PSEUDOPROCESSES ON A CIRCLE AND RELATED POISSON KERNELS

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Abstract. Pseudoprocesses, constructed by means of the solutions of higher-order heat-type equations have been developed by several authors and many related functionals have been analyzed by means of the Feynman-Kac functional or by means of the Spitzer identity. We here examine pseudoprocesses wrapped up on circles and derive their explicit signed density measures. We develop the Fourier series of their laws in the case of even-order and odd-order pseudoprocesses separately. We observe that circular even-order pseudoprocesses differ substantially from pseudoprocesses on the line because - for $t > \bar{t} > 0$, where $\bar{t}$ is a suitable n-dependent time value - they become real random processes. By composing the circular pseudoprocesses with positively-skewed stable processes we arrive at genuine circular processes whose distribution, in the form of Poisson kernels, is obtained. The distribution of circular even-order pseudoprocesses is similar to the Von Mises (or Fisher) circular normal and therefore to the wrapped up law of Brownian motion. The last section of the paper is related to the circular Fresnel pseudoprocess and we examine its composition with stable subordinators. This leads to combinations of Poisson kernels.

1. Introduction and preliminaries

Pseudoprocesses are connected with the fundamental solution of heat-type equations of the form

$$\frac{\partial}{\partial t} u_n(x,t) = c_n \frac{\partial^n}{\partial x^n} u_n(x,t), \qquad x \in \mathbb{R},\, t>0,\, n \in \mathbb{N}, \tag{1.1}$$

where

$$c_n = \begin{cases} (-1)^{\frac{n}{2}+1}, & \text{for even values of } n \\ \pm 1, & \text{for odd values of } n, \end{cases} \tag{1.2}$$

subject to the initial condition

$$u(x,0) = \delta(x). \tag{1.3}$$

For $n > 2$ the fundamental solutions to (1.1) are sign-varying. By means of a Wiener-type approach some authors (see for example Albeverio et al. [1], Daletsky [10], Daletsky and Fomin [11], Krylov [16], Ladohin [19]) have constructed pseudoprocesses which we denote by $X(t)$, $t > 0$ or $X_n(t)$, if we specify the order of the governing equation. In these papers the set of real functions $x : t \in [0,\infty) \to x(t)$ (sample paths) and the cylinders

$$C = \{x(t) : a_j \le x(t_j) \le b_j,\, j = 1, \cdots, n\} \tag{1.4}$$

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have been considered. By using the solutions u_n to (1.1) the measure of cylinders is given as

$$\mu_n(C) = \int_{a_1}^{b_1} dx_1 \cdots \int_{a_n}^{b_n} dx_n \prod_{j=1}^{n} u_n(x_j - x_{j-1}, t_j - t_{j-1}). \tag{1.5}$$

In (1.5) we denote by u_n

$$u_n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi x} e^{c_n(-i\xi)^n t}, \tag{1.6}$$

in case the integral above converges. The measure (1.5) is extended to the field generated by cylinders (1.4) for fixed $t_1 < \cdots < t_j < \cdots < t_n$. The signed measure obtained in this way is Markovian in the sense that

$$\mu_{x_0}\left\{X(t+T) \in \mathcal{B} \middle| \mathcal{F}_T\right\} = \mu_{X(T)}\left\{X(t) \in \mathcal{B}\right\}, \tag{1.7}$$

where $\mathcal{F}_T$ is the field generated as

$$\mathcal{F}_T = \sigma\left\{X(t_1) \in \mathcal{B}_1, \cdots, X(t_n) \in \mathcal{B}_n\right\}. \tag{1.8}$$

More information on properties of pseudoprocesses can be found in Cammarota and Lachal [9], Lachal [17] and Nishioka [22]. For pseudoprocesses with drift the reader can consult Lachal [18].

In this paper we consider pseudoprocesses on the ring $\mathcal{R}$ of radius one, denoted by $\Theta(t)$, $t > 0$, whose signed density measures are governed by

$$\begin{cases} \frac{\partial}{\partial t} v_n(\theta,t) = c_n \frac{\partial^n}{\partial \theta^n} v_n(\theta,t), & \theta \in [0,2\pi), t > 0, n \geq 2, \\ v_n(\theta,0) = \delta(\theta). \end{cases} \tag{1.9}$$

The signed measures of pseudoprocesses on the line $X(t)$, $t > 0$, and those on the unit-radius ring, $\Theta(t)$, $t > 0$, can be related by

$$\{\Theta(t) \in d\theta\} = \bigcup_{m=-\infty}^{\infty} \{X(t) \in d(\theta + 2m\pi)\}, \qquad 0 \leq \theta < 2\pi. \tag{1.10}$$

This means that the pseudoprocess Θ has sample paths which are obtained from those of X by wrapping them up around the circumference $\mathcal{R}$. Counterclockwise moving sample paths of Θ correspond to increasing sample paths of X.

For $n = 2$ we have in particular the circular Brownian motion studied by Hartman and Watson [14], Roberts and Ursell [27], Stephens [28]. The pseudoprocesses running on $\mathcal{R}$ are called circular pseudoprocesses and are denoted either by $\Theta(t)$, $t > 0$, or $\Theta_n(t)$ if we want to clarify the order of the equation governing their distribution. We analyze separately the even-order case and the odd-order one because they pose qualitatively different problems. In view of (1.10) we can write

$$v_{2n}(\theta,t) = \sum_{m=-\infty}^{\infty} u_{2n}(\theta + 2m\pi, t), \qquad 0 \leq \theta < 2\pi. \tag{1.11}$$

Equation (1.11) shows that the solution to (1.9) can be obtained by wrapping up the solution to (1.1) which reads

$$u_{2n}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi x} e^{-\xi^{2n} t}, \qquad x \in \mathbb{R}, t > 0. \tag{1.12}$$

The function (1.12) has been investigated in special cases by Hochberg [15], Krylov [16], Nishioka [22] and more in general by Lachal [17; 18]. The sign-varying structure of (1.12) has been discovered in special cases by Bernstein [7], Lévy [20], Polya [26], as early as at the beginning of the Twentieth century and has been more recently studied also by Li and Wong [21].

The Fourier series of (1.11) has the remarkably simple form

$$v_{2n}(\theta,t) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}e^{-k^{2n}t}\cos k\theta, \qquad \theta\in[0,2\pi)\,. \tag{1.13}$$

For $n=1$ we obtain the Fourier series of the law of the circular Brownian motion (see [14]). The function

$$v_{2}(\theta,t) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}e^{-k^{2}t}\cos k\theta \tag{1.14}$$

is similar to the Von Mises circular normal

$$v(\theta,k) \;=\; \frac{e^{k\cos\theta}}{2\pi I_0(k)} \;=\; \frac{1}{2\pi}\left(1+2\sum_{m=1}^{\infty}\frac{I_m(k)}{I_0(k)}\cos m\theta\right), \qquad \theta\in[0,2\pi)\,, \tag{1.15}$$

where

$$I_m(x) \;=\; \sum_{j=0}^{\infty}\left(\frac{x}{2}\right)^{2j+m}\frac{1}{k!\,\Gamma\left(m+j-1\right)} \tag{1.16}$$

is the m-th order Bessel function. The relationship between (1.14) and (1.15) is investigated in the paper by Hartman and Watson [14]. The Von Mises circular normal represents the hitting distribution of the circumference $\mathcal{R}$ of a Brownian motion with drift starting from the center of $\mathcal{R}$. The planar Brownian motion $(R(t),\,\Psi(t))$, $t>0$, with drift $\boldsymbol{k}=(k_1,k_2)$, $\|\boldsymbol{k}\|=k$, has transition function

$$\Pr\left\{R(t)\in d\rho,\,\Psi(t)\in d\varphi\right\} \;=\; \frac{\rho}{2\pi t}e^{-\frac{\rho^2}{2t}}e^{-\frac{k^2t}{2}}e^{\rho k\cos\varphi}d\rho d\varphi \tag{1.17}$$

and marginal

$$\Pr\left\{R(t)\in d\rho\right\} \;=\; \frac{\rho}{t}e^{-\frac{\rho^2}{2t}}e^{-\frac{k^2t}{2}}I_0\left(\rho k\right)d\rho. \tag{1.18}$$

Therefore

$$\Pr\left\{\Psi(t)\in d\varphi\middle|R(t)\in d\rho\right\} \;=\; \frac{e^{\rho k\cos\varphi}}{2\pi I_0\left(\rho k\right)}\,d\varphi \tag{1.19}$$

and for $\rho=1$ coincides with (1.15).

The analysis of the pictures of $v_{2n}(\theta,t)$ for different values of t and different values of the order $2n$, $n\in\mathbb{N}$, shows that the distributions (1.13) after a certain time become non-negative. This means that pseudoprocesses on the circle $\mathcal{R}$ behave differently from their counterparts on the line and rapidly become genuine processes. Furthermore we remark that in small initial intervals of time the circular pseudoprocesses have signed-valued distributions with a number of minima which rapidly unify into a single minumum (located at $\theta=\pi$) which for increasing t upcrosses the zero level. This is due to the fact that in a small initial interval of time the effect produced by the central bell of the distribution has not yet spread on the whole ring $\mathcal{R}$.

The value of the absolute minimum of $v_{2n}(\theta,t)$ for $t>\bar{t}$ has the form

$$v_{2n}(\pi,t) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k e^{-k^{2n}t}, \qquad t>\bar{t}. \tag{1.20}$$

The graph of functions $v_{2n}(\theta, t)$ slightly differ from that of the density of circular Brownian motion as shown in figures 1 and 2. The term $k = 1$ in (1.13) is the leading term of the series and the form of the distribution $v_{2n}(\theta, t)$ is very close to that of $\frac{1}{2\pi} + e^{-t} \cos\theta$.

FIGURE 1. The distributions of the fourth-order circular pseudoprocess for different values of t

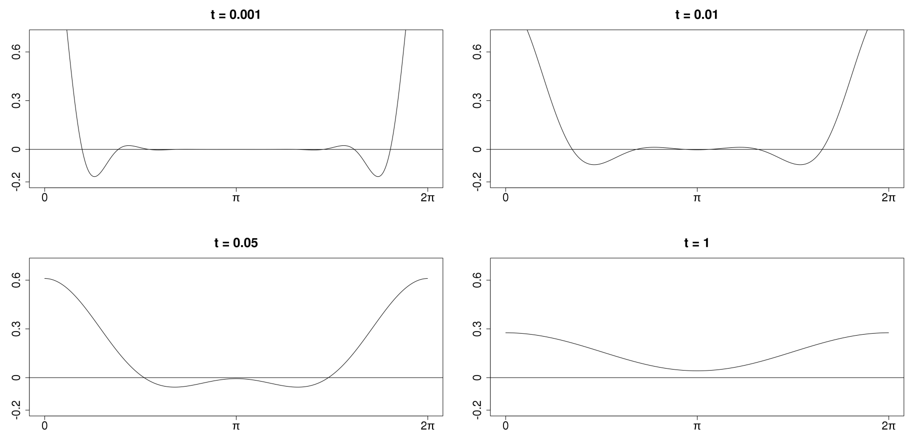
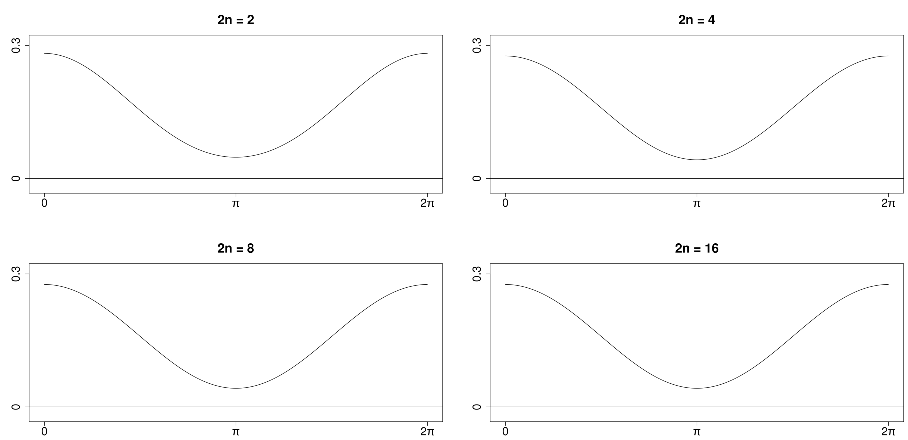


FIGURE 2. The distributions (for $t = 1$) of the circular pseudoprocesses of various order $2n$



The odd-order case is much more complicated because the solutions to equation (1.1) are asymmetric (with asymmetry decreasing for increasing values of the order n). Some properties of solutions to odd-order heat-type equations can be found in [17; 18]. In the present paper the wrapped up solution to (1.1) gives the fundamental solution of (1.9) as

$$v_{2n+1}(\theta,t) \;=\; \sum_{k=-\infty}^{\infty} u_{2n+1}(\theta+2k\pi,t), \qquad \theta\in[0,2\pi)\,, \tag{1.21}$$

whose Fourier series reads

$$v_{2n+1}(\theta,t) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k^{2n+1}t+k\theta). \tag{1.22}$$

We note that for $n=1$ the series (1.22) becomes a discrete version of the solution to (1.1) which reads

$$u_3(x,t) \;=\; \frac{1}{\sqrt[3]{3t}}\,\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right) \tag{1.23}$$

where

$$\mathrm{Ai}(x) \;=\; \frac{1}{\pi}\int_0^\infty \cos\left(\alpha x+\frac{\alpha^3}{3}\right)d\alpha \tag{1.24}$$

is the Airy function.

We consider also the wrapped up stable processes $\mathfrak{S}^{2\beta}(t)$, $t>0$, and the related governing space-fractional equation.In particular we show that the law of $\mathfrak{S}^{2\beta}(2^{-\beta}t)$, $t>0$, is the fundamental solution of the space-fractional equations

$$\begin{cases} \frac{\partial}{\partial t}v_2^\beta(\theta,t) \;=\; -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta v_2^\beta(\theta,t), & \theta\in[0,2\pi)\,,\, t>0, \beta\in(0,1]\,, \\ v_2^\beta(\theta,0) \;=\; \delta(\theta), \end{cases} \tag{1.25}$$

and has Fourier expansion

$$v_2^\beta(\theta,t) \;=\; \frac{1}{2\pi}\left[1+2\sum_{m=1}^{\infty}e^{-\left(\frac{m^2}{2}\right)^\beta t}\cos m\theta\right]. \tag{1.26}$$

The fractional operator appearing in (1.25) is the one-dimensional fractional Laplacian which can be defined by means of the Bochner representation (see, for example, Balakrishnan [3], Bochner [8])

$$-\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta \;=\; \frac{\sin\pi\beta}{\pi}\int_0^\infty\left(\lambda+\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)\right)^{-1}\lambda^\beta\,d\lambda, \qquad \beta\in(0,1). \tag{1.27}$$

We show that formula (1.26) coincides with the distribution of the subordinated Brownian motion on the circle, $\mathfrak{B}\left(H^\beta(t)\right)$, $t>0$, where $H^\beta(t)$, $t>0$, is a stable subordinator of order $\beta\in(0,1]$ (see, for example, Baeumer and Meerschaert [2]). Furthermore we notice that

$$\mathfrak{B}\left(2\,H^\beta(t)\right) \overset{\text{law}}{=} \mathfrak{S}^{2\beta}(t), \qquad t>0, \tag{1.28}$$

$\mathfrak{S}^{2\beta}(t)$, $t>0$, is a symmetric process on the ring $\mathcal{R}$ with distribution which can be obtained by its symmetric stable counterpart on the line as

$$\begin{aligned} p_{\mathfrak{S}^{2\beta}}(\theta,t) \;&=\; \sum_{m=\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\,e^{-i\xi(\theta+2m\pi)}e^{-t|\xi|^{2\beta}} \\ &= \frac{1}{2\pi}\left[1+2\sum_{k=1}^{\infty}e^{-k^{2\beta}t}\cos k\theta\right]. \end{aligned} \tag{1.29}$$

For $\beta = \frac{1}{2}$ we extract from (1.26) the Poisson kernel

$$v_2^1(\theta, t) \;=\; \frac{1}{2\pi}\frac{1-e^{-t\sqrt{2}}}{1+e^{-t\sqrt{2}}-2e^{-\frac{t}{\sqrt{2}}}\cos\theta}. \tag{1.30}$$

The composition of the circular pseudoprocesses $\Theta_n(t)$, $t>0$, with positively-skewed stable processes of order $\frac{1}{n}$, say $H^{\frac{1}{n}}(t)$, $t>0$, leads also to the Poisson kernel. In particular, we show that

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)\in d\theta\right\} \;=\; \frac{d\theta}{2\pi}\frac{1-e^{-2t}}{1+e^{-2t}-2e^{-t}\cos\theta}, \qquad \theta\in[0,2\pi)\,. \tag{1.31}$$

In the odd-order case the result is different, depends on n and has the following form for $\theta\in[0,2\pi)$

$$\begin{aligned}&\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in d\theta\right\} =\\ &=\frac{d\theta}{2\pi}\frac{1-e^{-2t\cos\frac{\pi}{2(2n+1)}}}{1+e^{-2t\cos\frac{\pi}{2(2n+1)}}-2e^{-t\cos\frac{\pi}{2(2n+1)}}\cos\left(\theta+t\sin\frac{\pi}{2(2n+1)}\right)}.\end{aligned} \tag{1.32}$$

The composition of pseudoprocesses with stable processes therefore produces genuine r.v.'s on the ring $\mathcal{R}$ as it happens on the line (see Orsingher and D'Ovidio [24]). We note that the distribution of the composition in the even order case is independent from n (formula (1.31)), while in the odd-order case the Poisson kernel obtained depends on n and has a rather complicated structure. For $n\to\infty$ the kernel (1.32) converges pointwise to (1.31) since the asymmetry of the fundamental solutions of (1.1) (as well as that of their wrapped up counterparts) decreases. The result (1.31) offers an interesting interpretation. The Poisson kernel (1.31) can be viewed as the probability that a planar Brownian motion starting from the point with polar coordinates $(e^{-t},0)$ hits the circumference $\mathcal{R}$ in the point $(1,\Theta)$ (see Fig. 4a). Therefore this distribution coincides with the law of an even-order pseudoprocess running on the circumference and stopped at time $H^{\frac{1}{2n}}(t)$, $t>0$. This result is independent from n and therefore is valid also for Brownian motion. A similar interpretation holds also for circular odd-order pseudoprocesses taken at the time $H^{\frac{1}{2n+1}}(t)$, $t>0$, but starting from the point with polar coordinates (e^{-a_nt}, b_nt), where $a_n=\cos\pi/(2(2n+1))$ and $b_n=\sin\pi/(2(2n+1))$.

A further generalization of the above framework is obtained by considering the fractional higher-order diffusion equations (studied on the whole real line by Beghin [4])

$$\begin{cases}\frac{\partial^\nu}{\partial t^\nu}v_{2n}^\nu(\theta,t) \;=\; c_{2n}\frac{\partial^{2n}}{\partial x^{2n}}v_{2n}^\nu(\theta,t), & \theta\in[0,2\pi)\,, t>0, n\in\mathbb{N},\\ v_{2n}^\nu(\theta,0) \;=\; \delta(\theta).\end{cases} \tag{1.33}$$

The solution to (1.33) can be written as

$$v_{2n}^\nu(\theta,t) \;=\; \frac{1}{2\pi}\left(1+2\sum_{m=1}^\infty E_{\nu,1}\left(-t^\nu m^{2n}\right)\cos m\theta\right), \tag{1.34}$$

where

$$L^\nu(t) \;=\; \inf\left\{s>0 : H^\nu(s)\geq t\right\}, \qquad t>0, \tag{1.35}$$

is the inverse of the stable subordinator $H^\nu(t)$, $t>0$. The fractional derivative in (1.33) must be meant in the Dzerbayshan-Caputo sense and $E_{\nu,1}$ represents the Mittag-Leffler function. Clearly for $\nu=1$ we retrive result (1.13).

The Fresnel pseudoprocess $F(t)$, $t>0$, related to the equation of vibrations of rods was introduced and investigated in Orsingher and D'Ovidio [23]. We here study the corresponding version on the ring $\mathcal{R}$, say $\mathfrak{F}(t)$, $t>0$, and obtain the fourth-order Poisson kernel representing the distribution of $\mathfrak{F}(\tau_t)$, $t>0$, where $\tau_t=\inf\{s>0: B(s)=t\}$. Circular Fresnel pseudoprocesses related to fractional equations of the form

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}}\mathfrak{f}^{\nu}(\theta,t) = -\frac{1}{2^2}\frac{\partial^4}{\partial\theta^4}\mathfrak{f}^{\nu}(\theta,t), \qquad \theta\in[0,2\pi)\,,\, t>0,\, \nu\in(0,1], \tag{1.36}$$

are analyzed and the corresponding densities

$$\mathfrak{f}^{\nu}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}E_{\nu,1}\left(-\frac{t^{2\nu}k^4}{2^2}\right)\cos k\theta \tag{1.37}$$

are obtained. The composition of the circular Fresnel pseudoprocess with Brownian first passage times leads to the superposition of Poisson kernels.

2. Pseudoprocesses on a ring

In this section we consider pseudoprocesses $\Theta(t)$, $t>0$, on the unit-radius circumference $\mathcal{R}$, whose density function $v_n(\theta,t)$, $\theta\in[0,2\pi)$, $t>0$, is governed by the higher order heat-type equation

$$\begin{cases}\frac{\partial}{\partial t}v_n(\theta,t) = c_n\frac{\partial^n}{\partial\theta^n}v_n(\theta,t), & \theta\in[0,2\pi)\,,\, t>0,\, n\geq 2,\\ v_n(\theta,0) = \delta(\theta).\end{cases} \tag{2.1}$$

The pseudoprocesses Θ_n have sample paths obtained by wrapping up the trajectories of pseudoprocesses on the line $\mathbb{R}$. Increasing sample paths on $\mathcal{R}$ correspond to counterclockwise moving motions on the ring $\mathcal{R}$. The structure of sample paths of pseudoprocesses has not been investigated in detail although some results by Lachal (Theorem 5.2, [18]) show that there is a sort of "slight" discontinuity in their behaviour (this is confirmed by Hochberg [15]) and the fact that the reflection principle fails (Beghin et al. [5], Lachal [17]).

It must be considered that the wrapping up of paths and the corresponding density measures produces in the long run genuine processes (with non-negative measure densities in the case n is even). Our first result concerns the distribution of $\Theta_n(t)$, $t>0$.

Theorem 2.1. *The solutions to the n-th order heat-type equations (2.1) reads*

$$\begin{cases}v_{2n}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}e^{-k^{2n}t}\cos k\theta, & \text{for } c_{2n}=(-1)^{n+1},\, n\geq 1,\\ v_{2n+1}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k^{2n+1}t+k\theta), & \text{for } c_{2n+1}=(-1)^{n},\, n\geq 1.\end{cases} \tag{2.2}$$

Proof. We can obtain the result (2.2) in two different ways. We start by considering the even-order case where the wrapping up of the solutions to (1.1) leads to

$$v_{2n}(\theta,t) = \sum_{m=-\infty}^{\infty}u_{2n}(\theta+2m\pi,t) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}d\xi\, e^{-i(\theta+2m\pi)\xi}e^{-\xi^{2n}t}. \tag{2.3}$$

The Fourier series expansion of the symmetric function $v_{2n}(\theta,t)$ has coefficients

$$a_k = \frac{1}{\pi}\int_0^{2\pi}d\theta\,\cos k\theta\left[\sum_{m=-\infty}^{\infty}u_{2n}(\theta+2m\pi,t)\right]$$

$$
\begin{aligned}
&= 2\sum_{m=-\infty}^{\infty}\int_m^{m+1} dy\; u_{2n}(2\pi y,t)\,\cos 2\pi k y \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} dz\,\cos kz\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi z}e^{-\xi^{2n}t}\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-\xi^{2n}t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\left(e^{iz(k-\xi)}+e^{-iz(k+\xi)}\right)\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-\xi^{2n}t}\left[\delta(\xi-k)+\delta(\xi+k)\right] \;=\; \frac{e^{-k^{2n}t}}{\pi}.
\end{aligned}
\tag{2.4}
$$

This shows that the first result of (2.2) holds. An alternative derivation of $v_{2n}(\theta,t)$ is based on the method of separation of variables. Thus under the assumption that $v_{2n}(\theta,t)=T(t)\psi(\theta)$ we get

$$
\frac{T^{(1)}(t)}{T(t)} \;=\; \frac{\psi^{(2n)}(\theta)}{\psi(\theta)}(-1)^{n+1} \;=\; -\beta^{2n}. \tag{2.5}
$$

In order to have periodic solutions we must take integer values of β and thus the general solution to (2.1) becomes

$$
v_{2n}(\theta,t) \;=\; \sum_{k=-\infty}^{\infty} A_k\, e^{-k^{2n}t}\cos k\theta \;=\; A_0 + 2\sum_{k=1}^{\infty} A_k\, e^{-k^{2n}t}\cos k\theta. \tag{2.6}
$$

The initial condition

$$
v_{2n}(\theta,0) \;=\; \delta(\theta) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos k\theta \tag{2.7}
$$

implies that $A_k=\frac{1}{2\pi}$. In the odd-order case, by proceeding as in the previous case we have that the Fourier coefficients of

$$
v_{2n+1}(\theta,t) \;=\; \sum_{m=-\infty}^{\infty} u_{2n+1}(\theta+2m\pi,t) \tag{2.8}
$$

become

$$
\begin{aligned}
a_k &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{(-1)^n(-i\xi)^{2n+1}t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\left(e^{i(k-\xi)z}+e^{-i(k+\xi)z}\right)\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi^{2n+1}t}\left[\delta(\xi-k)+\delta(\xi+k)\right] \\
&= \frac{1}{2\pi}\left[e^{-ik^{2n+1}}+e^{ik^{2n+1}}\right] \;=\; \frac{1}{\pi}\cos k^{2n+1}t.
\end{aligned}
\tag{2.9}
$$

In a similar way we have that

$$
\begin{aligned}
b_k &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{(-1)^n(-i\xi)^{2n+1}t}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\left(e^{i(k-\xi)z}-e^{-i(k+\xi)z}\right)\right] \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi^{2n+1}t}\left[\delta(\xi-k)-\delta(\xi+k)\right] \;=\; -\frac{1}{\pi}\sin k^{2n+1}t,
\end{aligned}
\tag{2.10}
$$

and thus the expression (2.2) of $v_{2n+1}(\theta,t)$ follows. □

In a recent paper by Orsingher and D'Ovidio [24] the authors have shown that the solutions to (1.1) have the following probabilistic representation

$$\begin{cases} u_{2n}(x,t) \ = \ \frac{1}{\pi x}\mathbb{E}\left\{\sin\left(x\, G^{2n}\left(\frac{1}{t}\right)\right)\right\}, & \text{for } c_{2n}=(-1)^{n+1},\ n\geq 1,\\ u_{2n+1}(x,t) \ = \ \frac{1}{\pi x}\mathbb{E}\left\{e^{-b_n x\, G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n x\, G^{2n+1}\left(\frac{1}{t}\right)\right)\right\}, & \text{for } c_{2n+1}=(-1)^{n},\ n\geq 1, \end{cases} \tag{2.11}$$

where $G^{\gamma}\left(t^{-1}\right)$ is a generalized gamma r.v. with density

$$g^{\gamma}(x,t) \ = \ \gamma\frac{x^{\gamma-1}}{t}e^{-\frac{x^{\gamma}}{t}}, \qquad x>0,\, t>0,\, \gamma>0, \tag{2.12}$$

and

$$a_n \ = \ \cos\frac{\pi}{2(2n+1)}, \qquad b_n \ = \ \sin\frac{\pi}{2(2n+1)}. \tag{2.13}$$

We now confirm the results of Theorem 2.1 by means of the wrapped up versions of the probabilistic representations (2.11) which become

$$\begin{cases} v_{2n}(\theta,t) \ = \ \sum_{m=-\infty}^{\infty}\frac{1}{\pi(\theta+2m\pi)}\mathbb{E}\left\{\sin\left((\theta+2m\pi)G^{2n}\left(\frac{1}{t}\right)\right)\right\},\\ v_{2n+1}(\theta,t) \ = \ \sum_{m=-\infty}^{\infty}\frac{1}{\pi(\theta+2m\pi)}\mathbb{E}\left\{e^{-b_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)\right)\right\}. \end{cases} \tag{2.14}$$

We prove that the Fourier series expansion of (2.14) coincides with (2.2). We start first with the even-order pseudoprocesses for which only the cosine Fourier coefficients a_k are necessary.

Proof of (2.15) *in the symmetric case.* We have that

$$\begin{aligned} a_k \ &= \frac{1}{\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{2\pi}\frac{d\theta}{\theta+2m\pi}\mathbb{E}\left\{\sin(\theta+2m\pi)G^{2n}\left(\frac{1}{t}\right)\right\}\cos k\theta \\ &= \frac{1}{\pi^2}\mathbb{E}\left\{\int_{-\infty}^{\infty}dz\frac{\cos kz}{z}\sin\left(z\,G^{2n}\left(\frac{1}{t}\right)\right)\right\} \\ &= \frac{1}{\pi^2}\mathbb{E}\left\{\int_0^{\infty}dz\left[\frac{\sin\left(kz+z\,G^{2n}\left(\frac{1}{t}\right)\right)}{z}+\frac{\sin\left(z\,G^{2n}\left(\frac{1}{t}\right)-kz\right)}{z}\right]\right\} \\ &= \frac{1}{\pi}\mathbb{E}\left\{\mathbb{I}_{\left[-G^{2n}\left(\frac{1}{t}\right)<k<G^{2n}\left(\frac{1}{t}\right)\right]}\right\} \ = \ \frac{1}{\pi}\Pr\left\{G^{2n}\left(\frac{1}{t}\right)>k\right\} \ = \ \frac{1}{\pi}e^{-k^{2n}t}, \end{aligned} \tag{2.15}$$

where we used the fact that

$$\int_0^{\infty}dx\,\frac{\sin\alpha x}{x} \ = \ \begin{cases}\frac{\pi}{2}, & \text{if } \alpha>0,\\ -\frac{\pi}{2}, & \text{if } \alpha<0.\end{cases} \tag{2.16}$$

The calculation (2.15) shows that the density of even-order circular pseudoprocesses can be viewed as the superposition of sinusoidal waves whose amplitude corresponds to the tails of a Weibull distribution. □

Proof of (2.14) *in the asymmetric case.* In the odd-order case we need both the sine and cosine coefficients of the Fourier expansion because the signed laws are asymmetric. For

$$v_{2n+1}(\theta,t) \ = \ \sum_{m=-\infty}^{\infty}\frac{1}{\pi(\theta+2m\pi)}\mathbb{E}\left\{e^{-b_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)\right)\right\} \tag{2.17}$$

the Fourier coefficients become

$$\begin{cases} a_k &= \frac{1}{\pi}\cos k^{2n+1}t, \\ b_k &= -\frac{1}{\pi}\sin k^{2n+1}t. \end{cases} \tag{2.18}$$

We give with some details the evaluation of (2.18)

$$\begin{aligned} a_k =& \frac{1}{\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\theta \frac{\cos k\theta}{\theta+2m\pi}\mathbb{E}\left\{ e^{-b_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)} \sin\left(a_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)\right)\right\} \\ =& \frac{1}{\pi^2}\mathbb{E}\left\{ \int_{-\infty}^{\infty} dz \frac{\cos kz}{z} e^{-b_n z G^{2n+1}\left(\frac{1}{t}\right)} \sin\left(a_n z G^{2n+1}\left(\frac{1}{t}\right)\right)\right\} \\ =& \frac{1}{2\pi^2}\mathbb{E}\left\{ \int_{-\infty}^{\infty} dz \frac{\sin\left(z\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)\right)+\sin\left(z\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)\right)}{z} e^{-b_n z G^{2n+1}\left(\frac{1}{t}\right)}\right\} \\ =& \frac{1}{2i2\pi^2}\mathbb{E}\bigg\{ \int_{-\infty}^{\infty} \frac{dz}{z}\left[e^{iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)-b_nzG^{2n+1}\left(\frac{1}{t}\right)} - e^{-iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)-b_nzG^{2n+1}\left(\frac{1}{t}\right)} \right. \\ & \left. + e^{iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)-b_nzG^{2n+1}\left(\frac{1}{t}\right)} - e^{-iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)-b_nzG^{2n+1}\left(\frac{1}{t}\right)} \right]\bigg\}. \end{aligned} \tag{2.19}$$

By considering the following integral representation of the Heaviside function

$$\mathcal{H}_y(x) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}} dw\, e^{-iwx}\frac{e^{iyw}}{iw} \;=\; \int_{\mathbb{R}} dw\, e^{iwx}\frac{e^{-iyw}}{iw} \tag{2.20}$$

the coefficients a_k in (2.19) become

$$\begin{aligned} a_k \;=& \frac{(2n+1)t}{2\pi}\int_0^{\infty} dw\, w^{2n}e^{-tw^{2n+1}} \left[\mathcal{H}_k(w(a_n-ib_n)) - \mathcal{H}_k(-w(a_n+ib_n)) \right. \\ & \left. +\mathcal{H}_k(w(a_n+ib_n)) - \mathcal{H}_k(-w(a_n-ib_n))\right] \\ =& \frac{i(2n+1)t}{2\pi}\int_0^{\infty} dw\, w^{2n}e^{-iw^{2n+1}t}\mathcal{H}_k(w) + \frac{i(2n+1)t}{2\pi}\int_0^{\infty} dw\, w^{2n}e^{iw^{2n+1}t}\mathcal{H}_k(-w) \\ & -\left[\frac{i(2n+1)t}{2\pi}\int_0^{\infty} dw\, w^{2n}e^{iw^{2n+1}t}\mathcal{H}_k(w) + \frac{i(2n+1)t}{2\pi}\int_0^{\infty} dw\, w^{2n}e^{-iw^{2n+1}t}\mathcal{H}_k(-w)\right] \\ =& \frac{i(2n+1)t}{2\pi}\left(\int_{-\infty}^{\infty} dw\, w^{2n}e^{-iw^{2n+1}t}\mathcal{H}_k(w) - \int_{-\infty}^{\infty} dw\, w^{2n}e^{iw^{2n+1}t}\mathcal{H}_k(w)\right) \\ =& \frac{1}{2\pi}\left(e^{ik^{2n+1}t}+e^{-ik^{2n+1}t}\right) \;=\; \frac{1}{\pi}\cos k^{2n+1}t. \end{aligned} \tag{2.21}$$

In order to justify the last step we can either take the Laplace transform with respect to t (see for example Orsingher and D'Ovidio [24]) or we can apply the following trick

$$a_k \;=\; \lim_{\zeta\to 0}\frac{i(2n+1)t}{2\pi}\left(\int_k^{\infty} dw\, e^{-\zeta w^{2n+1}} w^{2n}e^{-iw^{2n+1}t} - \int_k^{\infty} dw\, e^{-\zeta w^{2n+1}} w^{2n}e^{iw^{2n+1}t}\right). \tag{2.22}$$

The coefficients b_k (2.18) can be obtained by performing similar calculation. □

2.1. **Circular Brownian motion.** The circular Brownian motion $\mathfrak{B}(t)$, $t > 0$, has been analyzed by Roberts and Ursell [27], Stephens [28] and also by Hartman and Watson [14]. In a certain sense it can be viewed as a special case of symmetric pseudoprocesses on the ring $\mathcal{R}$. The distribution of $\mathfrak{B}(t)$, $t > 0$, has Fourier representation

$$p_{\mathfrak{B}}(\theta, t) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{\infty} e^{-\frac{k^2 t}{2}} \cos k\theta\right), \qquad \theta \in [0, 2\pi), \tag{2.23}$$

and can be also regarded as the wrapped up distribution of the standard Brownian motion

$$p_{\mathfrak{B}}(\theta, t) = \frac{1}{\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} e^{-\frac{(\theta+2m\pi)^2}{2t}}. \tag{2.24}$$

Formula (2.23) corresponds to $n = 1$ of (2.2) for the even-order case with a suitable adjustement of the time scale. The law (2.23) can be obtained directly by solving the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} p_{\mathfrak{B}}(\theta, t) = \frac{1}{2}\frac{\partial^2}{\partial \theta^2} p_{\mathfrak{B}}(\theta, t), & \theta \in [0, 2\pi), \, t > 0, \\ p_{\mathfrak{B}}(\theta, 0) = \delta(\theta). \end{cases} \tag{2.25}$$

or as the limit of a circular random walk as in [28]. The distribution of the circular Brownian motion is depicted in Figure 2 and looks like the Von Mises circular normal (this is the inspiring idea of the paper by Hartman and Watson [14] in which the connection between the two distributions is investigated). For $t \to \infty$ the distribution of $\mathfrak{B}(t)$, $t > 0$, tends to the uniform law.

We note that

$$\Pr\left\{-\frac{\pi}{2} < \mathfrak{B}(t) < \frac{\pi}{2}\right\} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} (-1)^k \frac{e^{-\frac{(2k+1)^2 t}{2}}}{2k+1} \tag{2.26}$$

and therefore

$$\Pr\left\{-\frac{\pi}{2} < \mathfrak{B}(t) < \frac{\pi}{2}\right\} \le \frac{1}{2} + \frac{2}{\pi} e^{-\frac{t}{2}}, \qquad \text{valid for } t > -2\log\frac{\pi}{4} \approx 0.209. \tag{2.27}$$

The relationship between circular Brownian motion $\mathfrak{B}(t)$, $t > 0$, and Brownian motion on the line $B(t)$, $t > 0$,

$$\{\mathfrak{B}(t) \in d\theta\} = \bigcup_{m=-\infty}^{\infty} \{B(t) \in d(\theta + 2m\pi)\}, \qquad \theta \in [0, 2\pi), \tag{2.28}$$

permits us to derive the distribution of

$$\max_{0 \le s \le t} |\mathfrak{B}(t)|, \qquad t > 0, \tag{2.29}$$

that is the distribution of the maximal distance reached by the circular Brownian motion from the starting point. Of course the sample paths overcoming the angular distance π at least once are assigned π as maximal distance which therefore has a positive probability (converging to 1 as time tends to infinity).

Proposition 2.2. *For the maximal distance (2.29) we have that*

$$\Pr\left\{\max_{0 \le s \le t} |\mathfrak{B}(s)| < \theta\right\} = \int_{-\theta}^{\theta} \Pr\left\{-\theta < \min_{0 \le s \le t} B(s) < \max_{0 \le s \le t} B(s) < \theta\right\}$$

$$= \int_{-\theta}^{\theta} dy \left(\sum_{m=-\infty}^{\infty} \frac{e^{-\frac{(y-4m\theta)^2}{2t}}}{\sqrt{2\pi t}} - \sum_{m=-\infty}^{\infty} \frac{e^{-\frac{(-y+2\theta(2m-1))^2}{2t}}}{\sqrt{2\pi t}} \right)$$

$$= \sum_{r=-\infty}^{\infty} (-1)^r \int_{-\frac{(1+2r)\theta}{\sqrt{t}}}^{\frac{(1-2r)\theta}{\sqrt{t}}} dw \, \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}. \tag{2.30}$$

The related first passage time of circular Brownian motion has density which has the following form

$$\begin{aligned} \Pr\{\mathcal{T}_\theta \in dt\} &= -\frac{d}{dt} \Pr\left\{ \max_{0\le s\le t} |\mathfrak{B}(s)| < \theta \right\} dt \\ &= \sum_{r=-\infty}^{\infty} \left[\frac{(-1)^r \, e^{-\frac{(1-2r)^2\theta^2}{2t}}}{2\sqrt{2\pi t^3}} \theta(1-2r) + \frac{(-1)^r \, e^{-\frac{(1+2r)^2\theta^2}{2t}}}{2\sqrt{2\pi t^3}} \theta(1+2r) \right] \\ &= \left(\frac{\theta \, e^{-\frac{\theta^2}{2t}}}{\sqrt{2\pi t^3}} \right) \sum_{r=-\infty}^{\infty} (-1)^r e^{-\frac{2r^2\theta^2}{r}} \left(\cosh \frac{2r\theta^2}{t} - 2r \sinh \frac{2r\theta^2}{t} \right) \end{aligned} \tag{2.31}$$

Curiously enough the factor $\theta e^{-\frac{\theta^2}{2t}} / \sqrt{2\pi t^3}$ coincides with the first passage time through θ of a Brownian motion on the line.

3. Fractional equations on the ring $\mathcal{R}$ and the related processes

In this section we consider various types of processes on the unit radius circumference $\mathcal{R}$.

3.1. Higher-order time-fractional equations.

We start by analyzing the processes related to the solutions of time-fractional higher-order heat-type equations. We consider the time-changed pseudoprocesses $\Theta_{2n}(L^\nu(t))$, $t > 0$, where

$$L^\nu(t) = \inf\{s > 0 : H^\nu(s) \ge t\} \tag{3.1}$$

and where $H^\nu(t)$, $t > 0$, is a positively skewed stable process of order $\nu \in (0, 1]$. We notice that the Laplace transform of the distribution $l_\nu(x,t)$ of (3.1) reads (see for example Orsingher and Toaldo [25])

$$\int_0^\infty dx \, e^{-\gamma x} l_\nu(x,t) = E_{\nu,1}(-\gamma t^\nu) \tag{3.2}$$

where

$$E_{\nu,1}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\nu j + 1)}, \qquad x \in \mathbb{R}, \nu > 0, \tag{3.3}$$

is the Mittag-Leffler function. For pseudoprocesses related to time-fractional equations we have the next theorem.

Theorem 3.1. *The solution to the problem, for $\nu \in (0,1]$, $n \in \mathbb{N}$,*

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} v_{2n}^\nu(\theta,t) = -\left(-\frac{\partial^2}{\partial\theta^2}\right)^n v_{2n}^\nu(\theta,t), & \theta \in [0,2\pi), \, t > 0, \\ v_{2n}^\nu(\theta,0) = \delta(\theta), \end{cases} \tag{3.4}$$

is the univariate (signed) distribution of $\Theta_{2n}(L^\nu(t))$, $t > 0$, which reads

$$v_{2n}^\nu(\theta,t) = \frac{1}{2\pi} \left(1 + 2\sum_{k=1}^{\infty} E_{\nu,1}\left(-k^{2n} t^\nu\right) \cos k\theta \right). \tag{3.5}$$

The time-fractional derivative in (3.4) must be understood in the Caputo sense, that is

$$\frac{\partial^\nu}{\partial t^\nu} v_{2n}^\nu(\theta,t) \;=\; \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{\frac{\partial}{\partial s} v_{2n}^\nu(\theta,s)}{(t-s)^\nu}\,ds, \qquad 0<\nu<1. \tag{3.6}$$

Proof. The law of $\Theta_{2n}\left(L^\nu(t)\right)$, $t>0$, is given by

$$\begin{aligned} v_{2n}^\nu(\theta,t) &= \frac{1}{2\pi}\int_0^\infty ds\, \left(1+2\sum_{k=1}^\infty e^{-k^{2n}s}\cos k\theta\right) l_\nu(s,t) \\ &= \frac{1}{2\pi}\left(1+2\sum_{k=1}^\infty E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta\right). \end{aligned} \tag{3.7}$$

Since, $\forall k\geq 1$, we have that

$$\frac{\partial^\nu}{\partial t^\nu}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta = -k^{2n}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta = (-1)^{n+1}\frac{\partial^{2n}}{\partial\theta^{2n}}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta, \tag{3.8}$$

and therefore we conclude that (3.7) satisfies the fractional equation (3.4). □

Remark 3.2. *For $n=1$, formula (3.7) becomes the distribution of subordinated Brownian motion $\mathfrak{B}\left(L^\nu(t)\right)$, $t>0$. For $\nu=1$ we retrieve from (3.5) the solutions (2.2) of the even-order heat-type equations on $\mathcal{R}$.*

3.2. **Space-fractional equations and wrapped up stable processes.** The following Theorem represents the counterpart on $\mathcal{R}$ of the Riesz statement on the relationship between space-fractional equations and symmetric stable laws (for the non-symmetric case see the paper by Feller [12]).

Theorem 3.3. *The law of the process $\mathfrak{B}\left(H^\beta(t)\right)$, $t>0$, is given by*

$$p_{\mathfrak{B}}^\beta(\theta,t) \;=\; \frac{1}{2\pi}\left[1+2\sum_{k=1}^\infty e^{-\left(\frac{k^2}{2}\right)^\beta t}\cos k\theta\right] \tag{3.9}$$

and solves the space-fractional equation, for $\beta\in(0,1]$,

$$\begin{cases} \frac{\partial}{\partial t}p_{\mathfrak{B}}^\beta(\theta,t) \;=\; -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta p_{\mathfrak{B}}^\beta(\theta,t), & \theta\in[0,2\pi)\,,\; t>0 \\ p_{\mathfrak{B}}^\beta(\theta,0) \;=\; \delta(\theta). \end{cases} \tag{3.10}$$

The fractional one-dimensional Laplacian in (3.10) is defined in (1.27) and $H^\beta(t)$, $t>0$, is a stable subordinator of order $\beta\in(0,1]$.

Proof. The law of $\mathfrak{B}\left(H^\beta(t)\right)$, $t>0$, is given by

$$p_{\mathfrak{B}}^\beta(\theta,t) \;=\; \int_0^\infty ds\, p_{\mathfrak{B}}(\theta,s)\,h_\beta(s,t) \;=\; \frac{1}{2\pi}\left[1+2\sum_{k=1}^\infty e^{-\left(\frac{k^2}{2}\right)^\beta t}\cos k\theta\right], \tag{3.11}$$

where $p_{\mathfrak{B}}$ is the law of circular Brownian motion and h_β is the density of a positively skewed random process of order $\beta\in(0,1]$. In order to check that (3.9) solves (3.10) it is convenient to write the fractional derivative appearing in (3.10) as

$$\begin{aligned} \left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta &= -\frac{\sin\pi\beta}{\pi}\int_0^\infty \left(\lambda+\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)\right)^{-1}\lambda^\beta\,d\lambda \\ &= -\frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^\infty \lambda^\beta\int_0^\infty e^{-u\lambda-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)}\,du\,d\lambda \end{aligned}$$

$$= \frac{1}{\Gamma(-\beta)} \int_0^\infty u^{-\beta-1} e^{-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)} \, du. \tag{3.12}$$

From (3.12) we have therefore that

$$\begin{aligned}
\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta \cos k\theta &= \frac{1}{\Gamma(-\beta)} \int_0^\infty du\, u^{-\beta-1} e^{-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)} \cos k\theta \\
&= \frac{1}{\Gamma(-\beta)} \int_0^\infty du\, u^{-\beta-1} \sum_{j=0}^\infty \frac{(-u)^j}{j!} \left(-\frac{1}{2}\right)^j \frac{\partial^{2j}}{\partial\theta^{2j}} \cos k\theta \\
&= \frac{1}{\Gamma(-\beta)} \int_0^\infty du\, u^{-\beta-1} \sum_{j=0}^\infty \frac{(-u)^j}{j!} \frac{k^{2j}}{2^j} \cos k\theta \\
&= \frac{1}{\Gamma(-\beta)} \int_0^\infty du\, u^{-\beta-1} e^{-u\frac{k^2}{2}} \cos k\theta \\
&= \left(\frac{k^2}{2}\right)^\beta \cos k\theta,
\end{aligned} \tag{3.13}$$

and this shows that (3.9) satisfies (3.10). □

Remark 3.4. *Another way to prove that*

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta \cos k\theta = \left(\frac{k^2}{2}\right)^\beta \cos k\theta \tag{3.14}$$

can be traced in the paragraph 4.6, page 428 of Balakrishnan [3], which confirms our result.

Theorem 3.5. *For the wrapped up version, say $\mathfrak{S}^{2\beta}(t)$, $t > 0$, of the symmetric stable processes $S^{2\beta}(t)$, $t > 0$, with characteristic function $\mathbb{E}e^{i\xi S^{2\beta}(t)} = e^{-t|\xi|^{2\beta}}$, we have the following equality in distribution*

$$\mathfrak{S}^{2\beta}(t) \overset{law}{=} \mathfrak{B}\left(2H^\beta(t)\right) \overset{law}{=} \mathfrak{B}\left(H^\beta\left(2^\beta t\right)\right), \qquad t > 0. \tag{3.15}$$

Proof. The density of $\mathfrak{S}^{2\beta}(t)$, $t > 0$, must be written as

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) = \frac{\Pr\left\{\mathfrak{S}^{2\beta}(t) \in d\theta\right\}}{d\theta} = \frac{1}{2\pi} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty d\xi\, e^{-i\xi(\theta+2m\pi)} e^{-t|\xi|^{2\beta}}. \tag{3.16}$$

The Fourier expansion of (3.16) becomes

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos k\theta \tag{3.17}$$

where

$$\begin{aligned}
a_k &= \frac{1}{\pi} \int_0^{2\pi} d\theta \frac{1}{2\pi} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty d\xi\, e^{-i\xi(\theta+2m\pi)} e^{-t|\xi|^{2\beta}} \cos k\theta \\
&= \frac{1}{\pi} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty d\xi\, e^{-t|\xi|^{2\beta}} \int_m^{m+1} d\theta\, e^{-i\xi 2\pi\theta} \cos 2k\pi\theta \\
&= \frac{1}{(2\pi)^2} \int_{-\infty}^\infty d\xi\, e^{-t|\xi|^{2\beta}} \int_{-\infty}^\infty dy\, e^{-i\xi y} \left(e^{iyk} + e^{-iyk}\right)
\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-t|\xi|^{2\beta}} \left[\delta(\xi - k) + \delta(\xi + k) \right] \; = \; \frac{1}{\pi} e^{-tk^{2\beta}}. \tag{3.18}$$

This permits us to conclude that

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) \; = \; \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^{2\beta} t} \cos k\theta. \tag{3.19}$$

□

While the integral in (3.16) (representing the Fourier inverse of symmetric stable laws) cannot carried out, its circular analogue can be explicitely worked out and leads to the Fourier expansion (3.19).

Corollary 3.6. *In view of the results of Theorems 3.1 and 3.3 we have that the solution to the space-time fractional equation, for $\beta \in (0, 1]$,*

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} p_{\mathfrak{B}}^{\nu,\beta}(\theta, t) \; = \; - \left(-\frac{1}{2} \frac{\partial^2}{\partial \theta^2} \right)^\beta p_{\mathfrak{B}}^{\nu,\beta}(\theta, t), \qquad \theta \in [0, 2\pi)\,, \, t > 0 \\ p_{\mathfrak{B}}^{\nu,\beta}(\theta, 0) \; = \; \delta(\theta). \end{cases} \tag{3.20}$$

can be written as

$$p_{\mathfrak{B}}^{\nu,\beta}(\theta, t) \; = \; \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} E_{\nu,1} \left(- \left(\frac{k^2}{2} \right)^\beta t^\nu \right) \cos k\theta, \tag{3.21}$$

and coincides with the law of the process

$$\mathcal{F}^{\nu,\beta}(t) \; = \; \mathfrak{B} \left(H^\beta \left(L^\nu(t) \right) \right), \qquad t > 0. \tag{3.22}$$

In (3.22) H^β is a stable subordinator of order $\beta \in (0, 1]$ and L^ν is the inverse of H^ν as defined in (3.1).

Proof. Here we only derive the distribution of $\mathcal{F}^{\nu,\beta}(t)$, $t > 0$. We have that

$$\begin{aligned} \Pr\left\{ \mathcal{F}^{\nu,\beta}(t) \in d\theta \right\} &= d\theta \int_0^\infty \Pr\left\{ \mathfrak{B}(s) \in d\theta \right\} \int_0^\infty \Pr\left\{ H^\beta(w) \in ds \right\} \Pr\left\{ L^\nu(t) \in dw \right\} \\ &= \frac{d\theta}{2\pi} + \frac{d\theta}{\pi} \int_0^\infty \sum_{k=1}^{\infty} e^{-\frac{k^2}{2} s} \cos k\theta \int_0^\infty \Pr\left\{ H^\beta(w) \in ds \right\} \Pr\left\{ L^\nu(t) \in dw \right\} \\ &= \frac{d\theta}{2\pi} + \frac{d\theta}{\pi} \int_0^\infty \sum_{k=1}^{\infty} e^{-\left(\frac{k^2}{2}\right)^\beta w} \cos k\theta \, \Pr\left\{ L^\nu(t) \in dw \right\} \\ &= \frac{d\theta}{2\pi} \left[1 + 2 \sum_{k=1}^{\infty} \cos k\theta \, E_{\nu,1} \left(- \left(\frac{k^2}{2} \right)^\beta t^\nu \right) \right], \end{aligned} \tag{3.23}$$

where in the last step we applied (3.2). □

4. From pseudoprocesses to Poisson kernels

In this section we show that the composition of pseudoprocesses of order n running on the circumference $\mathcal{R}$ with positively skewed stable processes of order $\frac{1}{n}$ leads to the Poisson kernel. This is the circular counterpart of the composition of pseudoprocesses with stable subordinators which leads to Cauchy processes. In both cases pseudoprocesses stopped at $H^{\frac{1}{n}}(t)$, $t > 0$, yield genuine random variables.

We distinguish the case where n is even from the case of odd-order pseudoprocesses. We have the first result in Theorem 4.1.

Theorem 4.1. *The composition* $\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)$, $t>0$, *of the pseudoprocess* Θ_{2n} *with the stable process* $H^{\frac{1}{2n}}(t)$, $t>0$, *has density*

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)\in d\theta\right\} = \frac{d\theta}{2\pi}\frac{1-e^{-2t}}{1+e^{-2t}-2e^{-t}\cos\theta}, \qquad n\in\mathbb{N}, \tag{4.1}$$

and distribution function

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)<\theta\right\} = \begin{cases}\frac{1}{\pi}\arctan\left(\frac{1+e^{-t}}{1-e^{-t}}\tan\frac{\theta}{2}\right), & \theta\in[0,\pi],\\ 1+\frac{1}{\pi}\arctan\left(\frac{1+e^{-t}}{1-e^{-t}}\tan\frac{\theta}{2}\right), & \theta\in(\pi,2\pi),\end{cases} \tag{4.2}$$

which are independent from n.

Proof. We have that

$$\begin{aligned}\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)\in d\theta\right\} &= d\theta\int_0^\infty ds\,\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty e^{-k^{2n}s}\cos k\theta\right]h_{\frac{1}{2n}}(s,t)\\ &= d\theta\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty \cos k\theta e^{-kt}\right]\\ &= \frac{d\theta}{2\pi}\left[1+\frac{e^{-t+i\theta}}{1+e^{-t+i\theta}}+\frac{e^{-t-i\theta}}{1-e^{-t-i\theta}}\right]\\ &= \frac{d\theta}{2\pi}\frac{1-e^{-2t}}{1+e^{-2t}-2e^{-t}\cos\theta}.\end{aligned} \tag{4.3}$$

The result (4.2) is derived by applying formula 2.552(3) page 172 of Gradshteyn and Ryzhik [13]

$$\int\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b}, \qquad a^2>b^2. \tag{4.4}$$

□

Remark 4.2. *The Poisson kernel* (4.1) *can be interpreted as the distribution of the process* $\boldsymbol{B}(\mathfrak{T}_{\mathcal{R}})$ *where* $\boldsymbol{B}$ *is a planar Brownian motion and* $\mathfrak{T}_{\mathcal{R}}=\inf\{t>0:\boldsymbol{B}(t)\in\mathcal{R}\}$. *In the case of Theorem 4.1 the planar Brownian motion starts from the point* $(e^{-t},0)$. *Therefore we have that*

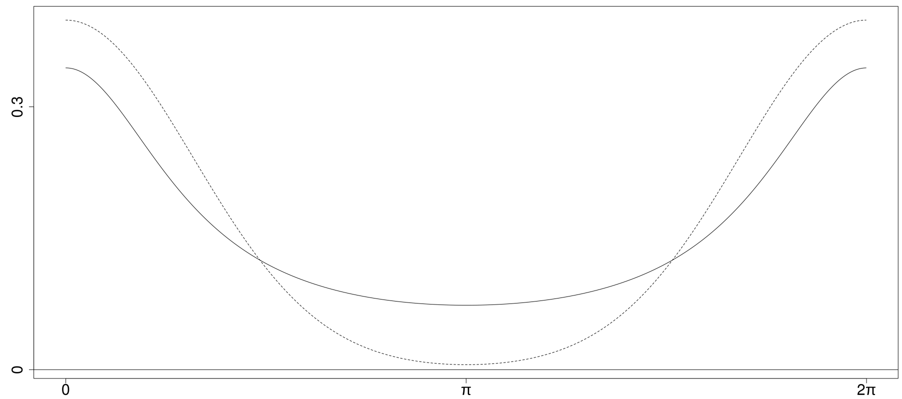
$$\boldsymbol{B}(\mathfrak{T}_{\mathcal{R}}) \overset{law}{=} \Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right), \qquad t>0. \tag{4.5}$$

This means that a pseudoprocess running on the ring $\mathcal{R}$ *and stopped at a stable time* $H^{\frac{1}{2n}}(t)$, $t>0$, *has the same distribution of a planar Brownian motion starting from* $(e^{-t},0)$ *at the first exit time from the unit-radius circle. The result* (4.5) *holds for all* $n\in\mathbb{N}$ *and represents the circular counterpart of the composition of pseudoprocesses on the line with stable subordinators* $H^{\frac{1}{n}}(t)$, $t>0$, *which possesses a Cauchy distributed law. As* $t\to\infty$ *the distribution* (4.1) *converges to the uniform law.*

Remark 4.3. *In view of* (4.2) *we note that for* $\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)$, $t>0$, *the probability of staying in the right-hand side of* $\mathcal{R}$ *has the remarkably simple form*

$$\Pr\left\{-\frac{\pi}{2}<\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\frac{\pi}{2}\right\} = \frac{1}{2}+\frac{2}{\pi}\arctan e^{-t}, \qquad \forall t>0. \tag{4.6}$$

FIGURE 3. In the picture the density of the circular Brownian motion (dotted line) and the kernel (4.1) are represented.



We now pass to the Poisson kernel associated to odd-order pseudoprocesses. The asymmetry implies that the density of the composition $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$, $t>0$, is bit more complicated than (4.1).

Theorem 4.4. *The composition* $\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)$, $t>0$, *has density*

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in d\theta\right\}=\frac{d\theta}{2\pi}\frac{1-e^{-2a_nt}}{1+e^{-2a_nt}-2e^{-a_nt}\cos\left(\theta+b_nt\right)},\tag{4.7}$$

and distribution function

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\theta\right\}=$$
$$=\begin{cases}\frac{1}{\pi}\left[\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2}-\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}\right], & 0<\frac{\theta+b_nt}{2}<\pi,\\ 1+\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2}-\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}, & \pi<\theta<2\pi-\frac{b_nt}{2}.\end{cases}\tag{4.8}$$

where

$$a_n=\cos\frac{\pi}{2(2n+1)},\qquad b_n=\sin\frac{\pi}{2(2n+1)}.\tag{4.9}$$

Proof. Let $h_{\frac{1}{2n+1}}(s,t)$, $s,t>0$, be the density of a positively skewed stable process of order $\frac{1}{2n+1}$. Then, in view of (2.2), we have that

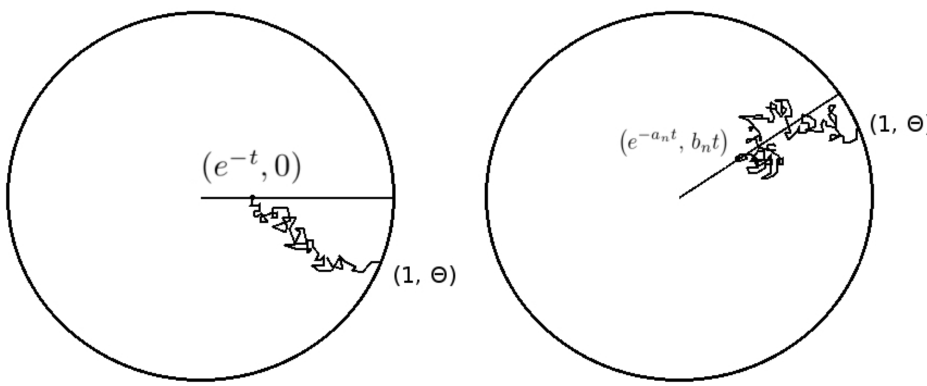
$$\begin{aligned}\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in d\theta\right\}&=d\theta\int_0^\infty ds\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty\cos\left(k\theta+k^{2n+1}s\right)\right)h_{\frac{1}{2n+1}}(s,t)\\&=d\theta\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty e^{-a_ntk}\cos\left(k(\theta+b_nt)\right)\right]\\&=\frac{d\theta}{2\pi}\left[1+\frac{e^{i\theta}e^{-t(a_n-ib_n)}}{1-e^{i\theta}e^{-t(a_n-ib_n)}}+\frac{e^{-i\theta}e^{-t(a_n+ib_n)}}{1-e^{-i\theta}e^{-t(a_n+ib_n)}}\right]\end{aligned}$$

$$= \frac{d\theta}{2\pi} \frac{1-e^{-2a_nt}}{1+e^{-2a_nt}-2e^{-a_nt}\cos(\theta+b_nt)}. \tag{4.10}$$

In view of (4.4) we can write

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\theta\right\} =$$
$$= \frac{1}{2\pi}\int_0^\theta dy\, \frac{1-e^{-2a_nt}}{1+e^{-2a_nt}-2e^{-a_nt}\cos(y+b_nt)}$$
$$= \begin{cases} \frac{1}{\pi}\left[\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2} - \arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}\right], & 0<\frac{\theta+b_nt}{2}<\pi, \\ 1+\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2} - \frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}, & \pi<\theta<2\pi-\frac{b_nt}{2}. \end{cases} \tag{4.11}$$

□



(A) (B)

FIGURE 4. The distribution of the hitting point of a planar Brownian motion is obtained as subordinated circular pseudoprocess in the even case (Fig. 4a) and odd case (Fig. 4b).

Remark 4.5. *From (4.11) we arrive at the following fine expression*

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\theta\right\}$$
$$= \frac{1}{\pi}\arctan\left[\frac{\left(1-e^{-2a_nt}\right)\tan\frac{\theta}{2}\left(1+\tan^2\frac{b_nt}{2}\right)}{\left(1-e^{-a_nt}\right)^2+4\tan\frac{\theta}{2}\tan\frac{b_nt}{2}+\left(1+e^{-a_nt}\right)^2\tan^2\frac{b_nt}{2}}\right], \qquad \theta\in[0,2\pi), \tag{4.12}$$

from which we are able to explicitely write for $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$ *the probability of staying in the interval* $(0,\pi)$ *as*

$$\Pr\left\{0<\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\pi\right\} = \frac{1}{\pi}\arctan\frac{\sinh a_nt}{\sin b_nt}, \qquad \forall t>0, \tag{4.13}$$

while for $\left(0,\frac{\pi}{2}\right)$ *we obtain*

$$\Pr\left\{0<\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\frac{\pi}{2}\right\}$$

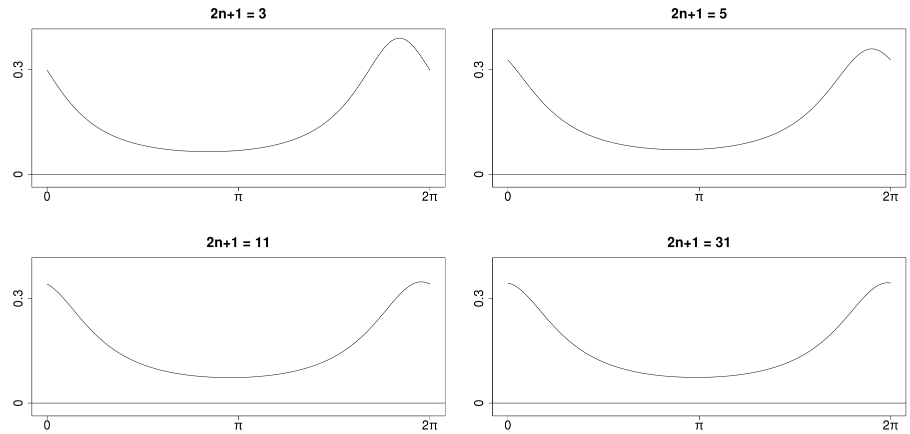


FIGURE 5. Distributions related to odd-order Poisson kernels (for $t = 1$)

$$
\begin{aligned}
&= \frac{1}{\pi} \arctan \frac{\left(1 - e^{-2a_n t}\right)\left(1 + \tan^2 \frac{b_n t}{2}\right)}{\left(1 - e^{-a_n t}\right)^2 + 4 \tan \frac{b_n t}{2} + \left(1 + e^{-a_n t}\right)^2 \tan^2 \frac{b_n t}{2}} \\
&= \frac{1}{\pi} \arctan \frac{\sinh a_n t}{2 \sinh^2 \frac{a_n t}{2} \cos^2 \frac{b_n t}{2} + e^{a_n t} \sin b_n t + 2 \cosh^2 \frac{a_n t}{2} \sin^2 \frac{b_n t}{2}} \\
&= \frac{1}{\pi} \arctan \frac{\sinh a_n t}{\cosh a_n t - \cos b_n t + e^{a_n t} \sin b_n t}. \qquad (4.14)
\end{aligned}
$$

By means of the same manipulations leading to (4.14) *we arrive at the alternative form of the distribution function for* $\theta \in [0, \pi]$,

$$
\Pr\left\{0 < \Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) < \theta\right\} = \frac{1}{\pi} \arctan \frac{\sinh a_n t \, \tan \frac{\theta}{2}}{\cosh a_n t - \cos b_n t + e^{a_n t} \sin b_n t \, \tan \frac{\theta}{2}}. \tag{4.15}
$$

Remark 4.6. *In the third-order case we can arrive at the Poisson kernel* (4.7) *for* $n = 1$ *by considering that (see [24])*

$$
\frac{\Pr\left\{X_3\left(H^{\frac{1}{3}}(t)\right) \in dx\right\}}{dx} = \int_0^\infty \frac{ds}{\sqrt[3]{3s}} Ai\left(\frac{x}{\sqrt[3]{3s}}\right) \frac{t}{s\sqrt[3]{3s}} Ai\left(\frac{t}{\sqrt[3]{3s}}\right) = \frac{\sqrt{3}t}{2\pi\left[\left(x + \frac{t}{2}\right)^2 + \frac{3t^2}{4}\right]}. \tag{4.16}
$$

The wrapped up counterpart of (4.16) *becomes for* $\theta \in [0, 2\pi)$

$$
\Pr\left\{\Theta_3\left(H^{\frac{1}{3}}(t)\right) \in d\theta\right\} = \sum_{m=-\infty}^{\infty} \Pr\left\{X_3\left(H^{\frac{1}{3}}(t)\right) \in d(\theta + 2m\pi)\right\} =
$$

$$
= d\theta \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{ds}{\sqrt[3]{3s}} Ai\left(\frac{\theta + 2m\pi}{\sqrt[3]{3s}}\right) \frac{t}{s\sqrt[3]{3s}} Ai\left(\frac{t}{\sqrt[3]{3s}}\right)
$$

$$
= d\theta \frac{\sqrt{3}t}{2\pi} \sum_{m=-\infty}^{\infty} \frac{t}{\left(\theta + 2m\pi + \frac{t}{2}\right)^2 + 3\frac{t^2}{4}} = \frac{d\theta\sqrt{3}t}{2} \sum_{m=-\infty}^{\infty} \int_0^\infty ds \frac{e^{-\frac{\left(\theta + 2m\pi + \frac{t}{2}\right)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{(\sqrt{3}t/2)^2}{2s}}}{\sqrt{2\pi s^3}}
$$

$$
\begin{aligned}
&= \frac{d\theta\sqrt{3}t}{2^2\pi}\int_0^\infty ds\left[1+2\sum_{k=1}^\infty \cos\left[k\left(\theta+\frac{t}{2}\right)\right]e^{-\frac{k^2 s}{2}}\right]\frac{e^{-\frac{(\sqrt{3}t/2)^2}{2s}}}{\sqrt{2\pi s^3}} \\
&= \frac{d\theta}{2\pi}\left[1+2\sum_{k=1}^\infty e^{-\frac{\sqrt{3}}{2}tk}\cos\left[k\left(\theta+\frac{t}{2}\right)\right]\right] = \frac{d\theta}{2\pi}\frac{1-e^{-\sqrt{3}t}}{1+e^{-\sqrt{3}t}-2e^{-\sqrt{3}t}\cos\left(\theta+\frac{t}{2}\right)}.
\end{aligned}
\tag{4.17}
$$

5. Circular Fresnel pseudoprocess

The equation of vibrations of rods

$$
\frac{\partial^2}{\partial t^2}f(x,t) = -\kappa^2\frac{\partial^4}{\partial x^4}f(x,t), \qquad x\in\mathbb{R},\, t>0, \tag{5.1}
$$

is a sort of biquadratic heat equation which has suggested the construction of a pseudoprocess called Fresnel pseudoprocess (see Orsingher and D'Ovidio [23]) and denoted by $F(t)$, $t>0$. For the standard equation of vibrations of rods

$$
\begin{cases}
\frac{\partial^2}{\partial t^2}f(x,t) = -\frac{1}{2^2}\frac{\partial^4}{\partial x^4}f(x,t), & x\in\mathbb{R},\, t>0,\\
f(x,0) = \delta(x),
\end{cases} \tag{5.2}
$$

the fundamental solution reads

$$
f(x,t) = \frac{1}{\sqrt{2\pi t}}\cos\left(\frac{x^2}{2t}-\frac{\pi}{4}\right). \tag{5.3}
$$

For cylinders $C=\{x : a_j\le x(t_j)\le b_j\}$, where $x: t\to x(t)$, $t>0$, are the sample paths of F, we construct a sign-varying non-markovian measure μ based on (5.3) as (see formula 4.1 of [23])

$$
\mu(C) = \int_{a_1}^{b_1}\dots\int_{a_n}^{b_n}\frac{(2\pi)^{-\frac{n}{2}}}{\prod_{j=1}^n\sqrt{t_j-t_{j-1}}}\cos\left(\sum_{j=1}^n\frac{(x_j-x_{j-1})^2}{2(t_j-t_{j-1})}-n\frac{\pi}{4}\right)\prod_{j=1}^n dx_j \tag{5.4}
$$

The extension of the measure μ on the field generated by cylinders is performed in the same way as in the case of pseudoprocesses related to (1.1). We note that the composition $F(|B(t)|)$, $t>0$, has density coinciding with the fundamental solution to the fourth-order heat equation (Benachour et al. [6]). Furthermore $F(\tau_t)$, $t>0$, $\tau_t=\inf\{s>0 : B(s)=t\}$ has distribution coinciding with a bimodal Cauchy-type r.v. (see Orsingher and D'Ovidio [23]).

We now study the counterpart on the ring, denoted by $\mathfrak{F}(t)$, $t>0$, of the Fresnel pseudoprocess F, and present its Fourier representation in the next theorem.

Theorem 5.1. *The density measure $\mathfrak{f}(\theta,t)$ of the Fresnel pseudoprocess $\mathfrak{F}(t)$, $t>0$, on the ring $\mathcal{R}$ is the wrapped up version of* (5.3)

$$
\mathfrak{f}(\theta,t) = \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\cos\left(\frac{(\theta+2m\pi)^2}{2t}-\frac{\pi}{4}\right) \tag{5.5}
$$

and has the Fourier expansion

$$
\mathfrak{f}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty\cos\frac{k^2 t}{2}\cos k\theta. \tag{5.6}
$$

The densities (5.5) *and* (5.6) *are solutions to*

$$\begin{cases} \frac{\partial^2}{\partial t^2}\mathfrak{f}(\theta,t) \;=\; -\frac{1}{2^2}\frac{\partial^4}{\partial\theta^4}\mathfrak{f}(\theta,t), & \theta\in[0,2\pi)\,,\, t>0, \\ \mathfrak{f}(\theta,0) \;=\; \delta(\theta). \end{cases} \tag{5.7}$$

Proof. The Fourier expansion of $\mathfrak{f}(\theta,t)$ is

$$\mathfrak{f}(\theta,t) \;=\; a_0+\sum_{k=1}^{\infty}a_k\,\cos k\theta \tag{5.8}$$

where

$$\begin{aligned} a_k \;&= \frac{1}{\pi}\int_0^{2\pi}\cos k\theta\left[\frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\cos\left(\frac{(\theta+2m\pi)^2}{2t}-\frac{\pi}{4}\right)\right] \\ &= \frac{2}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\int_m^{m+1}\cos 2k\pi\theta\,\cos\left(\frac{(2\pi\theta)^2}{2t}-\frac{\pi}{4}\right)d\theta \\ &= \frac{1}{\pi\sqrt{2\pi t}}\int_{-\infty}^{\infty}e^{ik\theta}\cos\left(\frac{\theta^2}{2t}-\frac{\pi}{4}\right)d\theta \;=\; \frac{1}{\pi}\cos\frac{k^2t}{2}. \end{aligned}\tag{5.9}$$

This immediately leads to (5.6). The check that (5.5) and (5.6) are solutions to the equation of vibrations of rods (5.7) is straightforward. □

Remark 5.2. *The law of the composition of the Fresnel pseudoprocess $\mathfrak{F}(t)$, $t>0$, with a positively-skewed stable process $H^\beta(t)$, $t>0$, $\beta\in(0,1]$, has the following representation*

$$\begin{aligned} \mathfrak{f}^\beta(\theta,t) \;&= \frac{1}{2\pi}+\frac{1}{2\pi}\sum_{k=1}^{\infty}\cos k\theta\,\left(e^{-t\left(-\frac{ik^2}{2}\right)^\beta}+e^{-t\left(\frac{ik^2}{2}\right)^\beta}\right) \\ &= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos k\theta e^{-\left(\frac{k^2}{2}\right)^\beta t\cos\frac{\pi\beta}{2}}\cos\left(\frac{k^{2\beta}t}{2^\beta}\,\sin\frac{\pi\beta}{2}\right). \end{aligned}\tag{5.10}$$

Instead the composition of the Fresnel pseudoprocess $\mathfrak{F}(t)$, $t>0$, with the inverse $L^\nu(t)$, $t>0$, of the stable process $H^\nu(t)$, $t>0$, has the Fourier representation

$$\mathfrak{f}^\nu(\theta,t) \;= \frac{1}{2\pi}+\frac{1}{2\pi}\sum_{k=1}^{\infty}\cos k\theta\,\left[E_{\nu,1}\left(i\frac{t^\nu k^2}{2}\right)+E_{\nu,1}\left(-i\frac{t^\nu k^2}{2}\right)\right] \tag{5.11}$$

$$= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos k\theta\,E_{2\nu,1}\left(-\frac{t^{2\nu}k^4}{2^2}\right) \tag{5.12}$$

In (5.10) *we used the characteristic function of positively skewed stable processes and in* (5.11) *we used* (3.2)*. Each term of* (5.11) *is the solution of a time fractional diffusion equation while $\mathfrak{f}^\nu(\theta,t)$ solves the fourth-order equation*

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}}\mathfrak{f}^\nu(\theta,t) \;=\; -\frac{1}{2^2}\frac{\partial^4}{\partial\theta^4}\mathfrak{f}^\nu(\theta,t), \qquad \theta\in[0,2\pi)\,,\, t>0. \tag{5.13}$$

For $\nu=\frac{1}{2}$ we extract from (5.12) *the Fourier expansion of the fourth-order heat equation* (1.13)*. For $\nu=1$, since $E_{2,1}(-x)=\cos\sqrt{x}$, we extract from* (5.12) *the fundamental solution* (5.6) *of* (5.7)*.*

5.1. **Poisson kernels and circular Fresnel pseudoprocess.** If we consider the Poisson kernel $\mathcal{P}(r,\Psi,R,\phi)$ where Ψ is a r.v. with $\Pr\{\Psi=\pm\psi\}=\frac{1}{2}$ we have that

$$\begin{aligned}\mathbb{E}\mathcal{P}(r,\Psi,R,\phi) &= \frac{1}{2\pi}\left[\frac{1}{2}\mathcal{P}(r,+\psi,R,\phi)+\frac{1}{2}\mathcal{P}(r,-\psi,R,\phi)\right] \\ &= \frac{1}{2\pi}\frac{\left(R^2-r^2\right)\left(R^2+r^2-2rR\cos\psi\right)}{\left(R^2+r^2-2rR\cos\psi\cos\phi\right)^2-\left(2rR\sin\psi\sin\phi\right)} \end{aligned} \tag{5.14}$$

The distribution (5.14) with respect to ϕ can be interpreted as the distribution of the hitting point on the circumference of radius R of a Brownian motion starting with probability $\frac{1}{2}$ either from $(r,+\psi)$ or from $(r,-\psi)$. The distribution (5.14) for $\Psi=\pm\frac{\pi}{2}$ takes the fine form

$$\mathbb{E}\mathcal{P}\left(r,\Psi=\pm\frac{\pi}{2},R,\phi\right) = \frac{1}{2\pi}\frac{R^4-r^4}{R^4+r^4+2r^2R^2\cos 2\phi}. \tag{5.15}$$

For arbitrary values of Ψ (5.14) can be written as

$$\mathbb{E}\mathcal{P}(r,\Psi,R,\phi) = \frac{1}{2\pi}\frac{R^2-r^2}{R^2-r^2-2rR\cos\psi}\sum_{k=0}^{\infty}\left(\frac{2rR\sin\psi\sin\phi}{R^2+r^2-2rR\cos\psi\cos\phi}\right)^{2k}. \tag{5.16}$$

Remark 5.3. *The composition*

$$\mathfrak{F}(\tau_t), \quad \textit{where} \quad \tau_t=\inf\{s>0:B(s)=t\}\overset{law}{=}H^{\frac{1}{2}}(\sqrt{2}t), \quad t>0 \tag{5.17}$$

can be viewed as

$$\begin{aligned}\mathfrak{F}(\tau_t) &= \frac{1}{2}\mathfrak{B}(i\tau_t)+\frac{1}{2}\mathfrak{B}(-i\tau_t) \\ &= \frac{1}{2}\mathfrak{B}\left(\tau_{\sqrt{i}t}\right)+\frac{1}{2}\mathfrak{B}\left(\tau_{-\sqrt{i}t}\right) \\ &= \frac{1}{2}\mathfrak{B}\left(H^{\frac{1}{2}}\left(\sqrt{2i}\,t\right)\right)+\frac{1}{2}\mathfrak{B}\left(H^{\frac{1}{2}}\left(-\sqrt{2i}\,t\right)\right) \\ &= \frac{1}{2}\Theta_2\left(\frac{1}{2}H^{\frac{1}{2}}\left(\sqrt{2i}\,t\right)\right)+\frac{1}{2}\Theta_2\left(\frac{1}{2}H^{\frac{1}{2}}\left(-\sqrt{2i}\,t\right)\right) \\ &= \frac{1}{2}\Theta_2\left(H^{\frac{1}{2}}\left(\sqrt{i}\,t\right)\right)+\frac{1}{2}\Theta_2\left(H^{\frac{1}{2}}\left(-\sqrt{i}\,t\right)\right), \end{aligned} \tag{5.18}$$

where $\mathfrak{B}$ and Θ_2 are circular Brownian motions differing only by a scaling factor. Thus by considering Theorem 4.1 we obtain for $\mathfrak{F}(\tau_t)$

$$\mathfrak{f}^{\tau_t}(\theta,t) = \frac{1}{2^2\pi}\left[\frac{1-e^{-\sqrt{2}t(1-i)}}{1-2e^{-\frac{t}{\sqrt{2}}(1-i)}\cos\theta+e^{-\sqrt{2}(1-i)t}}+\frac{1-e^{-\sqrt{2}t(1+i)}}{1-2e^{-\frac{t}{\sqrt{2}}(1+i)}\cos\theta+e^{-\sqrt{2}(1+i)t}}\right]. \tag{5.19}$$

Now we note that (5.19) presents the structure of the sum of two Poisson kernels written as

$$\mathfrak{f}^{\tau_t}(\theta,t) = \frac{1}{2^2\pi}\left[\frac{\left(1-r_1^2\right)}{1+r_1^2-2r_1\cos\theta}+\frac{1-r_2^2}{1+r_2^2-2r_2\cos\theta}\right] \tag{5.20}$$

where

$$r_1 = e^{-\frac{t}{\sqrt{2}}(1-i)}, \qquad r_2 = e^{-\frac{t}{\sqrt{2}}(1+i)}. \tag{5.21}$$

After some manipulations we obtain

$$\mathfrak{f}^{\tau_t}(\theta,t) \;=\; \frac{1}{2^2\pi}\left[\frac{8(1-r_1r_2)r_1r_2\left[1+r_1r_2-(r_1+r_2)\cos\theta\right]}{\left[2^2r_1r_2\cos\theta-(r_1+r_2)(1+r_1r_2)\right]^2-(r_1-r_2)^2(1-r_1^2r_2^2)}\right], \tag{5.22}$$

and by considering (5.21) *we arrive at*

$$\mathfrak{f}^{\tau_t}(\theta,t) \;=\; \frac{1}{2\pi}\frac{\left(1-e^{-\sqrt{2}t}\right)\left(1+e^{-\sqrt{2}t}-2e^{-\frac{t}{\sqrt{2}}}\cos\theta\cos\frac{t}{\sqrt{2}}\right)}{\left(2e^{-\frac{t}{\sqrt{2}}}\cos\theta-\cos\frac{t}{\sqrt{2}}\left(1+e^{-\frac{t}{\sqrt{2}}}\right)\right)^2+\left(1-e^{-2\sqrt{2}t}\right)\sin^2\frac{t}{\sqrt{2}}}. \tag{5.23}$$

We note that (5.22) for $r_1=-r_2=r$ coincides with (5.15) when $R=1$ and $\theta=\phi$.

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