Estimating the Risk-Adjusted Capital is an Affair in the Tails

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Davide Canestraro
SCOR SE, Zurich Branch
dcanestraro@scor.com

Michel M. Dacorogna
SCOR SE, Zurich Branch
mdacorogna@scor.com

Fabio Grasso
Sapienza - University of Rome
fabio.grasso@uniroma1.it

Abstract

(Re)insurance companies need to model their liabilities' portfolio to compute the risk-adjusted capital (RAC) needed to support their business. The RAC depends on both the distribution and the dependence functions that are applied among the risks in a portfolio. We investigate the impact of those assumptions on an important concept for (re)insurance industries: the diversification gain. Several copulas are considered in order to focus on the role of dependencies. To be consistent with the frameworks of both Solvency II and the Swiss Solvency Test, we deal with two risk measures: the Value-at-Risk and the expected shortfall. We highlight the behavior of different capital allocation principles according to the dependence assumptions and the choice of the risk measure.

Keywords

Capital Allocation, Copula, Dependence, Diversification Gain, Model Uncertainty, Monte Carlo Methods, Risk-Adjusted Capital, Risk Measure

1 Introduction

The new risk based solvency regulations require (re)insurance companies to model their liabilities to compute the risk-adjusted capital (RAC) needed to support their business. This concept relies heavily on the portfolio model that is at the heart of this computation. Particularly, it depends on both the distribution functions (dfs) used to model the individual risks and the dependence functions that are applied among those risks. Besides the debate among regulators on the type of risk measures that should be applied for estimating the RAC, there are few studies that systematically explore the impact of those assumptions on the results of the model, see e.g. Bagarry (2006) and Desmedt and Walhin (2008).

The aim of this paper is to explore various choices of models and to show how they influence an important concept for the (re)insurance industry: the diversification gain. Considering a simple portfolio composed of two risks, namely X and Y, we define the diversification gain according to Bürgi et al. (2009). This definition requires the choice of a risk measure for computing the risk-adjusted capital of a single risk and of the portfolio, see SCOR (2008). In our analysis we deal with two well known measures of risk, i.e. the Value-at-Risk (VaR) and the expected shortfall (ES), in agreement with Embrechts et al. (2005). To be consistent with the frameworks of both Solvency II and of the Swiss Solvency Test, we consider those risk measures at the 99.5% and 99% threshold, respectively. Regulators are expecting companies to model dependence between the various risk factors (see for instance FINMA (2008) on SST). This work is done to explore the effect of modeling choices on portfolio results and particularly on the diversification benefit measured in the model.

In order to focus on dependence and tail assumptions, the study is performed assuming identical marginal dfs for X and Y. Two typical distributions have been chosen: the lognormal df, which is very popular for modeling insurance risks and the Fréchet df, to explore extreme value distributions (see Embrechts et al. (1997)).

For computing the joint df, we use copulas to model dependencies. In particular, two families of copulas are taken into account: elliptical and Archimedean. Within the former family the Gaussian and the Student-t copula are considered, while from the latter family, we study the Clayton, the Frank and the Gumbel copula. Moreover, we flip Archimedean copulas in order to investigate the behavior of the portfolio in cases characterized by stronger/weaker tail dependencies.

In a first stage of the work, Monte Carlo methods are used to obtain estimations of expected value, VaR, ES and RAC, for both the single risks and the portfolio. With those values we can estimate the diversification gain obtained by combining both risks. To ensure a consistent comparison, different copulas are parameterized using the same value for Kendall's tau rank correlation coefficient. This allows us to impute differences due to the structure of dependence whilst its strength remains constant.

In a second stage, we consider the required capital, in terms of RAC, according to two different allocation principles related to the choice of the risk measure. The Euler principle and the haircut allocation principle are compared. We analyze both the change of the dependence strength, indicated by the value of Kendall's tau, and the change in the joint distribution, described by the choice of a specific copula. For ease of comparison, we limit ourselves to the aggregation of two underlying risks, but the method and the concept of this study can easily be extended to dimensions larger than two.

The rest of this paper is organized as follows. An overview on copulas and a description of the main families are given in Section 2. Section 3 introduces rank correlation and its link to copulas. In Section 4, we focus on the evaluation of the RAC and of the diversification gain. The impact of dependence on capital allocation is discussed in Section 5. Section 6 gives an outlook on future research and we conclude in Section 7.

2 Copulas and Structure of Dependence

Copulas were originally introduced as a useful mathematical tool to model dependence. An interesting review of the development of copula theory and its applications is found in Genest et al. (2008). A must-read literature on copulas is given in Embrechts (2009).

Definition 2.1. *A function $C : [0,1]^2 \to [0,1]$ is a copula if:*

1. $C(u,0) = C(0,v) = 0$ *for all* $u, v \in [0,1]$*;*
2. $C(u,1) = u$ *and* $C(1,v) = v$ *for all* $u, v \in [0,1]$*;*
3. C *is quasi-monotone, i.e., for any* $0 \le u_1 \le u_2 \le 1$ *and any* $0 \le v_1 \le v_2 \le 1$*,*

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \ge 0. \tag{2.1}$$

In other words, a bivariate copula is a cumulative distribution function (cdf) on $[0,1]^2$ whose marginals are standard uniform.

The usefulness of copulas for describing dependencies is revealed by Sklar's theorem (see Sklar (1959)). Sklar's result shows that copulas allow to separate the dependence structure from the behavior of the univariate marginals. This attractive feature of the copula representation invites to interpret a copula, associated with a random vector, as being its dependence structure.

There are essentially two types of copulas according to the way they are obtained:

- **Implicit copulas.** We can extract an implicit copula from any distribution with continuous marginal dfs; examples of this type are the elliptical copulas, i. e. the copulas derived from elliptically contoured (or elliptical) distributions.
- **Explicit copulas.** There are many copulas which we can write down in a simple closed form; the Archimedean family yields examples of this type.

Among elliptical copulas, in this paper we deal with the Gauss copula and the Student-t copula, extracted from the multivariate normal distribution and from the multivariate Student-t distribution, respectively.

- **Gauss.**
$$C^{Ga}_{\rho}(u,v)=\mathbf{\Phi}_{\rho}\left(\Phi^{-1}(u),\Phi^{-1}(v)\right),\qquad \rho\in(-1,1), \tag{2.2}$$
where Φ is the cdf of a standard univariate normal distribution and $\mathbf{\Phi}_{\rho}$ denotes the cdf of a bivariate normal with standard marginals and correlation coefficient ρ.

- **Student-t.**
$$C^{t}_{\nu,\rho}(u,v)=\mathbf{t}_{\nu,\rho}\left(t_{\nu}^{-1}(u),t_{\nu}^{-1}(v)\right),\qquad \rho\in(-1,1),\nu>0, \tag{2.3}$$
where t_{ν} is the cdf of a standard univariate Student-t distribution with ν degrees of freedom and $\mathbf{t}_{\nu,\rho}$ denotes the cdf of a bivariate Student-t with standard t_{ν} marginals and correlation coefficient ρ.

In contrast to the Gauss copula, the Student-t copula allows for joint heavy tails and an increased probability of joint extreme events. Moreover, the Student-t copula introduces an additional parameter, namely the degrees of freedom ν. Since the Student distribution tends to the Gaussian when $\nu\to\infty$, increasing the value of ν decreases the tendency to exhibit extreme co-movements.

Another interesting family of copulas are the Archimedean copulas. They are commutative, that is $C(u,v)=C(v,u)$ for all $u,v\in[0,1]$, and associative, that is $C(C(u,v),z)=C(u,C(v,z))$ for all $u,v,z\in[0,1]$. Archimedean copulas can be constructed through a function ϕ called generator (see Embrechts et al. (2005)). A generator characterizes a specific copula within the Archimedean family. If we choose as generator $\phi(t)=-\ln(t)$ we obtain the independence copula C^{Π}, which corresponds to the case of independent random variables. Other Archimedean copulas considered in this paper are the Clayton, the Frank and the Gumbel copula.

- **Clayton.** $\phi(t)=\frac{1}{\theta}(t^{-\theta}-1)$, hence
$$C^{Cl}_{\theta}(u,v)=\left[\max\left(u^{-\theta}+v^{-\theta}-1,0\right)\right]^{-1/\theta},\qquad \theta\in[-1,\infty)\backslash\{0\}. \tag{2.4}$$

- **Frank.** $\phi(t)=-\ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$, hence
$$C^{Fr}_{\theta}(u,v)=-\frac{1}{\theta}\ln\left(1+\frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}\right),\qquad \theta\in\mathbb{R}. \tag{2.5}$$

- **Gumbel.** $\phi(t)=(-\ln t)^{\theta}$, thus
$$C^{Gu}_{\theta}(u,v)=\exp\left(-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{1/\theta}\right),\qquad \theta\geq 1. \tag{2.6}$$

Both the Clayton and the Gumbel copula, unlike the above mentioned elliptical copulas, allow for asymmetries. The Clayton copula exhibits greater dependence in the lower tail than in the upper tail, while the opposite is valid for the Gumbel copula.

In insurance, as pointed out in Bürgi et al. (2009), the dependence ought to be stressed in the upper tail, because the few biggest claims are more relevant for the company than the large amount of small claims, which are often contractually cut-off. Hence we work also with a flipped Clayton copula, obtained by the transformation $(u,v)\to(1-u,1-v)$. We refer to this copula as the Clayton-M, indicating it with C^{Cl-M}_{θ}. For the sake of completeness, we consider also a flipped Gumbel copula. We refer to this copula as the Gumbel-M, indicating it with C^{Gu-M}_{θ}. We do not consider the flipped version of the Frank copula since this copula is the only Archimedean copula characterized by radial symmetry, like the Gauss and the Student-t. More on flipped copulas can be found in Venter (2002), pp. 90-91.

Both the Clayton and the Frank copulas are comprehensive copula families. They allow, depending on the value of their parameter, to describe countermonotonicity, full independence and comonotonicity.

An extensive list of Archimedean copulas can be found in Nelsen (2006), pp. 116-119, where 22 families are listed.

3 Rank Correlation and Strength of Dependence

Our next aim is to introduce a measure of dependence, namely the Kendall's tau rank correlation coefficient, which we adopt as a benchmark for choosing the strength of dependence. Kendall's tau was originally discussed by G. T. Fechner around 1900 and rediscovered by the British statistician Sir Maurice Kendall in Kendall (1938). For a complete historical review we refer to Kruskal (1958).

Definition 3.1. *For the random pair (X, Y) the Kendall's tau is defined as*

$$\rho_\tau(X, Y) = \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\}, \tag{3.1}$$

where $(\tilde{X}, \tilde{Y})$ is an independent copy of (X, Y).

As can be seen from its definition, Kendall's tau for (X, Y) is simply the probability of concordance minus the probability of discordance. For continuous marginal distributions and contrary to the linear correlation coefficient ρ, named after the British statistician Karl Pearson, Kendall's tau depends only on the unique copula of the risks.

Proposition 3.2. *Let (X, Y) be a vector of continuous random variables with copula C. Then Kendall's tau for (X, Y) is given by*

$$\rho_\tau(X, Y) = 4 \int\int_{[0,1]^2} C(u, v) dC(u, v) - 1. \tag{3.2}$$

This is equivalent to say

$$\rho_\tau(X, Y) = 4\mathbb{E}(C(U, V)) - 1, \tag{3.3}$$

where U,V are standard uniform.

The fact that ρ_τ is a copula-based measure implies that it inherits its property of invariance under strictly increasing transformations (see Embrechts et al. (2005), p. 188, for more on this property).

We collect some facts and useful considerations about Kendall's tau in the next theorem.

Theorem 3.3 (Kendall's tau)**.** *Let X and Y be random variables with continuous distributions F and G, joint distribution H and copula C. The following are true:*

- $\rho_\tau(X, Y) = \rho_\tau(Y, X)$.
- *If X and Y are independent then $\rho_\tau(X, Y) = 0$.*
- $-1 \le \rho_\tau(X, Y) \le 1$.
- *For $T : \mathbb{R} \to \mathbb{R}$ strictly monotone on $Ran(X)$, ρ_τ satisfies $\kappa(T(X), Y) = \kappa(X, Y)$ or $\kappa(T(X), Y) = -\kappa(X, Y)$ if T is increasing or decreasing, respectively.*
- $\rho_\tau(X, Y) = 1 \Leftrightarrow Y = T(X)$ *a.s. with T increasing.*
- $\rho_\tau(X, Y) = -1 \Leftrightarrow Y = T(X)$ *a.s. with T decreasing.*

Proof. See Embrechts et al. (2002), p. 196.

Moreover, for specific copula families, it is possible to establish simpler relationships between the copula itself and Kendall's tau.

Theorem 3.4 (Kendall's tau for Gauss copula)**.** *The Gauss copula defined as in (2.2) satisfies*

$$\rho_\tau\left(C_\rho^{Ga}\right) = \frac{2}{\pi} \arcsin \rho \tag{3.4}$$

Proof. See Embrechts et al. (2005), pp. 215, 216.

The relationship (3.4) holds more generally for all elliptical distribution, and hence also for the Student-t copula.

It is possible to link Kendall's tau to the generator of a specific Archimedean copula.

Copula	C_ρ^{Ga}	$C_{\nu,\rho}^{t}$	C_θ^{Cl}	C_θ^{Fr}	C_θ^{Gu}
ρ_τ	$\frac{2}{\pi}\arcsin\rho$	$\frac{2}{\pi}\arcsin\rho$	$\frac{\theta}{(\theta+2)}$	$1-\frac{4}{\theta}+\frac{4D_1(\theta)}{\theta}$	$1-\frac{1}{\theta}$

Table 1: Kendall's tau for specific copulas where $D_1(\theta)=\theta^{-1}\int_0^\theta t/(\exp(t)-1)dt$.

Theorem 3.5 (Kendall's tau for Archimedean copulas)**.** *Let X and Y be continuous random variables with unique Archimedean copula C and generator ϕ. Then*

$$\rho_\tau(X,Y) = 1+4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt, \tag{3.5}$$

$$= 1-4\int_0^\infty t\left(\frac{d}{dt}\phi^{[-1]}(t)\right)^2 dt. \tag{3.6}$$

Proof. For (3.5) see Genest and MacKay (1986), pp. 282-283; for (3.6) see Joe (1997).

Through this result explicit relationships between Kendall's tau and Archimedean copula parameters can be found. These are summarized in Table 1.

There are other copula-based dependence measures called coefficients of tail dependence. They specifically measure how the tails of the distribution, rather than the entire random variables, are correlated. As we are not using them in this study, we just mention that they exist and that their explicit formulas for various copulas can be found in Embrechts et al. (2005).

4 Dependence and Diversification Gain

In this section, we investigate through Monte Carlo methods the role of dependence on the risk-adjusted capital. Some preliminary concepts about risk measures are provided below.

Definition 4.1. *For a random variable X with $E(|X|)<\infty$ and df F, the VaR at confidence level $\alpha\in(0,1)$ is defined as*

$$VaR_\alpha(X) = \inf\{x\in\mathbb{R} : P(X>x)\le 1-\alpha\} = \inf\{x\in\mathbb{R} : F(x)\ge\alpha\}. \tag{4.1}$$

VaR is not a coherent measure of risk, according to the characterization given in Artzner et al. (1999).

A coherent measure of risk is the expected shortfall (ES). There is not a unique definition in the literature for the ES, see Acerbi and Tasche (2002). The nomenclature we adopt is in agreement with Embrechts et al. (2005).

Definition 4.2. *For a random variable X with $E(|X|)<\infty$ and df F, the expected shortfall at confidence level $\alpha\in(0,1)$ is defined as*

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F)du, \tag{4.2}$$

where $q_u(F)$ is the quantile function of F.

Hence the expected shortfall, contrary to VaR, takes (the shape of) the tail into account. ES is always greater or equal to VaR for a chosen confidence level α, i.e. $\mathrm{ES}_\alpha(X)\ge\mathrm{VaR}_\alpha(X)$.

For continuous distributions a more intuitive expression can be derived which shows that ES can be interpreted as the expected value given that VaR is exceeded.

Proposition 4.3. *For a random variable X with $E(|X|)<\infty$ and continuous df F, the expected shortfall at confidence level $\alpha\in(0,1)$ satisfies*

$$ES_\alpha(X) = E[X \mid X\ge VaR_\alpha(X)]. \tag{4.3}$$

Since we deal with lognormal and Fréchet marginal distributions, we provide for these dfs analytic formulas, which permit to compute both of the above mentioned risk measures.

As shown in Denuit et al. (2005), p. 98, for a lognormal random variable $X \sim \log\mathrm{N}(\mu, \sigma^2)$ we have

$$\mathrm{VaR}_\alpha(X) = \exp\big(\mu + \sigma\Phi^{-1}(\alpha)\big), \tag{4.4}$$

$$\mathrm{ES}_\alpha(X) = \exp\big(\mu + \sigma^2/2\big)\Big(\frac{1 - \Phi(\Phi^{-1}(\alpha) - \sigma)}{(1-\alpha)}\Big), \tag{4.5}$$

where Φ is the standard normal df.

For a Fréchet random variable X with shape parameter a and scale parameter s (see Appendix A), we have

$$\mathrm{VaR}_\alpha(X) = s(-\ln(\alpha))^{(-1/a)}, \tag{4.6}$$

$$\mathrm{ES}_\alpha(X) = s\frac{1}{1-\alpha}\Gamma\left(1 - \frac{1}{a}, -\ln(\alpha)\right)\Gamma\left(1 - \frac{1}{a}\right), \tag{4.7}$$

where $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function, i.e. $\Gamma(\gamma, x) = \int_0^x x^{\gamma-1}\exp(-x)dx/\Gamma(\gamma)$.

We use the notation $\varrho_\alpha(X)$ to indicate the value of a generic risk measure, at confidence level α, for a risk X. According to the Swiss Solvency Test (SST) guidelines, Swiss-based insurances have to adopt as risk measure the ES at 99%. In order to meet the solvency requirements under the Solvency 2 guidelines, European insurances will have to utilize as the risk measure the VaR calibrated to a confidence level of 99.5%. In our analysis we explore both cases.

The choice of the risk measure is preliminary to the evaluation of the RAC, as defined below.

Definition 4.4. *We define the RAC as the uncertainty around the expectation, i.e.*

$$RAC_{\varrho_\alpha}(X) = \varrho_\alpha(X) - E(X). \tag{4.8}$$

The main goal of this section is to provide a quantitative judgment about the diversification gain, as defined in Bürgi et al. (2009). The diversification gain represents the percentage of the RAC that a (re)insurance company can save in the management of its portfolio by taking into account the positive effect of aggregating various risks.

Definition 4.5. *We define the diversification gain for a portfolio Z, aggregating the risks X and Y, as*

$$D_{\varrho_\alpha}(Z) = 100\% - \frac{RAC_{\varrho_\alpha}(X+Y)}{RAC_{\varrho_\alpha}(X) + RAC_{\varrho_\alpha}(Y)}. \tag{4.9}$$

In a first stage, we study a simple portfolio Z, composed of two risks both lognormally distributed, namely $X, Y \sim \log\mathrm{N}(9.58, 0.83)$. The lognormal df is very popular for modeling insurance risks and we select μ and σ such that the coefficient of variation, that is the ratio of the standard deviation to the mean, is equal to one, which represents a high insurance risk. For computing the joint df, we use copulas to model the structure of dependence. According to the families introduced in Section 2, we examine two elliptical copulas, the Gauss and the Student-t, together with three Archimedean copulas, the Clayton, the Frank and the Gumbel. Moreover, as previously mentioned, we consider two flipped copulas, namely the Clayton-M and the Gumbel-M, to investigate the importance of tail dependencies. For the Student-t copula, if not specified, we assume $\nu = 1$.

Concerning the strength of the dependence and in order to ensure consistent comparison, different copulas are parameterized using the same value for Kendall's tau, through the equations given in Table 1. Figure 1 illustrates how different copulas imply different structures of dependence.

Using MATLAB, we implement ad-hoc procedures to compute all the values necessary to quantify the diversification gain. We simulate a realization (x, y) of the bivariate random vector (X, Y), according to the specific copula, to its parametrization and to the marginal distributions. Marshall and Olkin (1988) provide an algorithm for the simulation of Archimedean copulas. Algorithms for simulating from both the Gauss and the Student-t copula can be found in Embrechts et al. (2005). We repeat the simulation process 10^7 times for each characterization of the bivariate distribution. This allows us to derive precise Monte Carlo estimates for the portfolio of the expected value,

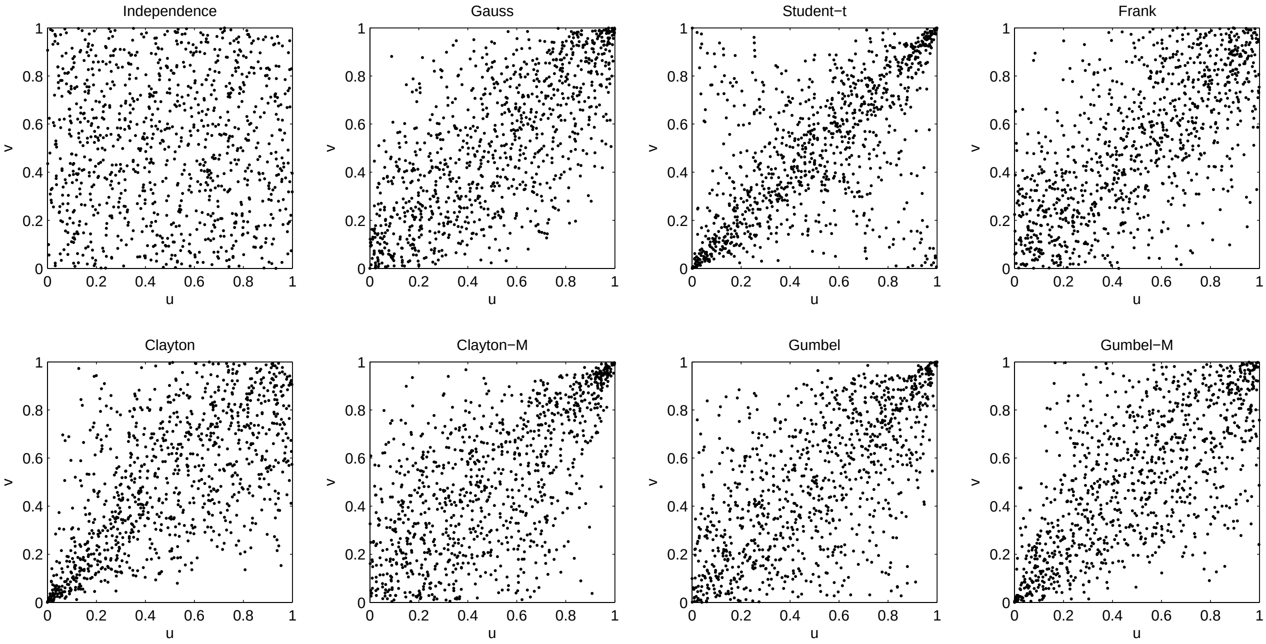

Figure 1: Rank scatter plots for copulas parameterized with $\rho_\tau = 0.5$.

$E(Z)$, the Value-at-Risk at 99.5%, $\mathrm{VaR}_{99.5\%}(Z)$, and the expected shortfall at 99%, $\mathrm{ES}_{99\%}(Z)$. For the sake of simplicity and without loss of generality, we drop percentiles from notation, keeping them fixed at the above mentioned levels.

From these results and from (4.8), we calculate the $\mathrm{RAC}_{\mathrm{VaR}}(Z)$ and the $\mathrm{RAC}_{\mathrm{ES}}(Z)$ according to the adopted measure of risk. Finally, in accordance to (4.9), we determine the $\mathrm{D}_{\mathrm{VaR}}(Z)$ and the $\mathrm{D}_{\mathrm{ES}}(Z)$.

In Table 2, we provide the results for three different values of Kendall's tau, namely 0.05, 0.35 and 0.7. For each determination of Kendall's tau, the corresponding values of the copula parameters are given. The Student-t copula requires an additional parameter, the degrees of freedom, that is not implied by the Kendall's tau. We select three values for ν, namely 1, 3 and 7, to have a more complete picture about the Student-t copula.

Looking at the figures displayed in Table 2, a first consideration can be drawn about risk measures. For this type of df, characterized by a (moderately) heavy tail, the difference between $\mathrm{VaR}_{99.5\%}(Z)$ and $\mathrm{ES}_{99\%}(Z)$ is relatively small. Nonetheless, due to this difference, the RAC increases, as expected, moving from VaR to ES. Regarding RAC, empirical results confirm the common intuition. The risk-adjusted capital increases as the strength of the dependence increases. For the Gauss copula, we have $\mathrm{RAC}_{\mathrm{ES}}(Z) = 153{,}605$ if $\rho_\tau = 0.05$, $\mathrm{RAC}_{\mathrm{ES}}(Z) = 186{,}401$ if $\rho_\tau = 0.35$ and $\mathrm{RAC}_{\mathrm{ES}}(Z) = 222{,}244$ if $\rho_\tau = 0.70$. Consequently, the diversification gain reflects these movements. Looking still at the Gauss copula, we obtain $\mathrm{D}_{\mathrm{ES}}(Z) = 34.31\%$ if $\rho_\tau = 0.05$, $\mathrm{D}_{\mathrm{ES}}(Z) = 20.27\%$ if $\rho_\tau = 0.35$ and $\mathrm{D}_{\mathrm{ES}}(Z) = 5.03\%$ if $\rho_\tau = 0.70$. A similar behavior is obtained in the case that VaR is considered instead of ES.

Not considering the limiting case of the independence copula and focusing on the structure of dependence, it is interesting to observe among the analyzed copulas a certain order with respect to conservativeness. Both when $\rho_\tau = 0.35$ and when $\rho_\tau = 0.70$, the Clayton copula provides the highest diversification gain, followed by the Frank, the Gumbel-M, the Gauss, the Student-t, the Gumbel and the Clayton-M copula. Further analysis have taken into account more values for rank correlations. The trend of the diversification gain is presented in case of ES in Figure 2 and in case of VaR in Figure 3. In Figure 2, we see clearly the behavior mentioned above. Except for very low levels of dependence, the Clayton-M copula is the most conservative in terms of diversification gains. This means that when a (re)insurance company has to assume a copula to model the dependence among the risks of its (bivariate) portfolio, the C_θ^{Cl-M} would guarantee a prudent choice. This is to be expected, since the Clayton-M copula concentrates the dependence in the right tail which is the one that matters for the computation of the RAC.

	Clayton-M	Gumbel	Student-t			Gauss	Gumbel-M	Frank	Clayton	Independence
$\boldsymbol{\rho_\tau = 0.05}$	$\theta = 0.1053$	$\theta = 1.0526$	$\rho = 0.0785$			$\rho = 0.0785$	$\theta = 1.0526$	$\theta = 0.4509$	$\theta = 0.1053$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$					
$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	143,013	143,401	164,739	150,505	143,192	136,844	135,210	134,990	133,778	132,316
$\mathrm{RAC}_{\mathrm{ES}}(Z)$	161,740	163,955	190,791	174,028	163,076	153,605	151,739	151,357	150,213	148,872
$\mathrm{D}_{\mathrm{VaR}}(Z)$	30.14%	29.98%	19.88%	26.42%	29.95%	33.09%	33.97%	33.87%	34.52%	35.32%
$\mathrm{D}_{\mathrm{ES}}(Z)$	30.83%	29.93%	18.75%	25.65%	30.27%	34.31%	35.17%	35.23%	35.75%	36.31%
$\boldsymbol{\rho_\tau = 0.35}$	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$	$\theta = 1.5385$	$\theta = 3.5088$	$\theta = 1.0769$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$					
$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	192,821	186,106	183,074	176,609	171,149	165,559	154,874	150,180	142,738	132,316
$\mathrm{RAC}_{\mathrm{ES}}(Z)$	221,479	213,935	210,572	203,735	195,191	186,401	173,540	167,197	159,379	148,872
$\mathrm{D}_{\mathrm{VaR}}(Z)$	5.81%	9.11%	10.43%	13.74%	16.39%	19.00%	24.30%	26.70%	30.19%	35.32%
$\mathrm{D}_{\mathrm{ES}}(Z)$	5.47%	8.62%	9.84%	13.23%	16.58%	20.27%	25.86%	28.73%	31.90%	36.31%
$\boldsymbol{\rho_\tau = 0.70}$	$\theta = 4.6667$	$\theta = 3.3333$	$\rho = 0.8910$			$\rho = 0.8910$	$\theta = 3.3333$	$\theta = 11.4115$	$\theta = 4.6667$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$					
$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	203,374	202,195	198,578	197,755	196,616	194,853	185,855	169,621	158,278	132,316
$\mathrm{RAC}_{\mathrm{ES}}(Z)$	232,333	231,268	227,053	226,362	225,281	222,244	209,851	186,778	175,205	148,872
$\mathrm{D}_{\mathrm{VaR}}(Z)$	0.44%	1.28%	2.77%	3.10%	3.87%	4.70%	9.10%	17.20%	22.59%	35.32%
$\mathrm{D}_{\mathrm{ES}}(Z)$	0.43%	1.24%	2.63%	3.01%	3.84%	5.03%	10.35%	20.23%	25.13%	36.31%

Table 2: Results for a portfolio composed of $X, Y \sim \mathrm{logN}(9.58, 0.83)$.

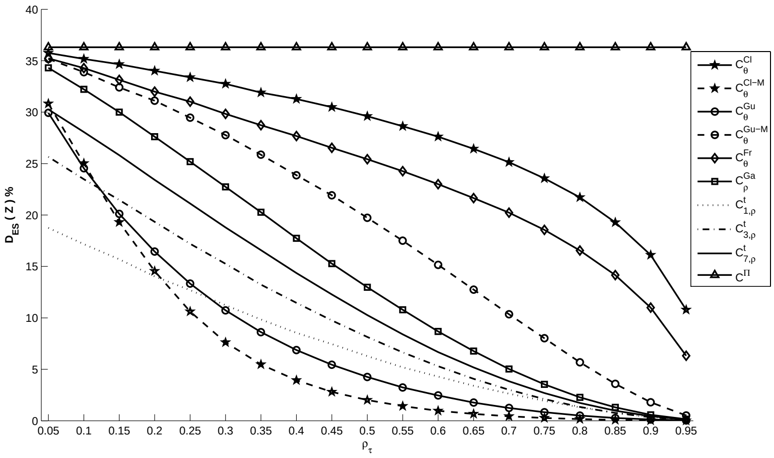

Figure 2: Diversification gain as a function of both the strength and the structure of dependence. The risk measure is the expected shortfall. X and Y are both lognormally distributed.

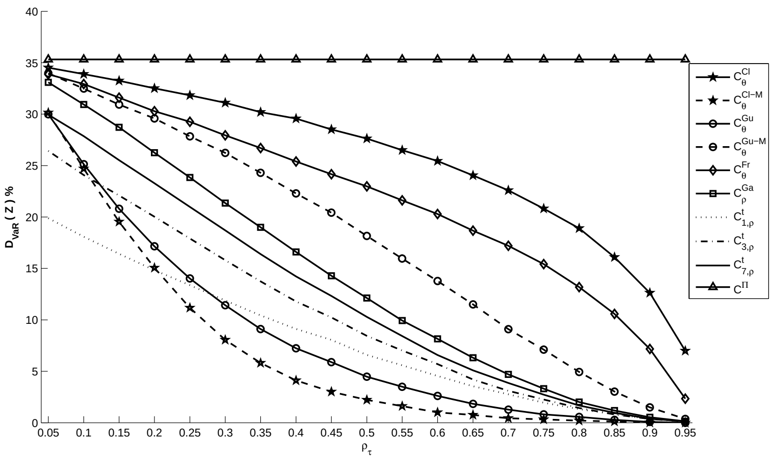

Figure 3: Diversification gain as a function of both the strength and the structure of dependence. The risk measure is the Value-at-Risk. X and Y are both lognormally distributed.

For instance, when $\rho_\tau = 0.35$, if the model is based on a Gauss copula, then the (re)insurance company can claim a diversification gain, $\mathrm{D}_{\mathrm{ES}}(Z)$, of more than 20% while if the real model would follow a Clayton-M copula, then the $\mathrm{D}_{\mathrm{ES}}(Z)$ would be only a bit more than 5%. Thus, a warning has to be sent with regard to the cautiousness of certain assumptions. Only for very low dependencies, the Student-t copula with $\nu = 1$ supplies the lowest diversification gain among the set of copulas considered. Another interesting feature illustrated in Figure 2 and anticipated by a theoretical property mentioned in Section 2, is that the results provided by the Student-t copula tend to those given by the Gauss copula, as the degrees of freedom increase. When $\nu = 7$, the divergence is already significantly reduced. Note that as $\rho_\tau \to 0$ then $\mathrm{D}_{\mathrm{ES}}(Z) \to\approx 36.30\%$ ($\mathrm{D}_{\mathrm{VaR}}(Z) \to\approx 35.30\%$) for all the copulas analyzed except the Student-t. This is not the case for the Student-t copula (if $\nu < \infty$) since this copula family gives asymptotic dependence in the tail even when $\rho = 0$ (i.e. when $\rho_\tau = 0$).

In Figure 3, based on VaR as risk measure, all the remarks valid for Figure 2 are confirmed. Hereafter, we use the ES to illustrate the results, pointing out differences with VaR, if any. Further, unreported, analysis have been conducted to check the behavior under other parameterizations of the lognormal dfs. No particular discrepancy has emerged.

In Figure 4, we can see the influence of dependence on the RAC. For each copula model, the darker column quantifies the $\mathrm{RAC}_{\mathrm{VaR}}(Z)$, and the lighter column represents the $\mathrm{RAC}_{\mathrm{ES}}(Z)$. The lines refer to the diversification gain. As noticed above, the difference among the two risk measures used is modest within this portfolio.

To explore extreme value distributions, we repeat all of the previous analysis with a new portfolio Z. Z is now composed from two risks, X and Y, both Fréchet distributed with $a = 1.5$ and $s = 4,657.15$. The shape parameter a is chosen to grant both an important tail and sufficient stability in the simulation process. In Table 3 we provide results for $\rho_\tau = 0.35$.

Focusing on $\mathrm{D}_{\mathrm{ES}}(Z)$, we observe an estimated value ranging from 4.38%, in case we assume a Clayton-M copula, to 18.41%, in case our model is based on a Clayton copula. The main difference, that has to be stressed with respect to the previous portfolio, is the relevance of the choice of the risk measure. For instance, when the copula is the Clayton-M, we have $\mathrm{RAC}_{\mathrm{VaR}}(Z) = 280,527$ and $\mathrm{RAC}_{\mathrm{ES}}(Z) = 547,802$. A similar discrepancy is present regardless of the model used for dependence. The divergence between the capital requirements, according to the risk measure applied, is due to the marginal distributions. The tail of the Fréchet df, contrary to the tail of the lognormal df, emphasizes the diversity between the two risk measures. Indeed, contrary to the VaR, the ES takes into account the shape of the tail, which is of course more important for extreme value distributions than for a lognormal df.

In Figure 5, we illustrate the trend of the diversification gain as a function of the dependence. Remarks provided for Figure 2 are valid here as well.

In Figure 6, we see clearly the influence of dependence on the RAC. The importance of the choice of the risk measure is eye-catching and it is preserved also varying the strength of the dependence.

(Unreported) analysis have been conducted to check the behavior under other parameterizations of the Fréchet dfs. No particular discrepancy has emerged, but the stability of the Monte Carlo process, for $a \to 1$, has to be supported by an increasing number of simulations.

As a further analysis, we mixed a new portfolio $Z = X + Y$ consisting of a moderately heavy tail df and an extreme value df. In particular, X is lognormally distributed with $\mu = 6.52$ and $\sigma = 2.15$, and Y is Fréchet distributed with $a = 1.5$ and $s = 4,657.15$.

The new parametrization of the lognormal df has been chosen such that the weights of both risks are almost equivalent in terms of capital allocation (see Section 5). In Table 4 we provide results for $\rho_\tau = 0.35$.

We observe once more the same order among dependence structures in terms of conservativeness. In Figure 7, the trend of the diversification gain as a function of the dependence is illustrated. No contradiction arose with respect to Figure 2.

In Figure 8, the influence of the dependence on the RAC is highlighted. The presence in the portfolio of one extreme value df is sufficient to emphasize the difference between the $\mathrm{RAC}_{\mathrm{VaR}}(Z)$ and the $\mathrm{RAC}_{\mathrm{ES}}(Z)$.

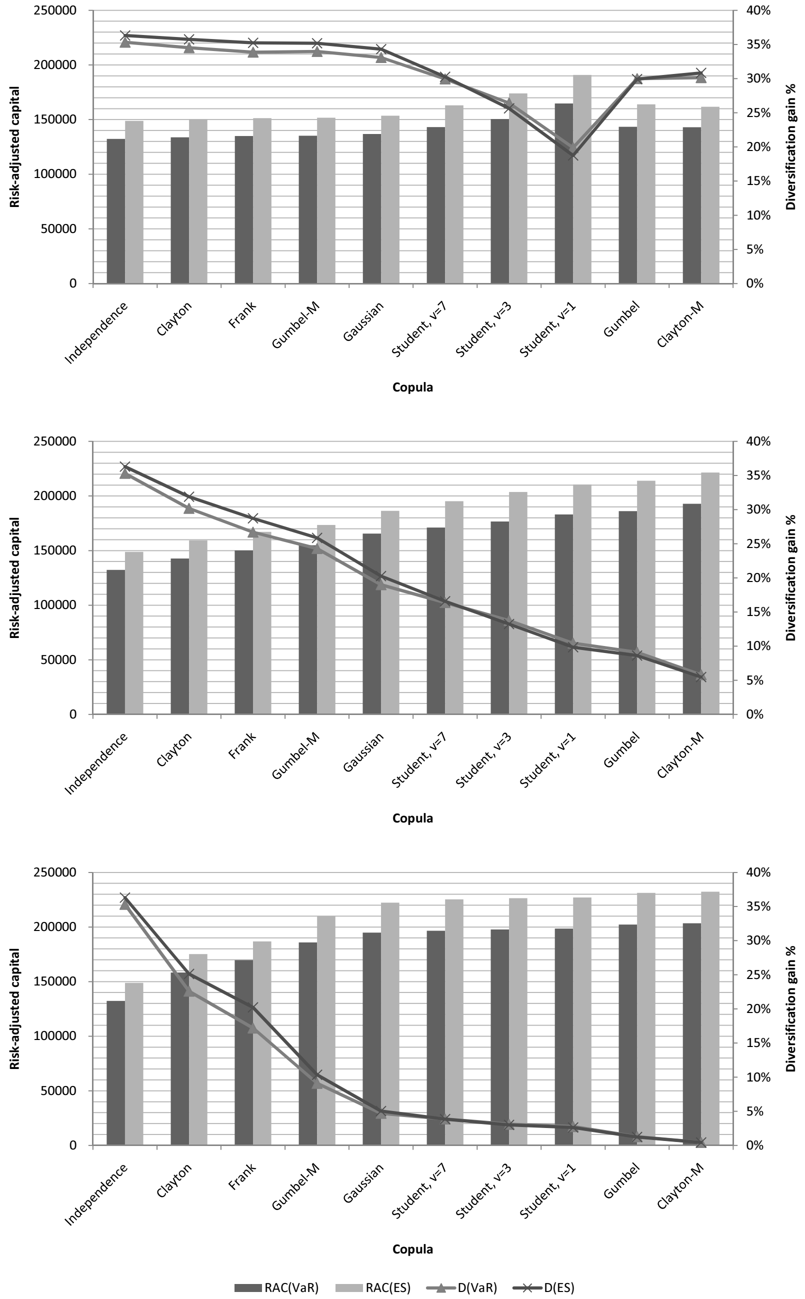

Figure 4: Risk-adjusted capital and diversification gain for different choices of the copula and rank correlations $\rho_\tau = 0.05$ (top), $\rho_\tau = 0.35$ (middle) and $\rho_\tau = 0.70$ (bottom). X and Y are both lognormally distributed.

	Clayton-M	Gumbel	Student-t			Gauss	Gumbel-M	Frank	Clayton	Independence
$\boldsymbol{\rho_\tau = 0.35}$	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$	$\theta = 1.5385$	$\theta = 3.5088$	$\theta = 1.0769$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$					
$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	280,527	273,321	273,036	266,257	265,606	263,426	255,347	252,047	245,257	238,857
$\mathrm{RAC}_{\mathrm{ES}}(Z)$	547,802	536,850	518,970	515,486	510,731	487,801	474,289	470,125	469,401	463,443
$\mathrm{D}_{\mathrm{VaR}}(Z)$	4.35%	6.65%	7.06%	8.79%	9.30%	9.81%	13.03%	14.23%	16.10%	18.34%
$\mathrm{D}_{\mathrm{ES}}(Z)$	4.38%	6.49%	6.87%	9.22%	11.18%	13.68%	16.35%	17.60%	18.41%	19.52%

Table 3: Results for a portfolio composed of X and Y both Fréchet distributed with $a = 1.5$ and $s = 4,657.15$.

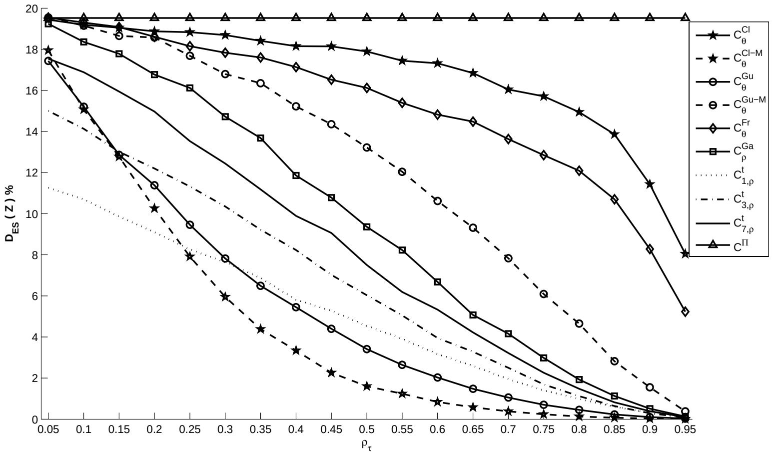

Figure 5: Diversification gain as a function of both the strength and the structure of dependence. The risk measure is the expected shortfall. X and Y are both Fréchet distributed.

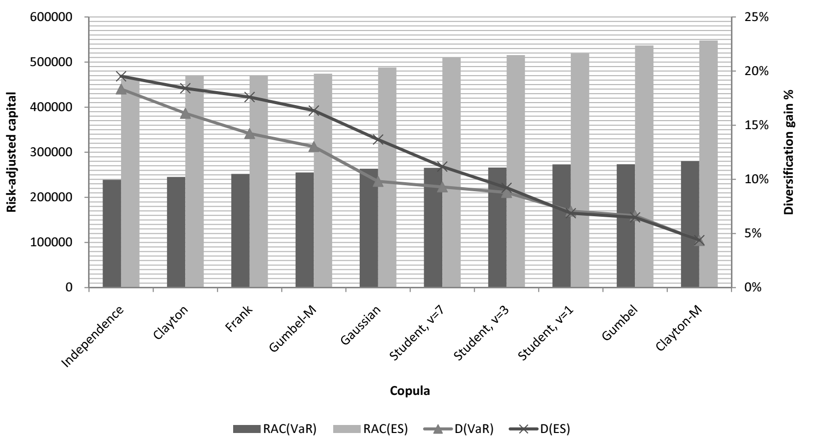

Figure 6: Risk-adjusted capital and diversification gain for different choices of the copula and rank correlations $\rho_\tau = 0.35$. X and Y are both Fréchet distributed.

	Clayton-M	Gumbel	Student-t			Gauss	Gumbel-M	Frank	Clayton	Independence
$\boldsymbol{\rho_\tau = 0.35}$	$\theta = 1.0769$	$\theta = 1.5385$	$\rho = 0.5225$			$\rho = 0.5225$	$\theta = 1.5385$	$\theta = 3.5088$	$\theta = 1.0769$	
			$\nu = 1$	$\nu = 3$	$\nu = 7$					
$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	298,496	292,262	291,237	285,075	282,340	280,407	274,275	267,400	261,589	258,160
$\mathrm{RAC}_{\mathrm{ES}}(Z)$	550,688	545,779	523,580	514,950	506,174	484,198	477,493	462,426	459,977	450,910
$\mathrm{D}_{\mathrm{VaR}}(Z)$	4.46%	6.60%	6.89%	9.06%	9.64%	9.84%	12.25%	14.16%	16.22%	17.53%
$\mathrm{D}_{\mathrm{ES}}(Z)$	4.73%	6.84%	7.15%	9.83%	12.01%	14.62%	17.11%	18.70%	19.63%	20.76%

Table 4: Results for a portfolio composed of X and Y, where X is lognormally distributed with $\mu = 6.52$ and $\sigma = 2.15$, and Y is Fréchet distributed with $a = 1.5$ and $s = 4,657.15$.

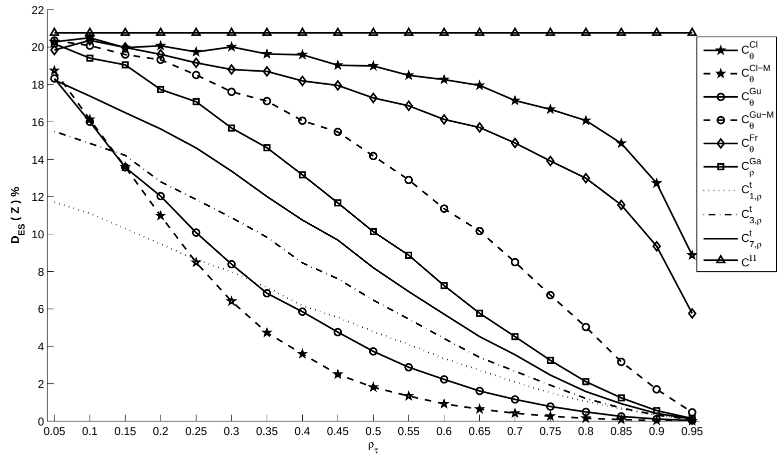

Figure 7: Diversification gain as a function of both the strength and the structure of dependence. The risk measure is the expected shortfall. X is lognormally distributed and Y is Fréchet distributed.

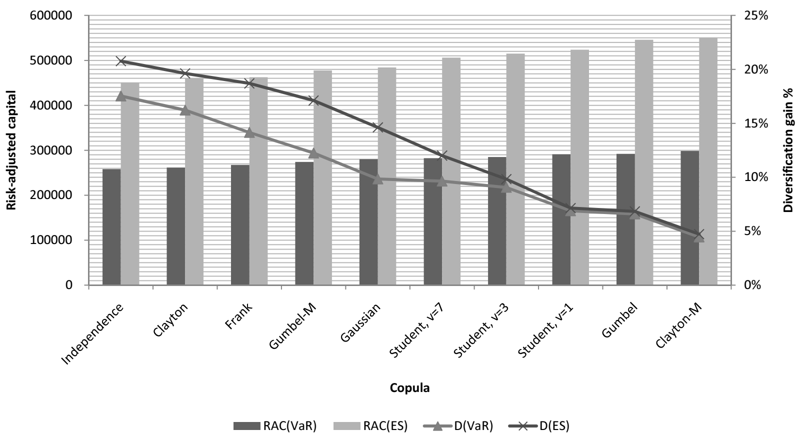

Figure 8: Risk-adjusted capital and diversification gain for different choices of the copula and rank correlations $\rho_\tau = 0.35$. X is lognormal distributed and Y is Fréchet distributed.

5 Dependence and Capital Allocation

In this section, we investigate by means of Monte Carlo simulations the role of dependence on capital allocation. We refer to Goovaerts et al. (2003) for a broad discussion on allocation principles. A capital allocation principle is a method to split the overall risk capital of a portfolio among its components. For the purpose of this analysis, two allocation principles are described. As in the previous section, we drop the α indicating the percentile from the notation.

- **Euler principle.** According to the Euler principle, the expected shortfall contribution of risk X to the portfolio $Z = X + Y$ is given by
$$\mathrm{ES}(X, Z) = E\left[X | Z \geq \mathrm{VaR}(Z)\right]. \tag{5.1}$$
Thus, the RAC allocated to risk X is equal to
$$\mathrm{RAC}_{\mathrm{ES}}(X, Z) = \mathrm{ES}(X, Z) - \mathrm{E}(X). \tag{5.2}$$
We denote by $\mathrm{RAC}_{\mathrm{ES}}(X|Z)$ the percentage of RAC allocated to risk X, i.e.
$$\mathrm{RAC}_{\mathrm{ES}}(X|Z) = \frac{\mathrm{RAC}_{\mathrm{ES}}(X, Z)}{\mathrm{RAC}_{\mathrm{ES}}(Z, Z)} = \frac{\mathrm{RAC}_{\mathrm{ES}}(X, Z)}{\mathrm{RAC}_{\mathrm{ES}}(Z)}. \tag{5.3}$$
The contribution of risk Y to the portfolio Z is obtained analogously. For more information on the Euler principle, we refer to Tasche (2008).

- **Haircut principle.** According to the haircut principle and in agreement with the above notation, the contribution of risk X to the portfolio $Z = X + Y$ is given by
$$\mathrm{VaR}(X, Z) = \frac{\mathrm{VaR}(X)}{\mathrm{VaR}(X) + \mathrm{VaR}(Y)}\mathrm{VaR}(Z). \tag{5.4}$$
Hence, the RAC allocated to risk X is equal to
$$\mathrm{RAC}_{\mathrm{VaR}}(X, Z) = \mathrm{VaR}(X, Z) - \mathrm{E}(X). \tag{5.5}$$
We denote by $\mathrm{RAC}_{\mathrm{VaR}}(X|Z)$ the percentage of RAC allocated to risk X, i.e.
$$\mathrm{RAC}_{\mathrm{VaR}}(X|Z) = \frac{\mathrm{VaR}(X)}{\mathrm{VaR}(X) + \mathrm{VaR}(Y)}. \tag{5.6}$$
A description of the haircut principle is offered in Dhaene et al. (2009), Section 2.

Both principles lead to a full allocation of the capital requirement, i.e. $\mathrm{RAC}_{\mathrm{ES}}(X|Z) + \mathrm{RAC}_{\mathrm{ES}}(Y|Z) = \mathrm{RAC}_{\mathrm{ES}}(Z)$ and $\mathrm{RAC}_{\mathrm{VaR}}(X|Z) + \mathrm{RAC}_{\mathrm{VaR}}(Y|Z) = \mathrm{RAC}_{\mathrm{VaR}}(Z)$. Concerning the haircut principle, the full allocation criterion may be not satisfied if we substitute the right hand side of (5.6) with
$$\frac{\varrho(X)}{\varrho(X + Y)}. \tag{5.7}$$
If $\varrho(\cdot) = \mathrm{VaR}(\cdot)$, in particular, this alternative implies in general that $\mathrm{VaR}(X) + \mathrm{VaR}(Y) > \mathrm{VaR}(X + Y)$, due to the lack of subadditivity, see Section 4.

Among capital allocation principles, Theorem 4.4 in Tasche (1999) proves that only the Euler principle is suitable for performance measurement. This feature is very important in steering the portfolio towards profitability through RORAC (return on risk-adjusted capital) optimization. The RORAC of a risk (portfolio) represents the ratio between the expected profit and the risk capital contribution necessary to run that risk within the portfolio. Roughly speaking, Tasche's theorem guarantees that if, according to the Euler principle, the RORAC of risk X_i is higher than the RORAC of the portfolio containing that risk, then an increase of the weight of risk X_i will improve the RORAC of the entire portfolio.

		Clayton-M			Gauss		
		Z'	Z''	Z'''	Z'	Z''	Z'''
		$\rho_\tau = 0.20$					
	$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	174,239	146,924	116,991	151,383	128,099	109,411
	$\mathrm{RAC}_{\mathrm{ES}}(Z)$	200,040	168,377	133,076	169,847	143,639	124,131
Euler	$\mathrm{RAC}_{\mathrm{ES}}(X\|Z)$	49.97%	63.88%	86.89%	50.21%	69.83%	93.25%
	$\mathrm{RAC}_{\mathrm{ES}}(Y\|Z)$	50.03%	36.12%	13.11%	49.79%	30.17%	6.75%
	$\mathrm{RAC}_{\mathrm{ES}}(X,Z)$	99,956	107,565	115,629	85,283	100,302	115,746
	$\mathrm{RAC}_{\mathrm{ES}}(Y,Z)$	100,084	60,812	17,447	84,564	43,337	8,385
Haircut	$\mathrm{RAC}_{\mathrm{VaR}}(X\|Z)$	50.00%	58.25%	75.17%	50.15%	58.26%	75.25%
	$\mathrm{RAC}_{\mathrm{VaR}}(Y\|Z)$	50.00%	41.75%	24.83%	49.85%	41.74%	24.75%
	$\mathrm{RAC}_{\mathrm{VaR}}(X,Z)$	87,122	85,584	87,939	75,919	74,627	82,332
	$\mathrm{RAC}_{\mathrm{VaR}}(Y,Z)$	87,117	61,340	29,052	75,464	53,472	27,078
		$\rho_\tau = 0.50$					
	$\mathrm{RAC}_{\mathrm{VaR}}(Z)$	201,787	168,429	125,686	180,858	151,972	118,774
	$\mathrm{RAC}_{\mathrm{ES}}(Z)$	231,055	190,749	141,822	204,212	171,353	134,091
Euler	$\mathrm{RAC}_{\mathrm{ES}}(X\|Z)$	50.04%	60.53%	82.07%	49.88%	63.35%	86.20%
	$\mathrm{RAC}_{\mathrm{ES}}(Y\|Z)$	49.96%	39.47%	17.93%	50.12%	36.65%	13.80%
	$\mathrm{RAC}_{\mathrm{ES}}(X,Z)$	115,617	115,453	116,389	101,871	108,551	115,582
	$\mathrm{RAC}_{\mathrm{ES}}(Y,Z)$	115,439	75,297	25,433	102,341	62,801	18,509
Haircut	$\mathrm{RAC}_{\mathrm{VaR}}(X\|Z)$	50.13%	58.31%	75.15%	49.99%	58.27%	75.14%
	$\mathrm{RAC}_{\mathrm{VaR}}(Y\|Z)$	49.87%	41.69%	24.85%	50.01%	41.73%	24.86%
	$\mathrm{RAC}_{\mathrm{VaR}}(X,Z)$	101,149	98,218	94,456	90,408	88,553	89,245
	$\mathrm{RAC}_{\mathrm{VaR}}(Y,Z)$	100,637	70,211	31,229	90,450	63,419	29,528

Table 5: Numerical results for the capital allocation. X and Y indicate the first and the second component of the general portfolio, denoted by Z.

In agreement with Section 4, we first study a portfolio Z' composed of two risks both lognormally distributed, namely $X, Y' \sim \mathrm{logN}(9.58, 0.83)$. The Value-at-Risk is computed at 99.5% and expected shortfall at 99%. Our goal is to study the Euler and the haircut capital allocation principles under several dependence assumptions. We also investigate how these principles react to changes in the riskiness of the portfolio components. Thus, we define two other portfolios, Z'' and Z''', composed of X and Y'' or X and Y''', respectively, where $Y'' \sim \mathrm{logN}(9.58, 0.70)$ and by $Y''' \sim \mathrm{logN}(9.58, 0.40)$. Hence, with respect to the original portfolio Z', the riskiness of the second component is reduced, varying its parameter σ. We implement ad-hoc procedures to quantify the allocated capital per risk. All the results are obtained with Monte Carlo methods that benefit of 10^7 iterations. In Table 5, we provide the results for portfolios where the assumed dependence structure is either a Clayton-M copula or a Gauss copula. Concerning the strength of dependence, which is characterized via the Kendall's tau, results are given for $\rho_\tau = 0.20$ and $\rho_\tau = 0.50$.

From (5.6), we notice that the denominator, $\mathrm{VaR}(X) + \mathrm{VaR}(Y)$, does not take into account the dependence among risks. Thus, as our empirical data emphasizes, the haircut allocation principle does not react neither to changes in the dependence structure nor to changes in the strength of dependence within the portfolio. For instance, considering Z''', both when we have a Clayton-M or a Gauss copula and $\rho_\tau = 0.20$ or $\rho_\tau = 0.50$, for any combination the $\mathrm{RAC}_{\mathrm{VaR}}(Y|Z) \cong 24.8\%$. Naturally, the capital requirement varies in terms of absolute amounts, but this is a consequence of the varied $\mathrm{RAC}_{\mathrm{VaR}}(Z)$ only. Hence, continuing the previous example, the $\mathrm{RAC}_{\mathrm{VaR}}(Y, Z)$ fluctuates from $29,052$ to $31,229$, and from $27,078$ to $29,528$. The alternative formulation of the haircut principle, using (5.7), would grant a sensibility to dependence assumptions, because of its denominator. Unfortunately, as noticed above, this alternative would not grant a full allocation of the capital.

The Euler principle, instead, is able to catch different dependence conditions directly in terms of the weights assigned to the portfolio components. For instance, considering again Z''', $\mathrm{RAC}_{\mathrm{ES}}(Y|Z)$ with a Gauss copula equals 6.75% when $\rho_\tau = 0.20$ and 13.80% when $\rho_\tau = 0.50$. Thus, to an increased strength of dependence, the Euler principle reacts assigning more weight to the less volatile risk. This behavior is in accordance with the intuition that if risks are linked by a stronger dependence, we have to pay higher attention to the less volatile risks as well, because the probability that something goes wrong for them too is increased by the stronger dependence with the more volatile risks. The same remark is valid considering a change in the structure of dependence. In particular, if we move from the Gauss to the Clayton-M assumption, we are heightening tail dependence and increasing conservativeness, as described in Section 4. Hence, continuing the previous example, we register an increase in the weight of the less volatile risk that moves from 6.75% to 13.11%, when $\rho_\tau = 0.20$, and from 13.80% to 17.93%, when $\rho_\tau = 0.50$.

(Unreported) extensive analysis have been conducted for all copulas listed in Table 1. The above mentioned order of conservativeness leads, in the Euler principle, to an equivalent order in terms of the importance of the weights assigned to less volatile risks. Hence the values of, say, $\mathrm{RAC}_{\mathrm{ES}}(Y|Z)$, will range from the minimum weight assigned by the Clayton copula, followed by the the Frank, the Gumbel-M, the Gauss, the t and the Gumbel copula, to the maximum weight assigned by the Clayton-M copula. The complete analysis on the capital allocation is repeated for two other portfolios. We consider a portfolio composed of two Fréchet distributed random variables, both originally with $a = 1.5$ and $s = 4,657.15$, and then we modify the second component. Similarly, we study a portfolio that mixes a Fréchet distributed random variable with $a = 1.5$ and $s = 4,657.15$ together with a lognormally distributed random variable with $\mu = 6.52$ and $\sigma = 2.15$, such that the weight of the two risks is almost equivalent in the original composition, and then we reduce the riskiness of the Fréchet rv increasing its shape parameter, a (see Appendix A). All these further analysis confirm the above remarks about the link between (structure and strength of) dependence and allocation of capital.

6 Proposals for Future Research

We dealt with bivariate copulas but most of the concept illustrated here can be easily extended to higher dimensions. The fair comparison among structures of dependence, granted by the link between the copulas' parameter and Kendall's tau, could be maintained without loss of generality in case of one-parameter copula families, like the Archimedean copulas here discussed. In case of elliptical copulas, instead, this fair comparison would require to deal with a subset of the specific copula family. For instance, if $d = 3$, we should define the Gauss copula such as it gives the same dependence among all the three portfolio components.

In Bürgi et al. (2009) the authors investigate the diversification gain using a reference model based on a Clayton-M copula. The Monte Carlo techniques employed in our analysis could be used to extend their study varying the reference model (both in terms of copula and parameter) and reviewing the diversification gain. This procedure would allow to judge the conservativeness of assumptions on dependence with respect to each possible reference model assumed to represent the reality.

7 Conclusion

In this paper, we pointed out the importance of dependence in the assessment of the capital requirements. The appraisal of RAC is heavily influenced by both the strength and the structure of dependence which are assumed in modeling the portfolio. Through the investigation of several copula models we identified an order among them in terms of conservativeness, with regard to the stated diversification gain. We observed robustness of this order varying both parametric and non parametric assumptions about marginal distributions. The results of our analysis send a warning concerning the cautiousness of certain assumptions. The main risk is to overestimate the diversification gain, and thus to underestimate the RAC of the portfolio, due to an improper choice of the copula model. Regarding risk measures, we noted the sizeable differences in terms of RAC based on VaR and on ES as the importance of the tail increases. Indeed, the tail of the distribution plays a fundamental role in determining the RAC, from a twofold perspective. In case the risk measure is the ES, it directly takes into account the shape of the tail for determining the capital requirements. Moreover, the dependence among risks ought to be modeled cautiously on the tail, as certain copulas allow to do.

Finally, we discussed capital allocation principles. We observed that the split of the overall risk capital of a portfolio among its components can either take dependence into account or not. The Euler principle reacts to changes in dependence assumptions. In particular, we showed that as dependence increases in strength or it is heightening with respect to tail dependence, the less volatile risk gains more weight coherently with the increased probability of a joint severe event.

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A Fréchet distribution

Definition A.1 (Fréchet distribution)**.** *A continuous random variable X is said to have a Fréchet distribution if its cdf is*

$$F_X(x) = \begin{cases} \exp\left(-(\frac{x}{s})^{-a}\right), & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{A.1}$$

where $a > 0$ is the shape parameter and s is the scale parameter.

The mean of the Fréchet distribution is given by $E(X) = \Gamma(1-1/a)\,s$ where $\Gamma(\gamma) = \int_0^\infty x^{\gamma-1}\exp(-x)dx$ is the Gamma function.