{i\beta x}p_4(x,t)dx = e^{-\frac{t}{\sqrt{2}}|\beta|}\cos\frac{\beta t}{\sqrt{2}}.$$

From (2.6) and (2.7) we have that

$$p_4(x,t) = \frac{t}{\sqrt{2}\pi}\frac{x^2+t^2}{x^4+t^4}. \qquad (2.13)$$

The law (2.13) has two maxima as Figure 1 shows.

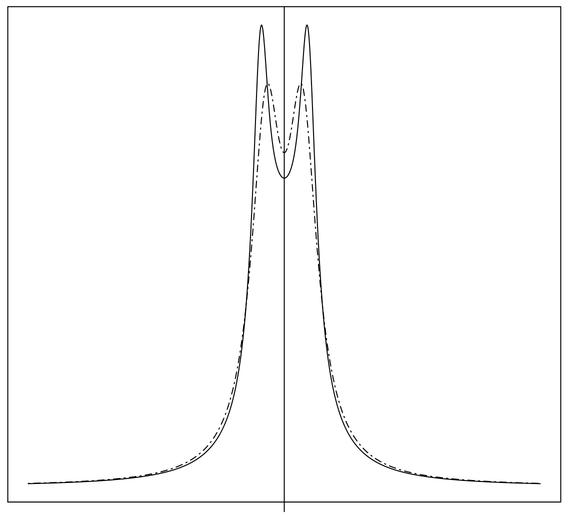
Remark 2.4. For $n = 3$, from (2.6), we have that

$$p_8(x,t) = \frac{t(x^2+t^2)}{2\pi(x^8+t^8)}\left[\left(x^4+t^4+2x^2t^2\cos\frac{\pi}{4}\right)\cos\frac{3\pi}{8}+\left(x^4+t^4+2x^2t^2\cos\frac{3\pi}{4}\right)\cos\frac{\pi}{8}\right].$$

From the fact that

$$\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} \quad \text{and} \quad \cos\frac{3\pi}{8} = \sin\frac{\pi}{8}$$

FIGURE 1. The profile of the functions p_4 (dotted line), formula (2.13) and p_8, formula (2.14).



we write

$$p_8(x,t) = \frac{t(x^2+t^2)}{2\pi(x^8+t^8)} \left[\left(x^4+t^4+\sqrt{2}x^2t^2\right)\sin\frac{\pi}{8} + \left(x^4+t^4-\sqrt{2}x^2t^2\right)\cos\frac{\pi}{8} \right].$$

From (2.7) we have also that

$$p_8(x,t) = \frac{t}{2\pi} \left[\frac{x^2+t^2}{x^4+t^4-\sqrt{2}x^2t^2}\sin\frac{\pi}{8} + \frac{x^2+t^2}{x^4+t^4+\sqrt{2}x^2t^2}\cos\frac{\pi}{8} \right]. \tag{2.14}$$

From (2.4) we obtain the characteristic function

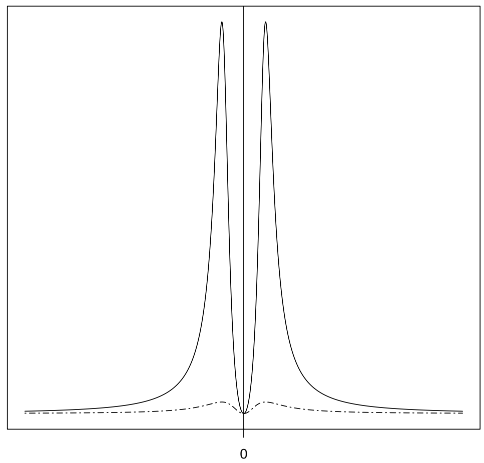
$$\int_{\mathbb{R}} e^{i\beta x} p_8(x,t)dx = \frac{1}{2^2}\left[e^{-t|\beta|\cos\frac{\pi}{8}}\cos\left(t\beta\sin\frac{\pi}{8}\right) + e^{-t|\beta|\sin\frac{\pi}{8}}\cos\left(t\beta\cos\frac{\pi}{8}\right)\right].$$

The density $p_8(x,t)$ is a bimodal curve as well as $p_4(x,t)$. The maxima of $p_8(x,t)$ are heigher than those of $p_4(x,t)$ as Figure 1 shows. Also $p_{2^n}(x,t)$ displays a bimodal structure with the height peaks increasing as n increases. The form of $p_{2^n}(x,t)$ reminds the structure of densities of fractional diffusions governed by equations

$$\frac{\partial^\nu u}{\partial t^\nu} = \lambda^2 \frac{\partial^2 u}{\partial x^2}$$

for $1<\nu<2$ (see [10]).

FIGURE 2. The profile of the function g_k for $n = 3$ and $k = 1$ (dotted line), $k = 3$.



Remark 2.5. The result (2.7) can conveniently be rewritten as

$$p_{2^n}(x,t) = \frac{t}{\pi(x^2+t^2)} \left[\frac{1}{2^{n-1}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{n-1}-1} \frac{x^4+t^4+2x^2t^2}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}} \cos\frac{k\pi}{2^n} \right]. \tag{2.15}$$

The factor in square parenthesis measures, in some sense, the disturbance of p_{2^n} on the classical Cauchy. For $n = 2$, we have in particular that

$$p_{2^2}(x,t) = \frac{t}{\pi(x^2+t^2)} \frac{1}{\sqrt{2}} \left[1 + \frac{2x^2t^2}{x^4+t^4} \right] = \frac{t}{\sqrt{2}\pi} \frac{x^2+t^2}{x^4+t^4}. \tag{2.16}$$

The density (2.16) has two symmetric maxima at $x = \pm t\sqrt{\sqrt{2}-1}$ and a minimum at $x = 0$ (see Fig. 6 of Orsingher and D'Ovidio [11]). The terms

$$g_k(x,t) = \frac{x^4+t^4+2x^2t^2}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}} \tag{2.17}$$

display two maxima at $x = \pm t$ with height depending on k and whose profile is depicted in Figure 2.

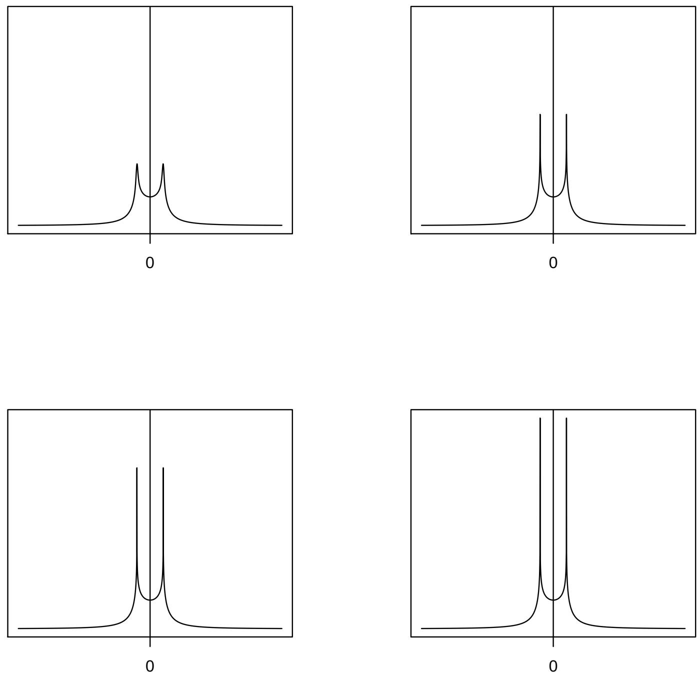
Remark 2.6. The density $p_{2^n}(x,t)$ can be written as

$$p_{2^n}(x,t) = \frac{t(x^2+t^2)}{2^{n-2}\pi(x^{2^n}+t^{2^n})} Q(x,t) \tag{2.18}$$

where $Q(x,t)$ is a polynomial of order $2^n - 2^2$. For $n = 2$ the function $Q(x,t)$ reduces to $\cos\frac{\pi}{4}$. For $n = 3$,

$$Q(x,t) = (x^4+t^4+\sqrt{2}x^2t^2)\sin\frac{\pi}{8} + (x^4+t^4-\sqrt{2}x^2t^2)\cos\frac{\pi}{8}.$$

FIGURE 3. The profile of the functions p_{2^n}, formula (2.7), for $n = 5, 10, 15, 20$.



The expression (2.18) shows that the probability law $p_{2^n}(x,t)$, $x \in \mathbb{R}$, $t > 0$ shares with the classical Cauchy density the property of non-existence of the mean value.

Remark 2.7. The density of the hyper Cauchy can also be presented in an alternative form by regrouping the terms in the right-hand side of (2.7) as

$$\begin{aligned} &\sum_{\substack{k=1 \\ \text{k odd}}}^{2^{n-2}-1} \left[\frac{\sin \frac{k\pi}{2^n}}{x^4 + t^4 - 2x^2t^2 \cos \frac{k\pi}{2^{n-1}}} + \frac{\cos \frac{k\pi}{2^n}}{x^4 + t^4 + 2x^2t^2 \cos \frac{k\pi}{2^{n-1}}} \right] \qquad (2.19) \\ &= \sum_{\substack{k=1 \\ \text{k odd}}}^{2^{n-2}-1} \frac{\left(x^4 + t^4 + 2x^2t^2 \cos \frac{k\pi}{2^{n-1}}\right) \sin \frac{k\pi}{2^n} + \left(x^4 + t^4 - 2x^2t^2 \cos \frac{k\pi}{2^{n-1}}\right) \cos \frac{k\pi}{2^n}}{x^8 + t^8 - 2x^4t^4 \cos \frac{k\pi}{2^{n-2}}}. \end{aligned}$$

For $n = 3$, from (2.19), we get again that

$$p_8(x,t) = \frac{t(x^2+t^2)}{2\pi(x^8+t^8)} \left[\left(x^4 + t^4 + \sqrt{2}x^2t^2\right) \sin \frac{k\pi}{8} + \left(x^4 + t^4 - \sqrt{2}x^2t^2\right) \cos \frac{k\pi}{8} \right].$$

Remark 2.8. The r.v.

$$W(t) = \left| C\left(t \cos \frac{\pi k}{2^n}\right) - t \sin \frac{\pi k}{2^n} \right| \qquad (2.20)$$

(where $C(t)$, $t>0$ is the Cauchy process) has probability density

$$f_k(w,t) = \frac{2t(w^2+t^2)\cos\frac{k\pi}{2^n}}{\pi(w^4+t^4+2w^2t^2\cos\frac{k\pi}{2^{n-1}})}, \quad w>0. \tag{2.21}$$

Indeed, we have that

$$Pr\left\{W(t)<w\right\} = \int_{-w+t\sin\frac{k\pi}{2^n}}^{+w+t\sin\frac{k\pi}{2^n}} dy \frac{t\cos\frac{k\pi}{2^n}}{\pi(y^2+t^2\cos^2\frac{k\pi}{2^n})} \tag{2.22}$$

and

$$\begin{aligned}
f_k(w,t) =& \frac{d}{dw} Pr\left\{\left|C\left(t\cos\frac{\pi k}{2^n}\right) - t\sin\frac{\pi k}{2^n}\right| < w\right\} \\
=& \frac{t\cos\frac{k\pi}{2^n}}{\pi\left((w+t\sin\frac{k\pi}{2^n})^2+t^2\cos^2\frac{k\pi}{2^n}\right)} + \frac{t\cos\frac{k\pi}{2^n}}{\pi\left((-w+t\sin\frac{k\pi}{2^n})^2+t^2\cos^2\frac{k\pi}{2^n}\right)} \\
=& \frac{t\cos\frac{k\pi}{2^n}}{\pi\left(w^2+2wt\sin\frac{k\pi}{2^n}+t^2\right)} + \frac{t\cos\frac{k\pi}{2^n}}{\pi\left(w^2-2wt\sin\frac{k\pi}{2^n}+t^2\right)} \\
=& \frac{2t(w^2+t^2)\cos\frac{k\pi}{2^n}}{\pi(w^2+t^2+2wt\sin\frac{k\pi}{2^n})(w^2+t^2-2wt\sin\frac{k\pi}{2^n})} \\
=& \frac{2t(w^2+t^2)\cos\frac{k\pi}{2^n}}{\pi(w^4+t^4+2w^2t^2\cos\frac{k\pi}{2^{n-1}})}
\end{aligned}$$

because

$$2\sin^2\frac{k\pi}{2^n} = 1-\cos\frac{k\pi}{2^{n-1}}.$$

By symmetrizing (2.20) as follows

$$Z(t) = \frac{W_1(t)-W_2(t)}{2}$$

where $W_1(t), W_2(t)$ are independent copies of $W(t)$ we obtain a distribution of the form

$$h_k(w,t) = \frac{t(w^2+t^2)\cos\frac{k\pi}{2^n}}{\pi(w^4+t^4+2w^2t^2\cos\frac{k\pi}{2^{n-1}})}, \quad w\in\mathbb{R} \tag{2.23}$$

which coincides with each term of (2.15). This construction explains the reason for which each term in (2.15) has two symmetric maxima at $w=\pm t\sqrt{2\sin\frac{k\pi}{2^n}-1}$ for $k: \sin\frac{\pi k}{2^n} > \frac{1}{2}$.

3. Higher-order Laplace-type equation

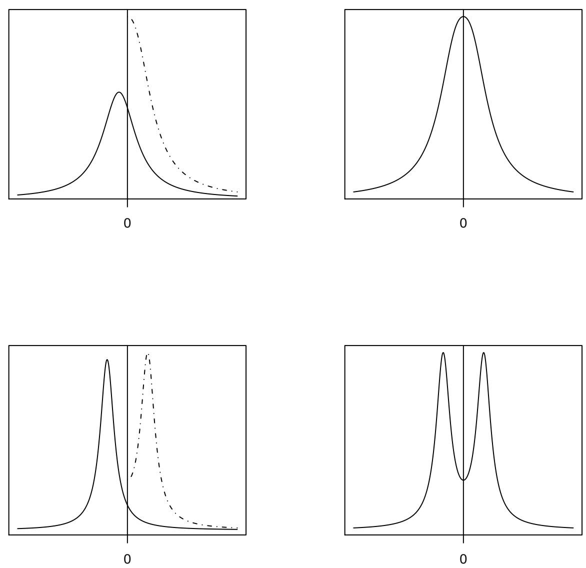
Let us consider the pseudo-processes related to higher-order heat-type equations

$$\frac{\partial u}{\partial t} = c_n \frac{\partial^n u}{\partial x^n}, \quad x\in\mathbb{R},\ t>0, \quad n>2. \tag{3.1}$$

where $c_n = (-1)^{\frac{n}{2}+1}$ for n even and $c_n=\pm 1$ for n odd.

Pseudo-processes constructed by exploiting the sign-varying measures obtained as fundamental solutions to (3.1) have been examined in many papers since the

Figure 4. The figure shows how the distribution (2.23) can be constructed from the Cauchy density by folding and symmetrizing, in the cases $n = 3$, $k = 1$ (top figures) and $k = 3$ (bottom figures). The dotted line gives the density of the folded distribution (3.13).



beginning of the Sixties. A description of the procedure of construction of pseudo-processes can be found, for example in Krylov [4], Ladokhin [6], Hochberg [3], Orsingher [9], Lachal [5]. In the case where $n = 2k+1$, $c_{2k+1} = -1$, the fundamental solution to (3.1) reads

$$u_{2k+1}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\beta x + i(-1)^k t\beta^{2k+1}} d\beta. \tag{3.2}$$

In particular, for $k = 1$

$$u_3(x,t) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\beta x + \beta^3 t\right) d\beta = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right) \tag{3.3}$$

where

$$Ai(x) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right]$$

is the Airy function (see for example Lebedev [7]).

In this section we study the composition of pseudo-processes with stable processes $S_\alpha(t)$, $t > 0$, $\alpha \in (0,1)$ whose characteristic function reads

$$\mathbb{E}e^{i\beta S_\alpha(t)} = \exp\left(-t|\beta|^\alpha e^{-i\frac{\pi\gamma}{2}\frac{\beta}{|\beta|}}\right) = \exp\left(-\sigma\, t|\beta|^\alpha\left(1 - i\theta\frac{\beta}{|\beta|}\tan\frac{\pi\alpha}{2}\right)\right) \tag{3.4}$$

where $\sigma = \cos \pi\gamma/2 > 0$ and

$$\theta = \cot\left(\frac{\pi\alpha}{2}\right)\tan\left(\frac{\pi\gamma}{2}\right).$$

The parameter γ must be chosen in such a way that $\theta \in [-1,1]$ for $\alpha \in (0,1)$. The skewness parameter $\theta = 1$ (that is $\gamma = \alpha$) corresponds to positively skewed stable distributions. For the density

$$p_\alpha(x,\gamma,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\beta x}\mathbb{E}e^{i\beta S_\alpha(t)}d\beta$$

we have the scaling property

$$p_\alpha(x,\gamma,t) = \frac{1}{t^{1/\alpha}} p_\alpha\left(\frac{x}{t^{1/\alpha}},\gamma,1\right). \tag{3.5}$$

For $\alpha \in (0,1)$, we have the series representation of stable density (see [10, page 245])

$$p_\alpha(x;\gamma,1) = \frac{\alpha}{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(\alpha(r+1))}{r!}x^{-\alpha(r+1)-1}\sin\left(\frac{\pi}{2}(\gamma+\alpha)(r+1)\right). \tag{3.6}$$

Theorem 3.1. *The composition of the pseudo-process $X_{2k+1}(t)$, $t > 0$ with the stable process $S_{\frac{1}{2k+1}}(t)$, $t > 0$, $k \in \mathbb{N}$, has a Cauchy probability distribution which can be written as*

$$Pr\{X_{2k+1}(S_{\frac{1}{2k+1}}(t) \in dx\}/dx = \frac{t\,\cos\frac{\pi}{2(2k+1)}}{\pi\left[\left(x+(-1)^{k+1}t\,\sin\frac{\pi}{2(2k+1)}\right)^2 + t^2\,\cos^2\frac{\pi}{2(2k+1)}\right]} \tag{3.7}$$

with $x \in \mathbb{R}$, $t > 0$. The density function (3.7) *is a solution to the higher-order Laplace equation*

$$\frac{\partial^{2k+1}u}{\partial t^{2k+1}} + \frac{\partial^{2k+1}u}{\partial x^{2k+1}} = 0, \quad x \in \mathbb{R},\ t > 0 \tag{3.8}$$

Proof. For $\theta = 1$, $\alpha = \gamma = 1/2k+1$, in view of (3.4) we have that

$$\begin{aligned}
U(\beta,t) &= \int_{-\infty}^{+\infty} e^{i\beta x} Pr\{X_{2k+1}(S_{\frac{1}{2k+1}}(t)) \in dx\} \\
&= \int_0^\infty Pr\{S_{\frac{1}{2k+1}}(t) \in ds\}\int_{-\infty}^{+\infty} e^{i\beta x}u_{2k+1}(x,s)\,dx \\
&= \int_0^\infty e^{is(-1)^k\beta^{2k+1}} Pr\{S_{\frac{1}{2k+1}}(t) \in ds\} \\
&= \exp\left(-t\left|(-1)^k\beta^{2k+1}\right|^{\frac{1}{2k+1}}\cos\frac{\pi}{2(2k+1)}\left(1 - i\ \mathrm{sgn}\left((-1)^k\beta^{2k+1}\right)\tan\frac{\pi}{2(2k+1)}\right)\right) \\
&= \exp\left(-t|\beta|\left(\cos\frac{\pi}{2(2k+1)} - i(-1)^k\frac{\beta}{|\beta|}\sin\frac{\pi}{2(2k+1)}\right)\right) \\
&= \exp\left(-t|\beta|\cos\frac{\pi}{2(2k+1)} - i(-1)^k t\beta\sin\frac{\pi}{2(2k+1)}\right).
\end{aligned} \tag{3.9}$$

This is the characteristic function of a Cauchy distribution with scale parameter $t\cos\frac{\pi}{2(2k+1)}$ and location parameter $t(-1)^{k+1}\sin\frac{\pi}{2(2k+1)}$. Formula (3.9) can also be rewritten as

$$\begin{aligned}U(\beta,t) &= \exp\left(-t|\beta|\left(\cos\frac{\pi}{2(2k+1)} - i(-1)^k\frac{\beta}{|\beta|}\sin\frac{\pi}{2(2k+1)}\right)\right)\\&= \exp\left(-t|\beta|\left(\cos\left(\frac{\pi}{2(2k+1)}(-1)^k\frac{\beta}{|\beta|}\right) - i\sin\left(\frac{\pi}{2(2k+1)}(-1)^k\frac{\beta}{|\beta|}\right)\right)\right)\\&= \exp\left(-t|\beta|e^{-i\frac{\pi}{2(2k+1)}(-1)^k\frac{\beta}{|\beta|}}\right). \qquad (3.10)\end{aligned}$$

The Fourier transform of equation (3.8) becomes

$$\frac{\partial^{2k+1}U}{\partial t^{2k+1}} + (-i\beta)^{2k+1}U = 0. \qquad (3.11)$$

The derivative of order $2k+1$ of (3.10) is

$$\frac{\partial^{2k+1}U}{\partial t^{2k+1}}(\beta,t) = (-|\beta|)^{2k+1}\left(e^{-i\frac{\pi}{2(2k+1)}(-1)^k\frac{\beta}{|\beta|}}\right)^{2k+1} U(\beta,t) \qquad (3.12)$$

and this shows that the Cauchy distribution (3.7) solves the higher-order Laplace equation (3.8). □

Remark 3.2. We notice that

$$\begin{aligned}\int_0^\infty Pr\{X_{2k+1}(S_{\frac{1}{2k+1}}(t)) \in dx\} =& \frac{1}{\pi}\int_{(-1)^{k+1}\tan\frac{\pi}{2(2k+1)}}^\infty \frac{dy}{1+y^2}\\=&\frac{1}{2}\left(1+\frac{(-1)^k}{2k+1}\right) \qquad (3.13)\end{aligned}$$

which is somehow in accord with Lachal [5]. The results (3.7) and (3.13) show that the mode of the Cauchy law (3.7) approaches the origin as k increases.

Let us consider the process of the form $X_3(S_{\frac{1}{3}}(t))$, $t>0$ where X_3 is a pseudo-process whose measure density is governed by the third-order heat equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3}, \quad x\in\mathbb{R},\ t>0 \qquad (3.14)$$

and $S_{\frac{1}{3}}$ is the stable process of order $1/3$. The distribution of $X_3(S_{\frac{1}{3}}(t))$, $t>0$ reads

$$Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\} = dx\int_0^\infty \frac{1}{\sqrt[3]{3s}}Ai\left(\frac{x}{\sqrt[3]{3s}}\right)\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right)ds \qquad (3.15)$$

where

$$Pr\{S_{\frac{1}{3}}(t)\in ds\} = ds\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right), \quad s\geq 0,\ t>0 \qquad (3.16)$$

for which

$$\begin{aligned}\int_0^\infty Pr\{S_{\frac{1}{3}}(t)\in ds\} =& \int_0^\infty ds\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right)\\=&(w=t/\sqrt[3]{3s}) = 3\int_0^\infty Ai(w)\,dw = 1.\end{aligned}$$

Corollary 3.3. *The law* (3.15) *solves the higher-order Laplace equation*

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial x^3} = 0, \quad x\in\mathbb{R},\ t>0 \qquad (3.17)$$

and can be written as

$$
\begin{aligned}
Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\} =& \frac{dx}{\pi} \frac{\frac{\sqrt{3}}{2}t}{\left(x+\frac{t}{2}\right)^2+\frac{3t^2}{4}} \\
=& \frac{dx}{\pi}\frac{3^{1/2}}{2}\frac{t}{x^2+xt+t^2} \\
=& dx\frac{3^{1/2}\,t}{2\pi}\frac{x-t}{x^3-t^3}.
\end{aligned} \tag{3.18}
$$

Proof. The Fourier transform of (3.15) becomes

$$
\int_{-\infty}^{\infty} e^{i\beta x} Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\} = \int_0^{\infty} e^{-i\beta^3 s} Pr\{S_{\frac{1}{3}}(t) \in ds\}. \tag{3.19}
$$

We show that (3.16) is a stable law of order $1/3$. In view of the representation of the the Airy function (4.10) of Orsingher and Beghin [10]

$$
Ai(w) = \frac{3^{-2/3}}{\pi}\sum_{k=0}^{\infty}\frac{(3^{1/3}w)^k}{k!}\sin\left(\frac{2\pi}{3}(k+1)\right)\Gamma\left(\frac{k+1}{3}\right) \tag{3.20}
$$

we can write that

$$
\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right) = \frac{t}{3\pi s\sqrt[3]{s}}\sum_{k=0}^{\infty}\left(\frac{t}{\sqrt[3]{s}}\right)^k\frac{1}{k!}\sin\left(\frac{2\pi}{3}(k+1)\right)\Gamma\left(\frac{k+1}{3}\right)
$$

We consider the series expansion (3.6) of the stable density (with $t=1$) for which (3.4) holds true. For $\alpha=\gamma=1/3$ (that is $\theta=+1$), $x=s/t^3$ in (3.6) we get that

$$
\begin{aligned}
p_{\frac{1}{3}}\left(\frac{s}{t^3};\frac{1}{3},1\right) =& \frac{1}{3\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{s}{t^3}\right)^{-\frac{k+1}{3}-1}\sin\left(\frac{\pi}{3}(k+1)\right)\Gamma\left(\frac{k+1}{3}\right) \\
=& (\text{by 4.5 of [10]}) \\
=& \frac{1}{3\pi}\frac{t^4}{s\sqrt[3]{s}}\sum_{k=0}^{\infty}\left(\frac{t}{\sqrt[3]{s}}\right)^k\frac{1}{k!}\sin\left(\frac{2\pi}{3}(k+1)\right)\Gamma\left(\frac{k+1}{3}\right) \\
=& t^3\left[\frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right)\right]
\end{aligned}
$$

and thus, from (3.5), we have that

$$
\frac{1}{t^3}p_{\frac{1}{3}}\left(\frac{s}{t^3};\frac{1}{3},1\right) = p_{\frac{1}{3}}\left(s;\frac{1}{3},t\right) = \frac{t}{s}\frac{1}{\sqrt[3]{3s}}Ai\left(\frac{t}{\sqrt[3]{3s}}\right), \quad s,t>0.
$$

We now evaluate the integral (3.19). We have that

$$
\begin{aligned}
&\int_0^{\infty} e^{-i\beta^3 s} Pr\{S_{\frac{1}{3}}(t) \in ds\} \\
=& \exp\left(-\cos\frac{\pi}{6}\,t|-\beta^3|^{\frac{1}{3}}\left(1-i\ \mathrm{sgn}\ (-\beta^3)\tan\frac{\pi}{6}\right)\right) \\
=& \exp\left(-\frac{\sqrt{3}}{2}t|\beta|\left(1+i\ \mathrm{sgn}\ (\beta)\frac{1}{\sqrt{3}}\right)\right) \\
=& \exp\left(-\frac{\sqrt{3}}{2}t|\beta| - i\frac{t}{2}\beta\right)
\end{aligned} \tag{3.21}
$$

since sgn $(-\beta^3) = \text{sgn}\,(-\beta) = -\text{sgn}\,(\beta) = -\frac{\beta}{|\beta|}$. From (3.21) we infer that

$$\begin{aligned} Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\} =& \frac{dx}{2\pi} \int_{-\infty}^{+\infty} e^{-i\beta x} \exp\left(-\frac{\sqrt{3}}{2} t|\beta| - i\frac{t}{2}\beta \right) d\beta \\ =& \frac{dx}{\pi} \frac{\frac{\sqrt{3}}{2}t}{\left(x+\frac{t}{2}\right)^2 + \frac{3t^2}{4}} = \frac{dx}{\pi}\frac{3^{1/2}}{2}\frac{t}{x^2+xt+t^2} \\ =& dx \frac{3^{1/2}\,t}{2\pi}\frac{x-t}{x^3-t^3} \end{aligned} \tag{3.22}$$

□

Remark 3.4. We observe that the r.v. $X_3(S_{\frac{1}{3}}(t))$ possesses Cauchy distribution with scale parameter $\sqrt{3}t/2$ and location parameter $-t/2$. Furthermore, it solves the third-order Laplace-type equation

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3.23}$$

Remark 3.5. From the fact that

$$\frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right) = \frac{1}{3\pi}\sqrt{\frac{x}{t}} K_{1/3}\left(\frac{2}{3^{3/2}}\frac{x^{3/2}}{\sqrt{t}}\right), \quad x,t>0 \tag{3.24}$$

we can write, for $x>0$,

$$Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\}/dx = \int_0^\infty \frac{1}{3\pi}\sqrt{\frac{x}{s}} K_{1/3}\left(\frac{2}{3^{3/2}}\frac{x^{3/2}}{\sqrt{s}}\right) \frac{t}{s}\frac{1}{3\pi}\sqrt{\frac{t}{s}} K_{1/3}\left(\frac{2}{3^{3/2}}\frac{t^{3/2}}{\sqrt{s}}\right) ds \tag{3.25}$$

$$= \frac{2\sqrt{xt^3}}{3^2\pi^2}\int_0^\infty s\, K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}s\right) K_{1/3}\left(\frac{2t^{3/2}}{3^{3/2}}s\right) ds. \tag{3.26}$$

In view of (see [2, formula 6.521])

$$\int_0^\infty s\, K_\nu(ys)\, K_\nu(zs)\, ds = \frac{\pi(yz)^{-\nu}(y^{2\nu}-z^{2\nu})}{2\sin\pi\nu\,(y^2-z^2)}, \quad \Re\{y+z\}>0,\ |\Re\{\nu\}|<1$$

we get that

$$Pr\{X_3(S_{\frac{1}{3}}(t)) \in dx\} = dx\,\frac{3^{1/2}\,t}{2\pi}\frac{x-t}{x^3-t^3}, \quad x,t>0 \tag{3.27}$$

which coincides with (3.18).

The Cauchy densities pertaining to the composition $X_{\frac{1}{2k+1}}(S_{\frac{1}{2k+1}}(t))$, $t>0$, solve also a second-order p.d.e. as we show in the next theorem.

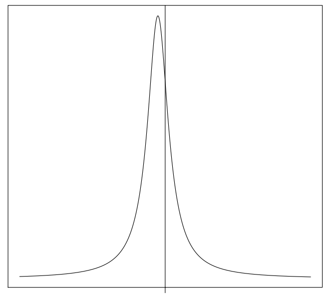
Theorem 3.6. *The Cauchy densities*

$$f(x,t;m) = \frac{1}{\pi}\frac{t\cos\frac{\pi}{2m}}{(x+t\sin\frac{\pi}{2m})^2 + t^2\cos^2\frac{\pi}{2m}}, \quad m\in\mathbb{N} \tag{3.28}$$

satisfy the following second-order equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 2\sin\frac{\pi}{2m}\frac{\partial^2 f}{\partial x\partial t}, \quad x\in\mathbb{R},\ t>0. \tag{3.29}$$

FIGURE 5. The profile of the function (3.18).



Proof. It is convenient to write (3.28) as a composed function

$$f(u,v)=\frac{1}{\pi}\frac{u}{u^2+v^2}$$

where

$$u=t\cos\frac{\pi}{2m},\qquad v=x+t\sin\frac{\pi}{2m}.$$

Since

$$\begin{aligned}
\frac{\partial f}{\partial t}&=\cos\frac{\pi}{2m}\frac{\partial f}{\partial u}+\sin\frac{\pi}{2m}\frac{\partial f}{\partial v}\\
\frac{\partial^2 f}{\partial t^2}&=\cos^2\frac{\pi}{2m}\frac{\partial^2 f}{\partial u^2}+2\cos\frac{\pi}{2m}\sin\frac{\pi}{2m}\frac{\partial^2 f}{\partial u\partial v}+\sin^2\frac{\pi}{2m}\frac{\partial^2 f}{\partial v^2}\\
\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial v}\qquad\text{and}\qquad\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial v^2}
\end{aligned}$$

and

$$\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}=0$$

we have that

$$\begin{aligned}
\frac{\partial^2 f}{\partial t^2}+\frac{\partial^2 f}{\partial x^2}=&\cos^2\frac{\pi}{2m}\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}+2\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}\frac{\partial^2 f}{\partial u\partial v}+\sin^2\frac{\pi}{2m}\frac{\partial^2 f}{\partial v^2}\\
=&\frac{\partial^2 f}{\partial v^2}\left[1-\cos^2\frac{\pi}{2m}+\sin^2\frac{\pi}{2m}\right]+2\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}\frac{\partial^2 f}{\partial u\partial v}\\
=&2\sin\frac{\pi}{2m}\frac{\partial}{\partial v}\left[\sin\frac{\pi}{2m}\frac{\partial f}{\partial v}+\cos\frac{\pi}{2m}\frac{\partial f}{\partial u}\right]\\
=&2\sin\frac{\pi}{2m}\frac{\partial}{\partial x}\frac{\partial f}{\partial t}
\end{aligned}$$

□

Remark 3.7. The characteristic function of (3.28) is

$$\int_{-\infty}^{+\infty}e^{i\beta x}f(x,t;m)dx=e^{-t|\beta|\cos\frac{\pi}{2m}-i\beta t\sin\frac{\pi}{2m}}$$

and can be obtained by considering the bounded solution to the Fourier transform of (3.29)

$$\frac{d^2F}{dt^2} + 2i\beta \sin\frac{\pi}{2m}\frac{dF}{dt} - \beta^2 F = 0.$$

For the even-order Laplace equations we have the following result.

Theorem 3.8. *The solution to the higher-order Laplace-type equation*

$$\frac{\partial^{2n} u}{\partial t^{2n}} = -\frac{\partial^{2n} u}{\partial x^{2n}}, \quad x \in \mathbb{R},\ t > 0 \tag{3.30}$$

subject to the initial conditions

$$\begin{cases} u(x,0) = \delta(x) \\ \frac{\partial^k u}{\partial t^k}(x,t)\Big|_{t=0^+} = \frac{(-1)^k\, k!}{\pi |x|^{k+1}} \cos\frac{\pi(k+1)}{2}, \quad 0 < k < 2n \end{cases} \tag{3.31}$$

is the classical Cauchy distribution given by

$$u(x,t) = Pr\{X_{2n}(S_{\frac{1}{2n}}(t)) \in dx\}/dx = \frac{t}{\pi(x^2+t^2)}, \quad x \in \mathbb{R},\ t > 0 \tag{3.32}$$

where $X_{2n}(t)$, $t > 0$ is a pseudo-process such that

$$\mathbb{E}e^{i\beta X_{2n}(t)} = e^{-t\beta^{2n}}.$$

Proof. The pseudo-process $X_{2n}(t)$, $t > 0$ related to the equation

$$\frac{\partial u}{\partial t} = (-1)^{n+1}\frac{\partial^{2n} u}{\partial t^{2n}}$$

has fundamental solution whose Fourier transform reads

$$\int_{-\infty}^{+\infty} e^{i\beta x} u(x,t)dx = e^{-t\beta^{2n}}.$$

If $S_{\frac{1}{2n}}(t)$, $t > 0$ is a stable subordinator with Laplace transform

$$\mathbb{E}\exp\left(-\lambda S_{\frac{1}{2n}}(t)\right) = \exp\left(-t\lambda^{\frac{1}{2n}}\right), \quad \lambda > 0,\ t > 0 \tag{3.33}$$

the characteristic function of $X_{2n}(S_{\frac{1}{2n}}(t))$, $t > 0$ becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{i\beta x} Pr\{X_{2n}(S_{\frac{1}{2n}}(t)) \in dx\} &= \int_0^{\infty} e^{-s\beta^{2n}} Pr\{S_{\frac{1}{2n}}(t) \in ds\} \\ &= \exp\left(-t|\beta| e^{i\frac{\pi r}{n}}\right), \quad r = 0, 1, \dots, 2n-1 \end{aligned} \tag{3.34}$$

For $r = 0$, we have the characteristic function of the Cauchy symmetric law. For $r \neq 0$ and $n \le r \le 2n-1$ we have a function which is not absolutely integrable and, for $0 < r < n-1$ is not a characteristic function (but can be regarded as a Cauchy r.v. at a complex time). The functions

$$F_r(\beta,t) = e^{-t|\beta| e^{i\frac{\pi r}{2n}}}$$

for all $0 \le r \le 2n-1$ are solutions to

$$\frac{\partial^{2n} F_r}{\partial t^{2n}} = (-1)^{n+1} F_r.$$

We now check that for $0 \le k \le 2n-1$ the initial conditions (3.31) are verified by the Cauchy distribution. Indeed,

$$\begin{aligned}\frac{\partial^k u}{\partial t^k}(x,t)\Big|_{t=0} =& \frac{\partial^k}{\partial t^k}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\beta x} e^{-t|\beta|} d\beta\right)\Bigg|_{t=0} \\ =& \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\beta x}(-1)^k|\beta|^k d\beta \\ =& \frac{(-1)^k k!}{\pi|x|^{k+1}}\cos\left(\frac{\pi(k+1)}{2}\right).\end{aligned}$$

□

Remark 3.9. We notice that for $n=1$ the problem above becomes

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t>0$$

subject to the initial conditions

$$\begin{cases} u(x,0) = \delta(x) \\ \frac{\partial u}{\partial t}(x,t)\Bigg|_{t=0^+} = \frac{-1}{\pi|x|^2}\cos\pi \end{cases}$$

which is in accord with

$$\frac{\partial}{\partial t}\frac{t}{\pi(x^2+t^2)}\Big|_{t=0^+} = \frac{1}{\pi x^2}.$$

The connection between wave equations and the composition of two independent Cauchy processes $C^1(|C^2(t)|)$, $t>0$ has been investigated in D'Ovidio and Orsingher [1] and more general results involving the Cauchy process have been presented in Nane [8].

Remark 3.10. We finally notice that the equation

$$\frac{\partial^6 u}{\partial t^6} + \frac{\partial^6 u}{\partial x^6} = 0 \tag{3.35}$$

can be decoupled as

$$\left(\frac{\partial^3}{\partial t^3} + i\frac{\partial^3}{\partial x^3}\right)\left(\frac{\partial^3}{\partial t^3} - i\frac{\partial^3}{\partial x^3}\right)u = 0. \tag{3.36}$$

Form the Corollary 3.3, the solution to (3.36) can be therefore written as

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\left[\frac{\frac{\sqrt{3}}{2}te^{i\frac{\pi}{6}}}{\left(x+\frac{te^{i\frac{\pi}{6}}}{2}\right)^2 + \frac{3}{4}t^2e^{i\frac{\pi}{3}}} + \frac{\frac{\sqrt{3}}{2}te^{-i\frac{\pi}{6}}}{\left(x+\frac{te^{-i\frac{\pi}{6}}}{2}\right)^2 + \frac{3}{4}t^2e^{-i\frac{\pi}{3}}}\right] \\ =& \frac{\sqrt{3}}{2^2\pi}t\Bigg[\frac{e^{i\frac{\pi}{6}}\left(x^2+\frac{t}{4}e^{-i\frac{\pi}{3}}+xte^{-i\frac{\pi}{6}}+\frac{3}{4}t^2e^{-i\frac{\pi}{3}}\right)}{\left(x^2+\frac{t}{4}e^{-i\frac{\pi}{3}}+xte^{-i\frac{\pi}{6}}+\frac{3}{4}t^2e^{-i\frac{\pi}{3}}\right)\left(x^2+\frac{t}{4}e^{i\frac{\pi}{3}}+xte^{i\frac{\pi}{6}}+\frac{3}{4}t^2e^{i\frac{\pi}{3}}\right)} \\ &+ \frac{e^{-i\frac{\pi}{6}}\left(x^2+\frac{t}{4}e^{i\frac{\pi}{3}}+xte^{i\frac{\pi}{6}}+\frac{3}{4}t^2e^{i\frac{\pi}{3}}\right)}{\left(x^2+\frac{t}{4}e^{-i\frac{\pi}{3}}+xte^{-i\frac{\pi}{6}}+\frac{3}{4}t^2e^{-i\frac{\pi}{3}}\right)\left(x^2+\frac{t}{4}e^{i\frac{\pi}{3}}+xte^{i\frac{\pi}{6}}+\frac{3}{4}t^2e^{i\frac{\pi}{3}}\right)}\Bigg] \\ =& \frac{\sqrt{3}}{2^2\pi}t\frac{(x^2+t^2)\cos\frac{\pi}{6}+xt}{\left(x^2+te^{-i\frac{\pi}{3}}+xte^{-i\frac{\pi}{6}}+\right)\left(x^2+te^{i\frac{\pi}{3}}+xte^{i\frac{\pi}{6}}\right)}\end{aligned}$$

$$=\frac{\sqrt{3}}{2^2\pi}t\frac{(x^2+t^2)\cos\frac{\pi}{6}+xt}{\left(x^2+t^2+xt\cos\frac{\pi}{6}\right)^2+2x^2t^2\cos\frac{\pi}{3}}. \tag{3.37}$$

Equation (3.36) is satisfied by the Cauchy density and therefore by the probability law (3.37) which however is no longer a Cauchy distribution but is unimodal and asymmetric.

References

[1] M. D'Ovidio and E. Orsingher. Composition of processes and related partial differential equations. *J. Theor. Probab.*, 24:342 – 375, 2011.

[2] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series and products.* Academic Press, 2007. Seventh edition.

[3] K. J. Hochberg. A signed measure on path space related to Wiener measure. *Ann. Probab.*, 6:433 – 458, 1978.

[4] V. Yu. Krylov. Some properties of the distribution corresponding to the equation $\frac{\partial u}{\partial t}=(-1)^{q+1}\frac{\partial u^{2q}}{\partial x^{2q}}$. *Soviet Math. Dokl.*, 1:760 – 763, 1960.

[5] A. Lachal. Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations. *Elect. J. Probab.*, 8 (20):1 – 53, 2003.

[6] V.I. Ladokhin. On the measure on functional spaces corresponding to complex and diffusion coefficients. *Uchenye Zapiski Kazan Univ.*, 123:36 – 42, 1963. in Russian.

[7] N. N. Lebedev. *Special functions and their applications.* Dover, New York, 1972.

[8] E. Nane. Higher order pde's and iterated processes. *Trans. Amer. Math. Soc.*, 360(5):2681 – 2692, 2008.

[9] E. Orsingher. Processes governed by signed measures connected with third-order "heat-type" equations. *Litovsk. Mat. Sb.*, 31(2):323 – 336, 1991. Translation in Lithuanian Math. J. 31 (1991), no. 2, 220 - 231 (1992).

[10] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.*, 37:206 – 249, 2009.

[11] E. Orsingher and M. D'Ovidio. Vibrations and fractional vibrations of rods, plates and Fresnel pseudo-processes. *J. Stat. Phys.*, 145:143 – 174, 2011. Published online: 01 September 2011.

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