Large deviation principles for telegraph processes and random flights

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Abstract

The aim of this paper is to derive large deviation results for random motions with finite velocity. We start with large deviation principles for the telegraph process with drift: we consider both non-conditional distributions and conditional distributions, and we compare the speeds of convergence with inequalities between rate functions. The same results are presented for the random flights in $\mathbb{R}^2$ and in $\mathbb{R}^4$ which represent a multidimensional version of the telegraph process. Other large deviation principles in this paper concern the non-conditional distributions for an inhomogeneous telegraph process and for a planar random motion with orthogonal directions.

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1 Introduction

The stochastic processes are often used to describe the real motions. An important example is given by the Brownian motion, which was introduced after the fruitful observations of botanist Robert Brown on the movement of the particles in a liquid. However the Brownian motion is not free from defects such as the unbounded first variation. Therefore, over the years, many authors have introduced random models having the main characteristics of the real movements: finite velocity and persistence.

A typical example in this direction is the telegraph process, which is represented by a particle moving on the real line, alternatively forward and backward, with finite speed. Furthermore, one suppose that the changes of direction are governed by a homogeneous Poisson process. The telegraph process is also connected with the theory of the differential equations because its density law satisfies an hyperbolic partial differential equation. The probabilistic properties of this random model have been analyzed, for example, in [25], [10], [26] and [2]. Other authors have also studied generalizations of the telegraph process, concerning the waiting times between two consecutive changes of direction or the randomization of the velocity (see [7], [32], [33], [8], [4]).

It is interesting to observe that applications of the telegraph process emerge in different fields. Indeed, in physics the propagation of a damped wave along a wire is described by the telegraph equation. In ecology the telegraph walk has been exploited to model the movements of the animals on the soil (see [11]). In order to model the dynamics of the price of risky assets, in [9] was considered the geometric telegraph process $\{S(t) : t \geq 0\}$, namely $S(t) = s_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t - \sigma X(t)\right)$, where $S(0) = s_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $\{X(t) : t \geq 0\}$ is the standard telegraph motion. A model of

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financial market based on a telegraph process with drift, that is with two different velocities and switching rates, is introduced in [29] and [30].

Some extensions of the telegraph motion in higher dimensions have been proposed over the years. A random motion in $\mathbb{R}^2$ with four possible orthogonal directions appears in [27] and the exact probability distribution of the process is obtained by means of a suitable representation of the random motion in terms of independent telegraph processes. Planar random motions with an infinite number of directions are analyzed in [31], [21] and [14] and the directions are independent random variables with uniform distribution on a disk with radius one. In [28] this type of random walks, also known as random flights, is studied in $\mathbb{R}^d$ for $d \geq 2$. It is worth to observe that it is possible to derive the explicit conditional (on the number of the changes of direction) and non-conditional distributions for the random flights only in $\mathbb{R}^2$ and $\mathbb{R}^4$. Some possible applications of the random flights concern the displacements of the microorganisms on laboratory slides and the mechanics of the gas particles.

The aim of this paper is to present asymptotic results for telegraph processes (possibly with drift) and some of its multidimensional versions in the literature recalled above. Here we refer to the theory of large deviations which gives an asymptotic computation of small probabilities on exponential scale. The most interesting results in this paper concern the cases where we investigate both the non-conditional distributions and conditional distributions for the same model; in these cases we can compare the speeds of convergence with inequalities between large deviation rate functions. Large deviation results for the telegraph process can be derived from the ones for Markov additive processes (see e.g. [13], [22], [23] and [24]); the computations specified for the telegraph process can be found in [18] (section 2) and [20]. We are not aware of any other work in the literature on large deviations for conditional distributions of the telegraph process and for some multidimensional versions of the telegraph process such as the random flights.

The outline of the paper is the following. We start with some preliminaries on large deviations in section 2. In section 3 we present the results for the telegraph process with drift, for an inhomogeneous case studied in [12], and for the planar random motion in [27]. Finally section 4 is devoted to the results for random flights.

We conclude with some standard notation used throughout the paper: We write $f(x) \sim g(x)$ as $x \to \ell$ to mean that $\lim_{x\to\ell} \frac{f(x)}{g(x)} = 1$ (throughout this paper we have $\ell = 0$ and $\ell = \infty$); for a given set S, we write $\overline{S}$ and S^c for the closure and the complementary of S, respectively.

2 Preliminaries on large deviations

We recall the basic definitions in [6] (pages 4–5). Let $\mathcal{Z}$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_{\mathcal{Z}}$. A lower semi-continuous function $I : \mathcal{Z} \to [0, \infty]$ is called rate function. A family of probability measures $\{\nu_t : t > 0\}$ on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ satisfies the *large deviation principle* (LDP for short), as $t \to \infty$, with rate function I if

$$\limsup_{t\to\infty} \frac{1}{t} \log \nu_t(F) \leq -\inf_{z\in F} I(z) \quad \text{for all closed sets } F$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log \nu_t(G) \geq -\inf_{z\in G} I(z) \quad \text{for all open sets } G.$$

A rate function I is said to be good if all the level sets $\{\{z \in \mathcal{Z} : I(z) \leq \gamma\} : \gamma \geq 0\}$ are compact. In what follows we use condition (b) with equation (1.2.8) in [6], which is equivalent to the lower bound for open sets:

$$\liminf_{t\to\infty} \frac{1}{t} \log \nu_t(G) \geq -I(z) \qquad \begin{array}{l}\text{for all } z \in \mathcal{Z} \text{ such that } I(z) < \infty \text{ and} \\ \text{for all open sets } G \text{ such that } z \in G.\end{array} \tag{1}$$

A known large deviation result used throughout the paper is the contraction principle (see e.g. Theorem 4.2.1 in [6]). Roughly speaking, if a family of probability measures $\{\nu_t : t > 0\}$ on $(\mathcal{Z}, \mathcal{B}_\mathcal{Z})$ satisfies the LDP with a good rate function I and $f : \mathcal{Z} \to \mathcal{Y}$ is a continuous function, then the family of probability measures $\{\nu_t \circ f^{-1} : t > 0\}$ on $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ satisfies the LDP with a good rate function J defined by $J(y) = \inf\{I(z) : f(z) = y\}$.

In some cases we talk about LDP for a family of $\mathcal{Z}$-valued random variables $\{Z_t : t > 0\}$; in such a case we consider the above definition with $\nu_t(\cdot) = P(Z_t \in \cdot)$.

Remark 2.1. *Assume that $\{\nu_t^{(1)} : t > 0\}$ and $\{\nu_t^{(2)} : t > 0\}$ satisfy the LDP with the rate functions I_1 and I_2, respectively; moreover assume that I_1 and I_2 uniquely vanish at the same point z_0. Then, if we have $I_1(z) > I_2(z)$ for all z in a neighborhood U of z_0 (except z_0 because $I_1(z_0) = I_2(z_0) = 0$), we can say that $\{\nu_t^{(1)} : t > 0\}$ converges to z_0 faster than $\{\nu_t^{(2)} : t > 0\}$, as $t \to \infty$. Indeed, for all $\varepsilon > 0$, there exists t_ε such that $\frac{\nu_t^{(1)}(U^c)}{\nu_t^{(2)}(U^c)} \leq e^{-t(I_1(U^c) - I_2(U^c) + 2\varepsilon)}$ for all $t > t_\varepsilon$, where $I_k(U^c) = \inf_{z \in U^c} I_k(z)$ for $k \in \{1, 2\}$; thus, since $I_1(U^c) > I_2(U^c) > 0$, we have $\frac{\nu_t^{(1)}(U^c)}{\nu_t^{(2)}(U^c)} \to 0$ as $t \to \infty$.*

3 Results for telegraph processes and for a planar random motion

The telegraph process with drift is a random motion $\{X(t) : t \geq 0\}$ on the real line which starts at the origin of $\mathbb{R}$ and moves with a two-valued integrated telegraph signal, i.e., for some $\lambda_1 \lambda_2, c_1, c_2 > 0$, we have a rightward velocity c_1, a leftward velocity $-c_2$, and the rates of the occurrences of velocity switches are λ_1 and λ_2, respectively. More precisely we have

$$X(t) = V(0) \int_0^t (-1)^{N(s)} ds$$

where: the random variable $V(0)$ is such that $P(V(0) \in \{-c_2, c_1\}) = 1$; the process $\{N(t) : t \geq 0\}$ which counts the number of changes of direction is an inhomogeneous Poisson process such that $N(t) = \sum_{n \geq 1} 1_{\{S_1 + \cdots S_n \leq t\}}$, the random variables $\{S_n : n \geq 1\}$ are conditionally independent given $V(0)$, and the conditional distribution is the following:

$$\text{if } V(0) = c_1, \text{ then } \left\{ \begin{array}{l} \{S_{2k-1} : k \geq 1\} \text{ are exponentially distributed with mean } \frac{1}{\lambda_1} \\ \{S_{2k} : k \geq 1\} \text{ are exponentially distributed with mean } \frac{1}{\lambda_2}; \end{array} \right.$$
$$\text{if } V(0) = -c_2, \text{ then } \left\{ \begin{array}{l} \{S_{2k-1} : k \geq 1\} \text{ are exponentially distributed with mean } \frac{1}{\lambda_2} \\ \{S_{2k} : k \geq 1\} \text{ are exponentially distributed with mean } \frac{1}{\lambda_1}. \end{array} \right.$$

We remark that we do not have drift if $c_1 = c_2$ and $\lambda_1 = \lambda_2$. We often assume that $\lambda_1 = \lambda_2 = \lambda$ for some $\lambda > 0$; in such a case $\{N(t) : t \geq 0\}$ is a Poisson process with intensity λ and, moreover, $V(0)$ and $\{N(t) : t \geq 0\}$ are independent (because $V(0)$ and $\{S_n : n \geq 1\}$ are independent).

It is useful to consider a different point of view. The process $\{X(t) : t \geq 0\}$ can be seen as the additive component of a very simple continuous time Markov additive process driven by a Markov chain $J = \{J(t) : t \geq 0\}$, with state space $\{1, 2\}$ and intensity matrix

$$\begin{pmatrix} -\lambda_1 & +\lambda_1 \\ +\lambda_2 & -\lambda_2 \end{pmatrix}$$

(for the illustration of this point of view, see e.g. [1], chapter 2, section 5, page 40). We remark that $\{J(0) = k\} = \{V(0) = (-1)^{k+1} c_k\}$ for $k \in \{1, 2\}$; moreover $\{N(t) : t \geq 0\}$ is the transition number process concerning the Markov chain J.

In subsection 3.1 we present two LDPs and we compare the two rate functions: firstly we recall the known LDP for the non-conditional distributions $\left\{P\left(\frac{X(t)}{t}\in\cdot\right):t>0\right\}$; secondly, assuming that $\lambda_1=\lambda_2$, we prove the LDP for the conditional distributions $\left\{P\left(\frac{X(t)}{t}\in\cdot\left|\frac{N(t)}{t}=w_t\right.\right):t>0\right\}$ assuming that w_t converges to some $w\in(0,\infty)$ as $t\to\infty$. A connection between a result in subsection 3.1 and a result in [19] is discussed in subsection 3.2. In subsection 3.3 we prove the LDP for the non-conditional distributions $\left\{P\left(\frac{Y_\theta(t)}{t}\in\cdot\right):t>0\right\}$, where $\{Y_\theta(t):t\geq 0\}$ is the inhomogeneous telegraph process in Theorem 2.1 in [12]; more precisely the process $\{N_\theta(t):t\geq 0\}$ which counts the number of changes of direction is driven by the intensity $\{\lambda_\theta:t\geq 0\}$ defined by $\lambda_\theta(t):=\theta\tanh(\theta t)$ for $\theta>0$. Finally, in subsection 3.4, we prove the LDP for the non-conditional distributions $\left\{P\left(\left(\frac{Y_1(t)}{t},\frac{Y_2(t)}{t}\right)\in\cdot\right):t>0\right\}$, where $\{(Y_1(t),Y_2(t)):t\geq 0\}$ is the process in [27] which describes the planar random motion of a particle moving with orthogonal directions.

3.1 The telegraph process with drift

In this subsection we allow a general initial distribution of J, i.e. we set

$$(P(J(0)=1),P(J(0)=2))=(p,1-p)$$

for some $p\in[0,1]$; as we shall see the LDPs proved here do not depend on the value p.

We start with Proposition 3.1 which provides the LDP for the non-conditional distributions. It is a known result and there are several references for this LDP (see the references recalled in the Introduction which refers to Markov additive processes). In view of what follows it is useful to refer to the proof in [20] (section 3) which provides an explicit expression of the rate function: this proof is based on the LDP for the empirical laws

$$\left\{\left(\frac{\int_0^t 1_{\{J(s)=1\}}ds}{t},\frac{\int_0^t 1_{\{J(s)=2\}}ds}{t}\right):t>0\right\} \tag{2}$$

of $\{J(t):t\geq 0\}$ and a very easy application of contraction principle.

Proposition 3.1. *The family* $\left\{P\left(\frac{X(t)}{t}\in\cdot\right):t>0\right\}$ *satisfies the LDP with good rate function* $I^X_{\lambda_1,\lambda_2,c_1,c_2}$ *defined by*

$$I^X_{\lambda_1,\lambda_2,c_1,c_2}(x)=\begin{cases}\left(\sqrt{\lambda_1\frac{x+c_2}{c_1+c_2}}-\sqrt{\lambda_2\frac{c_1-x}{c_1+c_2}}\right)^2 & \text{if } x\in[-c_2,c_1]\\ \infty & \text{otherwise.}\end{cases}$$

Remark 3.2. *Note that, if* $\lambda_1=\lambda_2=\lambda$ *and* $c_1=c_2=c$ *for some* $\lambda,c>0$*, i.e.* $\{X(t):t\geq 0\}$ *is a standard telegraph process without drift, the rate function in Proposition 3.1 becomes*

$$I^X_{\lambda,\lambda,c,c}(x)=\begin{cases}\lambda\left(1-\sqrt{1-\frac{x^2}{c^2}}\right) & \text{if } x\in[-c,c]\\ \infty & \text{otherwise.}\end{cases}$$

Now we prove the LDP for the conditional distributions assuming that $\lambda_1=\lambda_2$.

Proposition 3.3. *Assume that* $\lambda_1=\lambda_2=\lambda$ *for some* $\lambda>0$*. If* $w_t\to w\in(0,\infty)$*, then* $\left\{P\left(\frac{X(t)}{t}\in\cdot\left|\frac{N(t)}{t}=w_t\right.\right):t>0\right\}$ *satisfies the LDP with good rate function* $I^{X|N}_{\lambda,\lambda,c_1,c_2}(\cdot;w)$ *defined by*

$$I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)=\begin{cases}w\log\left(\frac{c_1+c_2}{2\sqrt{(c_2+x)(c_1-x)}}\right) & \text{if } -c_2<x<c_1\\ \infty & \text{otherwise.}\end{cases}$$

Proof. We consider the extension of equations (2.17)-(2.18) in [5] which concern the case $c_1 = c_2 = c$ (for some $c > 0$) and $p = \frac{1}{2}$; see Appendix for details. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log P\left(\frac{X(t)}{t} \in G \Big| \frac{N(t)}{t} = w_t\right) \geq -w \log\left(\frac{c_1 + c_2}{2\sqrt{(c_2 + x)(c_1 - x)}}\right).$$

for all x such that $-c_2 < x < c_1$ and for all open sets G such that $x \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $(x - \varepsilon, x + \varepsilon) \subset G \cap (-c_2, c_1)$. Moreover, by taking into account the conditional distributions derived in the Appendix, for some $y_\varepsilon \in (x - \varepsilon, x + \varepsilon)$, we have

$$\begin{aligned}
&P\left(\frac{X(t)}{t} \in G \Big| \frac{N(t)}{t} = w_t\right) \geq P\left(\frac{X(t)}{t} \in (x - \varepsilon, x + \varepsilon) \Big| \frac{N(t)}{t} = w_t\right) \\
&= \begin{cases} \frac{(tw_t)!}{\left(\left(\frac{tw_t - 1}{2}\right)!\right)^2 ((c_1+c_2)t)^{tw_t}} \int_{(x-\varepsilon)t}^{(x+\varepsilon)t} \{(c_2 t + y)(c_1 t - y)\}^{\frac{tw_t - 1}{2}} dy & \text{if } tw_t \in \{1, 3, 5, \dots\} \\ \frac{(tw_t)!}{\left(\frac{tw_t}{2}\right)!\left(\frac{tw_t}{2} - 1\right)!((c_1+c_2)t)^{tw_t}} \int_{(x-\varepsilon)t}^{(x+\varepsilon)t} \{(c_2 t + y)(c_1 t - y)\}^{\frac{tw_t}{2} - 1} & \\ \cdot\{(c_2 t + y)p + (c_1 t - y)(1 - p)\} dy & \text{if } tw_t \in \{2, 4, 6, \dots\} \end{cases} \\
&= \begin{cases} \frac{(tw_t)!}{\left(\left(\frac{tw_t - 1}{2}\right)!\right)^2 ((c_1+c_2)t)^{tw_t}} t^{tw_t} \{(c_2 + y_\varepsilon)(c_1 - y_\varepsilon)\}^{\frac{tw_t - 1}{2}} 2\varepsilon & \text{if } tw_t \in \{1, 3, 5, \dots\} \\ \frac{(tw_t)!}{\left(\frac{tw_t}{2}\right)!\left(\frac{tw_t}{2} - 1\right)!((c_1+c_2)t)^{tw_t}} t^{tw_t} \{(c_2 + y_\varepsilon)(c_1 - y_\varepsilon)\}^{\frac{tw_t}{2} - 1} & \\ \cdot\{(c_2 + y_\varepsilon)p + (c_1 - y_\varepsilon)(1 - p)\} 2\varepsilon & \text{if } tw_t \in \{2, 4, 6, \dots\}. \end{cases}
\end{aligned}$$

Then, since $\lim_{n\to\infty} \frac{\log(n!)}{n \log(n)} = 1$, we obtain

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log P\left(\frac{X(t)}{t} \in G \Big| \frac{N(t)}{t} = w_t\right) \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \left(tw_t \log(tw_t) - 2\frac{tw_t}{2} \log\left(\frac{tw_t}{2}\right) - tw_t \log(c_1 + c_2) + \frac{tw_t}{2} \log\{(c_2 + y_\varepsilon)(c_1 - y_\varepsilon)\}\right) \\
&= w \log 2 - w \log(c_1 + c_2) + \frac{w}{2} \log\{(c_2 + y_\varepsilon)(c_1 - y_\varepsilon)\} = -w \log\left(\frac{c_1 + c_2}{2\sqrt{(c_2 + y_\varepsilon)(c_1 - y_\varepsilon)}}\right),
\end{aligned}$$

and we complete the proof of the lower bound letting ε go to zero.
2) Proof of the upper bound for closed sets. We have to check

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(\frac{X(t)}{t} \in F \Big| \frac{N(t)}{t} = w_t\right) \leq -\inf_{x\in F} I_{\lambda,\lambda,c_1,c_2}^{X|N}(x; w) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\frac{c_1 - c_2}{2} \in F$ and if $F \cap (-c_2, c_1) = \emptyset$. Thus, from now on, we assume that $\frac{c_1 - c_2}{2} \notin F$ and $F \cap (-c_1, c_2) \neq \emptyset$. We also assume that $F \cap (\frac{c_1 - c_2}{2}, c_1), F \cap (-c_2, \frac{c_1 - c_2}{2}) \neq \emptyset$; otherwise the proof below can be readily adapted. We remark that $I_{\lambda,\lambda,c_1,c_2}^{X|N}(\cdot; w)$ is decreasing in $(-c_2, \frac{c_1 - c_2}{2}]$ and $I_{\lambda,\lambda,c_1,c_2}^{X|N}(\cdot; w)$ is increasing in $[\frac{c_1 - c_2}{2}, c_1)$; then, if we set $\underline{x}_F := \sup(F \cap (-c_2, \frac{c_1 - c_2}{2}))$ and $\overline{x}_F := \inf(F \cap (\frac{c_1 - c_2}{2}, c_1))$, we have $F \subset (-\infty, \underline{x}_F] \cup [\overline{x}_F, \infty)$, where $\underline{x}_F \in (-c_2, \frac{c_1 - c_2}{2})$ and $\overline{x}_F \in (\frac{c_1 - c_2}{2}, c_1)$. Then, by taking into account again the conditional distributions derived in the Appendix, we have

$$
\begin{aligned}
&P\left(\frac{X(t)}{t}\geq\overline{x}_F\Big|\frac{N(t)}{t}=w_t\right)\\
=&\begin{cases}
\frac{(tw_t)!}{\left((\frac{tw_t-1}{2})!\right)^2((c_1+c_2)t)^{tw_t}}\int_{\overline{x}_Ft}^{c_1t}\{(c_2t+y)(c_1t-y)\}^{\frac{tw_t-1}{2}}dy & \text{if } tw_t\in\{1,3,5,\dots\}\\
\frac{(tw_t)!}{(\frac{tw_t}{2})!(\frac{tw_t}{2}-1)!((c_1+c_2)t)^{tw_t}}\int_{\overline{x}_Ft}^{c_1t}\{(c_2t+y)(c_1t-y)\}^{\frac{tw_t}{2}-1} & \\
\cdot\{(c_2t+y)p+(c_1t-y)(1-p)\}dy & \text{if } tw_t\in\{2,4,6,\dots\}
\end{cases}\\
\leq&\begin{cases}
\frac{(tw_t)!}{\left((\frac{tw_t-1}{2})!\right)^2((c_1+c_2)t)^{tw_t}}t^{tw_t}\{(c_2+\overline{x}_F)(c_1-\overline{x}_F)\}^{\frac{tw_t-1}{2}}(c_1-\overline{x}_F) & \text{if } tw_t\in\{1,3,5,\dots\}\\
\frac{(tw_t)!}{(\frac{tw_t}{2})!(\frac{tw_t}{2}-1)!((c_1+c_2)t)^{tw_t}}t^{tw_t}\{(c_2+\overline{x}_F)(c_1-\overline{x}_F)\}^{\frac{tw_t}{2}-1} & \\
\cdot(c_1+c_2)(c_1-\overline{x}_F) & \text{if } tw_t\in\{2,4,6,\dots\}
\end{cases}
\end{aligned}
$$

and, by taking into account that $\lim_{n\to\infty}\frac{\log(n!)}{n\log(n)}=1$ as in the first part of the proof, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{X(t)}{t}\geq\overline{x}_F\Big|\frac{N(t)}{t}=w_t\right)\\
&\quad\leq\limsup_{t\to\infty}\frac{1}{t}\left(tw_t\log(tw_t)-2\frac{tw_t}{2}\log\left(\frac{tw_t}{2}\right)-tw_t\log(c_1+c_2)+\frac{tw_t}{2}\log\{(c_2+\overline{x}_F)(c_1-\overline{x}_F)\}\right)\\
&\quad=w\log 2-w\log(c_1+c_2)+\frac{w}{2}\log\{(c_2+\overline{x}_F)(c_1-\overline{x}_F)\}=-I^{X|N}_{\lambda,\lambda,c_1,c_2}(\overline{x}_F;w).
\end{aligned}
$$

Finally, since we can prove the inequality

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{X(t)}{t}\leq\underline{x}_F\Big|\frac{N(t)}{t}=w_t\right)\leq-I^{X|N}_{\lambda,\lambda,c_1,c_2}(\underline{x}_F;w)$$

in a similar way, we complete the proof of the upper bound by taking into account $F\subset(-\infty,\underline{x}_F]\cup[\overline{x}_F,\infty)$, by applying Lemma 1.2.15 in [6] and noting that $\max\{-I^{X|N}_{\lambda,\lambda,c_1,c_2}(\underline{x}_F;w),-I^{X|N}_{\lambda,\lambda,c_1,c_2}(\overline{x}_F;w)\}$ coincides with $-\inf_{x\in F}I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)$. □

In Remark 3.5 at the end of this subsection we compare the LDPs of non-conditional distributions and of conditional distributions in the spirit of Remark 2.1; in view of this we refer to the inequalities between the rate functions $I^{X}_{\lambda,\lambda,c_1,c_2}$ and $I^{X|N}_{\lambda,\lambda,c_1,c_2}(\cdot;w)$ proved in the next proposition (see Figure 1 for the case $\lambda=c_1=1$ and $c_2=2$).

Proposition 3.4. *We have two cases. (i) For $w\geq\lambda$, we have $I^{X}_{\lambda,\lambda,c_1,c_2}(x)\leq I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)$ for all $x\in\mathbb{R}$; moreover the inequality is strict for $x\in[-c_2,c_1]\setminus\{\frac{c_1-c_2}{2}\}$. (ii) For $w\in(0,\lambda)$ set $\alpha_\pm=\frac{1}{2}\left[c_1-c_2\pm\sqrt{1-\frac{w^2}{\lambda^2}}(c_1+c_2)\right]$; then there exist $\beta_+\in(\alpha_+,c_1)$ and $\beta_-\in(-c_2,\alpha_-)$ such that $I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)>I^{X}_{\lambda,\lambda,c_1,c_2}(x)$ for $x\in[-c_2,\beta_-)\cup(\beta_+,c_1]$, $I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)<I^{X}_{\lambda,\lambda,c_1,c_2}(x)$ for all $x\in(\beta_-,\beta_+)\setminus\{\frac{c_1-c_2}{2}\}$ and $I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)=I^{X}_{\lambda,\lambda,c_1,c_2}(x)$ otherwise.*

Proof. Firstly note that, for all $w>0$, we have: $I^{X|N}_{\lambda,\lambda,c_1,c_2}(\frac{c_1-c_2}{2};w)=I^{X}_{\lambda,\lambda,c_1,c_2}(\frac{c_1-c_2}{2})=0$; $I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)=I^{X}_{\lambda,\lambda,c_1,c_2}(x)=\infty$ for $x\notin[-c_2,c_1]$; $\lambda=I^{X}_{\lambda,\lambda,c_1,c_2}(x)<I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)=\infty$ for $x\in\{-c_2,c_1\}$. Moreover let us consider the difference function (for $x\in(-c_2,c_1)$) defined by

$$\Delta(x):=I^{X|N}_{\lambda,\lambda,c_1,c_2}(x;w)-I^{X}_{\lambda,\lambda,c_1,c_2}(x)=w\log\left(\frac{c_1+c_2}{2\sqrt{(c_2+x)(c_1-x)}}\right)-\lambda\left(1-\frac{2\sqrt{(c_2+x)(c_1-x)}}{c_1+c_2}\right).$$

Then the derivative is $\Delta'(x) = \frac{x-\frac{c_1-c_2}{2}}{(c_2+x)(c_1-x)}\left(w - \frac{2\lambda\sqrt{(c_2+x)(c_1-x)}}{c_1+c_2}\right)$, and we complete the proof as follows.

Statement (i) can be proved noting that, for $w \geq \lambda$, the function Δ is decreasing in the interval $(-c_2, \frac{c_1-c_2}{2})$, is increasing in the interval $(\frac{c_1-c_2}{2}, c_1)$, and its global minimum (uniquely attained at $x = \frac{c_1-c_2}{2}$) is equal to zero.

Statement (ii) can be proved noting that, for $w \in (0, \lambda)$ and $\alpha_\pm = \frac{1}{2}\left[c_1 - c_2 \pm \sqrt{1 - \frac{w^2}{\lambda^2}}(c_1 + c_2)\right]$, we have what follows: the function Δ is decreasing in the intervals $(-c_2, \alpha_-)$ and $(\frac{c_1-c_2}{2}, \alpha_+)$, and is increasing in the intervals $(\alpha_-, \frac{c_1-c_2}{2})$ and (α_+, c_1); a local maximum of Δ is attained at $x = \frac{c_1-c_2}{2}$ and is equal to zero; the minimum of Δ is attained for $x \in \{\alpha_-, \alpha_+\}$ and is equal to $-\left[w \log\left(\frac{w}{\lambda}\right) - w + \lambda\right] < 0$; $\Delta(x) \uparrow \infty$ as $x \uparrow c_1$ and as $x \downarrow -c_2$. $\square$

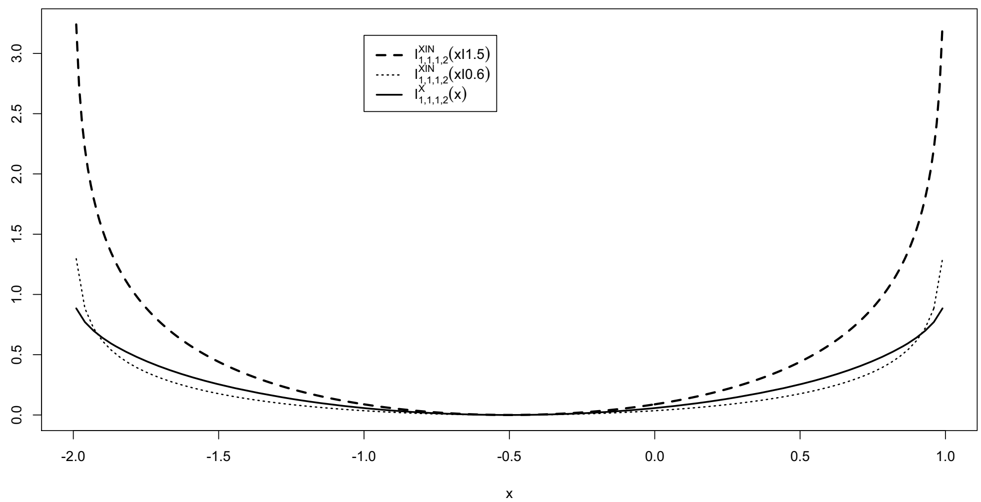

Figure 1: The functions $I^X_{1,1,1,2}(x)$ and $I^{X|N}_{1,1,1,2}(x;w)$, where $x \in (-2, 1)$. Two choices for w: $w = 1.5 \geq 1$; $w = 0.6 \in (0, 1)$.

Remark 3.5. *In the LDP presented for conditional distributions we require that $\frac{N(t)}{t}$ converges to some w (as $t \to \infty$). One can expect that, for w large enough, the convergence of the conditional distributions to the limit $\frac{c_1-c_2}{2}$ is faster than the convergence of the non-conditional distributions. Actually, by taking into account Remark 2.1 and the inequalities between $I^{X|N}_{\lambda,\lambda,c_1,c_2}(\cdot; w)$ and $I^X_{\lambda,\lambda,c_1,c_2}$ proved in Proposition 3.4, we can say that the convergence of the conditional distributions is always faster than the convergence of the non-conditional distributions if and only if $w \geq \lambda$.*

3.2 A connection between Proposition 3.3 and another LDP in the literature

The empirical laws in (2) are consistent estimators of the stationary distribution $(\pi_J^{(1)}, \pi_J^{(2)}) = \left(\frac{\lambda_2}{\lambda_1+\lambda_2}, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$ of $\{J(t) : t > 0\}$. Proposition 3.2 in [19] provides the LDP for another family $\{(\hat{\pi}_n^{(1)}, \hat{\pi}_n^{(2)}) : n \geq 1\}$ of consistent estimators of $(\pi_J^{(1)}, \pi_J^{(2)})$ defined by

$$\hat{\pi}_n^{(i)} = \frac{\sum_{k=0}^{n-1}(T_{k+1} - T_k)1_{\{J_k=i\}}}{T_n},$$

where $T_0 = 0$ and $\{T_n : n \geq 1\}$ are the epochs of the occurrences of the transition number process, i.e. $N(t) = \sum_{n\geq 1} 1_{\{T_n \leq t\}}$ (see eq. (3) in [19]). This family is derived from another family of consistent estimators presented in [16] for the stationary distribution of a general semi-Markov process with finite state space and irreducible embedded Markov chain. The large deviation rate function for $\{(\hat{\pi}_n^{(1)}, \hat{\pi}_n^{(2)}) : n \geq 1\}$ is H defined by (see subsection 4.3 in [19])

$$H(\pi_1, \pi_2) = \begin{cases} \log\left(\frac{1}{2}\left(\sqrt{\frac{\lambda_1\pi_1}{\lambda_2\pi_2}} + \sqrt{\frac{\lambda_2\pi_2}{\lambda_1\pi_1}}\right)\right) & if\ \pi_1, \pi_2 \in (0,1) \text{ and } \pi_1 + \pi_2 = 1 \\ \infty & \text{otherwise.} \end{cases}$$

Then, proceeding as in the proof of Proposition 3.1 illustrated in [20] (section 3), $\{c_1\hat{\pi}_n^{(1)} - c_2\hat{\pi}_n^{(2)} : n \geq 1\}$ satisfies the LDP with the rate function $\tilde{I}^X_{\lambda_1,\lambda_2,c_1,c_2}$ defined by

$$\tilde{I}^X_{\lambda_1,\lambda_2,c_1,c_2}(x) := \inf\{H(\pi_1, \pi_2) : c_1\pi_1 - c_2\pi_2 = x\},$$

and we have

$$\tilde{I}^X_{\lambda_1,\lambda_2,c_1,c_2}(x) = \begin{cases} \log\left(\frac{(\lambda_1-\lambda_2)x+\lambda_1c_2+\lambda_2c_1}{2\sqrt{\lambda_1\lambda_2(c_1-x)(x+c_2)}}\right) & \text{if } x \in (-c_2, c_1) \\ \infty & \text{otherwise.} \end{cases}$$

Finally we remark that $\tilde{I}^X_{\lambda,\lambda,c_1,c_2} = I^{X|N}_{\lambda,\lambda,c_1,c_2}(\cdot|1)$, where the right hand side is a particular case of the rate function in Proposition 3.3; a possible explanation of this equality is that $\hat{\pi}_n^{(k)}$ is the normalized occupation time in the state $k \in \{1, 2\}$ after n transitions of $\{J(t) : t > 0\}$.

3.3 An inhomogeneous case

In [12] it was shown that the finite dimensional law of an inhomogeneous telegraph process driven by a Poisson process with intensity $\{\lambda(t) : t \geq 0\}$ is a solution of the telegraph equation with nonconstant coefficients. Here we consider an inhomogeneous telegraph process $\{Y_\theta(t) : t \geq 0\}$, for $\theta > 0$, which is a rare example for which the finite dimensional law can be provided explicitly. We set

$$Y_\theta(t) = V(0)\int_0^t (-1)^{N_\theta(s)} ds \tag{3}$$

where: the random variable $V(0)$ is such that $P(V(0) = c) = P(V(0) = -c) = \frac{1}{2}$; the process $\{N_\theta(t) : t \geq 0\}$ is an inhomogeneous Poisson process with intensity $\{\lambda_\theta(t) : t \geq 0\}$ defined by $\lambda_\theta(t) := \theta\tanh(\theta t)$; $V(0)$ and $\{N_\theta(t) : t \geq 0\}$ are independent. We remark that $\Lambda_\theta(t) := \int_0^t \lambda_\theta(s)ds = \log\cosh(\theta t)$ and we have $\lim_{t\to\infty} \frac{\Lambda_\theta(t)}{t} = \theta$ because $\theta t - \log 2 \leq \Lambda_\theta(t) \leq \theta t$ for all $t \geq 0$.

Proposition 3.6. *The family* $\left\{P\left(\frac{Y_\theta(t)}{t} \in \cdot\right) : t > 0\right\}$ *satisfies the LDP with good rate function* $I^X_{\theta,\theta,c,c}$ *as in Proposition in 3.1 (see Remark 3.2).*

Proof. In this proof we consider the law of $Y_\theta(t)$ for any fixed $t > 0$ (see e.g. Theorem 2.1 in [12]):

$$P(Y_\theta(t) \in E) = \int_{E\cap(-ct,ct)} \frac{\theta t}{\cosh(\theta t)} \frac{I_1(\frac{\theta}{c}\sqrt{c^2t^2 - x^2})}{2\sqrt{c^2t^2 - x^2}} dx + \frac{1}{2\cosh(\theta t)} 1_E(ct) + \frac{1}{2\cosh(\theta t)} 1_E(-ct)$$

for any measurable set E, where I_1 is the modified Bessel function of the first kind with $\nu = 1$. We recall that, for $\nu \in \mathbb{R}$, the modified Bessel function of the first kind is $I_\nu(r) := \sum_{k=0}^\infty \frac{(r/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$ (see e.g. eq. (5.7.1) in [15]); moreover we have $I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ as $x \to \infty$ (see e.g. eq. (5.11.10) in

[15]), and therefore $\lim_{x\to\infty}\frac{\log I_1(x)}{x}=1$. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{Y_\theta(t)}{t}\in G\right)\geq-\theta\left(1-\sqrt{1-\frac{x^2}{c^2}}\right)$$

for all $x\in[-c,c]$ and for all open sets G such that $x\in G$. Firstly we can take $\varepsilon>0$ small enough to have $(x-\varepsilon,x+\varepsilon)\subset G$; moreover, if $x\in(-c,c)$, we also require that $(x-\varepsilon,x+\varepsilon)\subset(-c,c)$. We consider the function f defined by $f(r):=\frac{I_1(r)}{r}$ for $r\neq 0$; moreover we set $f(0):=\frac{1}{2}$ and, since $I_\nu(x)\sim\frac{x^\nu}{2^\nu\Gamma(\nu+1)}$ as $x\to 0$ (see e.g. eq. (5.16.4) in [15]), f is continuous on $\mathbb{R}$. Then, for some $y_\varepsilon\in(x-\varepsilon,x+\varepsilon)\cap[-c,c]$, we have

$$\begin{aligned}
&P\left(\frac{Y_\theta(t)}{t}\in G\right)\geq P\left(\frac{Y_\theta(t)}{t}\in(x-\varepsilon,x+\varepsilon)\right)\\
&\geq\int_{((x-\varepsilon)t,(x+\varepsilon)t)\cap[-ct,ct]}\frac{\theta t}{\cosh(\theta t)}\frac{\theta}{2c}f\left(\frac{\theta}{c}\sqrt{c^2t^2-y^2}\right)dy\\
&=\int_{(x-\varepsilon,x+\varepsilon)\cap[-c,c]}\frac{\theta^2t^2}{2c\cdot\cosh(\theta t)}f\left(\frac{\theta t}{c}\sqrt{c^2-y^2}\right)dy\\
&=\frac{\theta^2t^2}{2c\cdot\cosh(\theta t)}f\left(\theta t\sqrt{1-\frac{y_\varepsilon^2}{c^2}}\right)\underbrace{\text{measure}((x-\varepsilon,x+\varepsilon)\cap[-c,c])}_{>0};
\end{aligned}$$

thus we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{Y_\theta(t)}{t}\in G\right)\geq-\theta\left(1-\sqrt{1-\frac{y_\varepsilon^2}{c^2}}\right),$$

and we complete the proof of the lower bound letting ε go to zero.
2) Proof of the upper bound for open sets. We have to check

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{Y_\theta(t)}{t}\in F\right)\leq-\inf_{x\in F}I^X_{\theta,\theta,c,c}(x)\quad\text{for all closed sets }F.$$

Firstly note that this condition trivially holds if $0\in F$ and if $F\cap[-c,c]=\emptyset$. Thus, from now on, we assume that $0\notin F$ and $F\cap[-c,c]\neq\emptyset$. We also assume that $F\cap(0,c],F\cap[-c,0)\neq\emptyset$; otherwise the proof below can be readily adapted. We remark that $I^X_{\theta,\theta,c,c}$ is decreasing in $[-c,0]$ and $I^X_{\theta,\theta,c,c}$ is increasing in $[0,c]$; then, if we set $\underline{x}_F:=\sup(F\cap[-c,0))$ and $\overline{x}_F:=\inf(F\cap(0,c])$, we have $F\subset(-\infty,\underline{x}_F]\cup[\overline{x}_F,\infty)$, where $\underline{x}_F\in[-c,0)$ and $\overline{x}_F\in(0,c]$. Then, since $I_1(r)$ is increasing for $r>0$, we have

$$\begin{aligned}
&P\left(\frac{Y_\theta(t)}{t}\geq\overline{x}_F\right)=\int_{\overline{x}_Ft}^{ct}\frac{\theta t}{\cosh(\theta t)}\frac{I_1\left(\frac{\theta}{c}\sqrt{c^2t^2-y^2}\right)}{2\sqrt{c^2t^2-y^2}}dy+\frac{1}{2\cosh(\theta t)}\\
&=\int_{\overline{x}_F}^{c}\frac{\theta t}{\cosh(\theta t)}\frac{I_1\left(\frac{\theta t}{c}\sqrt{c^2-y^2}\right)}{2t\sqrt{c^2-y^2}}tdy+\frac{1}{2\cosh(\theta t)}\\
&\leq\frac{\theta t}{\cosh(\theta t)}\frac{I_1\left(\frac{\theta t}{c}\sqrt{c^2-\overline{x}_F^2}\right)}{2}\int_{\overline{x}_F}^{c}\frac{1}{\sqrt{c^2-y^2}}dy+\frac{1}{2\cosh(\theta t)};
\end{aligned}$$

actually, if $\overline{x}_F=c$, we have $P\left(\frac{Y_\theta(t)}{t}\geq\overline{x}_F\right)=\frac{1}{2\cosh(\theta t)}$. In conclusion we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{Y_\theta(t)}{t}\geq\overline{x}_F\right)\leq-I^X_{\theta,\theta,c,c}(\overline{x}_F)$$

(if $\overline{x}_F \in (0,c)$ and $p>0$, this is a consequence of Lemma 1.2.15 in [6]). Finally, since we can prove the inequality

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{Y_\theta(t)}{t}\le \underline{x}_F\right)\le -I^X_{\theta,\theta,c,c}(\underline{x}_F)$$

in a similar way, we complete the proof of the upper bound following the lines of the second part of the proof of Proposition 3.3. □

We are able to provide an alternative proof of Proposition 3.6 based on the *exponential equivalence* condition (see e.g. Definition 4.2.10 in [6])

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\left\{\left|\frac{X_\theta(t)}{t}-\frac{Y_\theta(t)}{t}\right|>\delta\right\}\right)=-\infty \text{ for all } \delta>0 \tag{4}$$

for a suitable version of the processes $\left\{\frac{Y_\theta(t)}{t}:t>0\right\}$ and $\left\{\frac{X_\theta(t)}{t}:t>0\right\}$ defined on the same probability space, where $\{X_\theta(t):t\ge 0\}$ is the process $\{X(t):t\ge 0\}$ in Proposition 3.1 with $\lambda_1=\lambda_2=\theta$ and $c_1=c_2=c$; indeed, if condition (4) holds, the LDP in Proposition 3.6 is an immediate consequence of Theorem 4.2.13 in [6] and the LDP for $\left\{\frac{X_\theta(t)}{t}:t>0\right\}$ (i.e. Proposition 3.1 with $\lambda_1=\lambda_2=\theta$ and $c_1=c_2=c$).

In what follows we check (4) assuming that both the processes $\{Y_\theta(t):t\ge 0\}$ and $\{X_\theta(t):t\ge 0\}$ have the same random initial velocity $V(0)$ such that $P(V(0)\in\{-c,c\})=1$; thus we have (3) and

$$X_\theta(t)=V(0)\int_0^t(-1)^{N^*_\theta(s)}ds$$

for a homogeneous Poisson process $\{N^*_\theta(t):t\ge 0\}$ with intensity θ. Then, for a homogeneous Poisson process $\{M(t):t\ge 0\}$ with intensity 1, we set $N^*_\theta(t):=M(\theta t)$ and $N_\theta(t):=M(\Lambda_\theta(t))$, whence we obtain

$$\frac{X_\theta(t)}{t}:=\frac{V(0)}{t}\int_0^t(-1)^{M(\theta s)}ds=\frac{V(0)}{\theta t}\int_0^{\theta t}(-1)^{M(r)}dr$$

and

$$\frac{Y_\theta(t)}{t}:=\frac{V(0)}{t}\int_0^t(-1)^{M(\Lambda_\theta(s))}ds=\frac{V(0)}{\theta t}\int_0^{\Lambda_\theta(t)}(-1)^{M(r)}\frac{e^r\sqrt{e^{2r}-1}+e^{2r}}{e^r\sqrt{e^{2r}-1}+e^{2r}-1}dr;$$

the latter equality can be derived by considering the change of variable $r=\Lambda_\theta(s)$, whence we obtain $s=\Lambda_\theta^{-1}(r):=\frac{1}{\theta}\log(e^r+\sqrt{e^{2r}-1})$. Thus we get

$$\begin{aligned}\left|\frac{X_\theta(t)}{t}-\frac{Y_\theta(t)}{t}\right|=&\frac{c}{\theta t}\left|\int_0^{\theta t}(-1)^{M(r)}dr-\int_0^{\Lambda_\theta(t)}(-1)^{M(r)}\frac{e^r\sqrt{e^{2r}-1}+e^{2r}}{e^r\sqrt{e^{2r}-1}+e^{2r}-1}dr\right|\\ \le&\frac{c}{\theta t}\left(\int_0^{\Lambda_\theta(t)}\frac{1}{e^r\sqrt{e^{2r}-1}+e^{2r}-1}dr+\theta t-\Lambda_\theta(t)\right)=:a_{c,\theta}(t)\end{aligned}$$

(where $a_{c,\theta}(t)$ is deterministic), and we have

$$\begin{aligned}\lim_{t\to\infty}a_{c,\theta}(t)=&\frac{c}{\theta}\lim_{t\to\infty}\left\{\frac{\Lambda'_\theta(t)}{e^{\Lambda_\theta(t)}\sqrt{e^{2\Lambda_\theta(t)}-1}+e^{2\Lambda_\theta(t)}-1}+\theta-\Lambda'_\theta(t)\right\}\\ =&\frac{c}{\theta}\lim_{t\to\infty}\theta\left\{1+\tanh(\theta t)\left\{\frac{1}{\cosh(\theta t)\sqrt{\cosh^2(\theta t)-1}+\cosh^2(\theta t)-1}-1\right\}\right\}=0\end{aligned}$$

because $\Lambda'_\theta(t)=\theta\tanh(\theta t)$. In conclusion (4) holds because $\left\{\left|\frac{X_\theta(t)}{t}-\frac{Y_\theta(t)}{t}\right|>\delta\right\}\subset\{a_{c,\theta}(t)>\delta\}$ and the event $\{a_{c,\theta}(t)>\delta\}$ is empty for t large enough.

3.4 LDP for a (normalized) planar random motion

In this subsection we prove the LDP for the non-conditional laws of a planar random motion $\{(Y_1(t), Y_2(t)) : t \geq 0\}$ studied in [27]. Such a random motion consists of the displacements of a particle having four possible directions along the directions of $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ with velocity c; moreover, when a Poisson event occurs with rate λ, the motion takes with probability $\frac{1}{2}$ one of the two directions orthogonal to the previous one. It is interesting that we can avoid to handle the non-conditional laws of the process (provided explicitly in section 3 in [27]); indeed we can easily prove the LDP with a straightforward application of the contraction principle.

Proposition 3.7. *The family* $\left\{P\left(\left(\frac{Y_1(t)}{t}, \frac{Y_2(t)}{t}\right) \in \cdot\right) : t > 0\right\}$ *satisfies the LDP with good rate function* $I_{\lambda,c}^{(Y_1,Y_2)}$ *defined by*

$$I_{\lambda,c}^{(Y_1,Y_2)}(y_1, y_2) = \begin{cases} \frac{\lambda}{2}\left(1 - \sqrt{1 - \frac{(y_1+y_2)^2}{c^2}}\right) + \frac{\lambda}{2}\left(1 - \sqrt{1 - \frac{(y_1-y_2)^2}{c^2}}\right) & if\ (y_1, y_2) \in Q_c \\ \infty & otherwise, \end{cases}$$

where $Q_c := \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 + y_2| \leq c, |y_1 - y_2| \leq c\}$.

Proof. Let $\{X_1(t) : t \geq 0\}$ and $\{X_2(t) : t \geq 0\}$ be the processes defined by

$$X_k(t) = V_k(0) \int_0^t (-1)^{N_k(s)} ds$$

where, for $k \in \{1, 2\}$, the random variable $V_k(0)$ is such that $P(V_k(0) = \frac{c}{2}) = P(V_k(0) = -\frac{c}{2}) = \frac{1}{2}$ and $\{N_k(t) : t \geq 0\}$ is a Poisson process with intensity $\frac{\lambda}{2}$. We assume that $V_1(0)$, $V_2(0)$, $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent, and therefore $\{X_1(t) : t \geq 0\}$ and $\{X_2(t) : t \geq 0\}$ are independent. Then, by eq. (3.2) in [27], we have

$$\begin{cases} Y_1(t) = X_1(t) + X_2(t) \\ Y_2(t) = X_1(t) - X_2(t). \end{cases}$$

Moreover, by Proposition 3.1 with $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$ and $c_1 = c_2 = \frac{c}{2}$ and by Corollary 2.9 in [17], the family of laws $\left\{P\left(\left(\frac{X_1(t)}{t}, \frac{X_2(t)}{t}\right) \in \cdot\right) : t > 0\right\}$ satisfies the LDP with good rate function $J_{\lambda,c}^{(X_1,X_2)}$ defined by

$$\begin{aligned} J_{\lambda,c}^{(X_1,X_2)}(x_1, x_2) :=& I^X_{\frac{\lambda}{2},\frac{\lambda}{2},\frac{c}{2},\frac{c}{2}}(x_1) + I^X_{\frac{\lambda}{2},\frac{\lambda}{2},\frac{c}{2},\frac{c}{2}}(x_2) \\ =& \begin{cases} \frac{\lambda}{2}\left(1 - \sqrt{1 - \frac{x_1^2}{(\frac{c}{2})^2}}\right) + \frac{\lambda}{2}\left(1 - \sqrt{1 - \frac{x_2^2}{(\frac{c}{2})^2}}\right) & \text{if } (x_1, x_2) \in [-\frac{c}{2}, \frac{c}{2}] \times [-\frac{c}{2}, \frac{c}{2}] \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Then, since $(x_1, x_2) \mapsto (y_1, y_2) = f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ is a continuous function, the family of laws $\left\{P\left(\left(\frac{Y_1(t)}{t}, \frac{Y_2(t)}{t}\right) \in \cdot\right) : t > 0\right\}$ satisfies the LDP with good rate function $I_{\lambda,c}^{(Y_1,Y_2)}$ defined by

$$I_{\lambda,c}^{(Y_1,Y_2)}(y_1, y_2) := \inf\left\{J_{\lambda,c}^{(X_1,X_2)}(x_1, x_2) : f(x_1, x_2) = (y_1, y_2)\right\}$$

by the contraction principle. Finally it is easy to check that $I_{\lambda,c}^{(Y_1,Y_2)}(y_1, y_2) = J_{\lambda,c}^{(X_1,X_2)}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$ and this meets the expression of the rate function in the statement. □

4 Results for random flights

We start introducing some standard notation: the norm of $\underline{z}_d = (z_1, \ldots, z_d) \in \mathbb{R}^d$ is denoted (as usual) by $\|\underline{z}_d\| := \sqrt{z_1^2 + \cdots + z_d^2}$; moreover, for $\delta > 0$, we set

$$B_\delta(\underline{z}_d) := \{\underline{y}_d \in \mathbb{R}^d : \|\underline{y}_d - \underline{z}_d\| < \delta\}.$$

In this section we deal with the random flights in $\mathbb{R}^d$ presented in [28]. Roughly speaking, we consider a random motion $\{\underline{X}_d(t) = (X_1(t), \ldots, X_d(t)) : t \geq 0\}$ which starts at the origin, moves with constant velocity c, changes direction of motion at any occurrence of a homogeneous Poisson process $\{N(t) : t \geq 0\}$ with intensity λ, and chooses the directions uniformly on the d-dimensional hypersphere with radius 1. Note that the starting point could be different from the origin $\underline{0}_d = (0, \ldots, 0) \in \mathbb{R}^d$ (as, for instance, in [3] for the case $d = 2$); the results presented below could be easily adapted with some slight changes.

We treat the case $d = 2$ in subsection 4.1, and the case $d = 4$ in subsection 4.2. In subsection 4.3 we discuss the differences between the cases $d = 2$ and $d = 4$; some of them concern their connections with the one-dimensional case studied in subsection 3.1 with $\lambda_1 = \lambda_2 = \lambda$ and $c_1 = c_2 = c$.

4.1 The case $d = 2$

We start with the LDP for the non-conditional distributions.

Proposition 4.1. *The family* $\left\{P\left(\frac{\underline{X}_2(t)}{t} \in \cdot\right) : t > 0\right\}$ *satisfies the LDP with good rate function* $I_{\lambda,c}^{\underline{X}_2}$ *defined by*

$$I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2) = \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{\|\underline{x}_2\|^2}{c^2}}\right) & if\ \|\underline{x}_2\| \leq c \\ \infty & otherwise. \end{cases}$$

Proof. In this proof we consider the law of $\underline{X}_2(t)$ for any fixed $t > 0$ (see e.g. Theorem 2 in [14]): we have an absolutely continuous part given by

$$\frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\underline{y}_2\|^2}}}{\sqrt{c^2t^2 - \|\underline{y}_2\|^2}} 1_{B_{ct}(\underline{0}_2)}(\underline{y}_2) dy_1 dy_2,$$

and a singular part uniformly distributed on the boundary of $B_{ct}(\underline{0}_2)$ with weight $e^{-\lambda t}$. The proof is divided in two parts.

1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log P\left(\frac{\underline{X}_2(t)}{t} \in G\right) \geq -\lambda\left(1 - \sqrt{1 - \frac{\|\underline{x}_2\|^2}{c^2}}\right)$$

for all $\underline{x}_2 \in \mathbb{R}^2$ such that $\|\underline{x}_2\| \leq c$ and for all open sets G such that $\underline{x}_2 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{x}_2) \subset G$; moreover, if $\underline{x}_2 \in B_c(\underline{0}_2)$, we also require that

$B_\varepsilon(\underline{x}_2) \subset B_c(\underline{0}_2)$. Then we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_2(t)}{t} \in G\right) \geq P\left(\frac{\underline{X}_2(t)}{t} \in B_\varepsilon(\underline{x}_2)\right) \\
&\geq \int_{B_{\varepsilon t}(\underline{x}_2 t)\cap B_{ct}(\underline{0}_2)} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\underline{y}_2\|^2}}}{\sqrt{c^2t^2 - \|\underline{y}_2\|^2}} dy_1 dy_2 \\
&\geq \int_{B_\varepsilon(\underline{x}_2)\cap B_c(\underline{0}_2)} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda t}{c}\sqrt{c^2 - \|\underline{y}_2\|^2}}}{ct} t^2 dy_1 dy_2 \\
&\geq \frac{\lambda t}{2\pi c} \frac{e^{-\lambda t + \lambda t\sqrt{1 - \frac{\sup\{\|\underline{y}_2\|^2 : \underline{y}_2 \in B_\varepsilon(\underline{x}_2)\cap B_c(\underline{0}_2)\}}{c^2}}}}{c} \cdot \underbrace{\text{measure}(B_\varepsilon(\underline{x}_2)\cap B_c(\underline{0}_2))}_{>0},
\end{aligned}$$

and we conclude by taking $\liminf_{t\to\infty} \frac{1}{t}\log$ (for both the left hand side and the right hand side) and by letting ε go to zero.

2) Proof of the upper bound for open sets. We have to check

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{\underline{X}_2(t)}{t} \in F\right) \leq -\inf_{\underline{x}_2 \in F} I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_2 \in F$ and if $F \cap \overline{B_c(\underline{0}_2)} = \emptyset$. Thus, from now on, we assume that $\underline{0}_2 \notin F$ and $F \cap \overline{B_c(\underline{0}_2)} \neq \emptyset$. We can find $\underline{x}_2^F \in F$ such that $\|\underline{x}_2^F\| = \inf\{\|\underline{x}_2\| : \underline{x}_2 \in F \cap \overline{B_c(\underline{0}_2)}\}$; note that $r_F := \|\underline{x}_2^F\| \in (0, c]$. Then, since $F \subset (B_{r_F}(\underline{0}_2))^c$, we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_2(t)}{t} \in F\right) \leq P\left(\frac{\underline{X}_2(t)}{t} \in (B_{r_F}(\underline{0}_2))^c\right) \\
&= \int_{r_F t}^{ct} \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \rho^2}}}{\sqrt{c^2t^2 - \rho^2}} 2\pi\rho d\rho + e^{-\lambda t} \\
&= \left[-e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \rho^2}}\right]_{\rho = r_F t}^{\rho = ct} + e^{-\lambda t} = \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r_F^2 t^2}\right),
\end{aligned}$$

whence we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{\underline{X}_2(t)}{t} \in F\right) \leq -\lambda + \frac{\lambda}{c}\sqrt{c^2 - r_F^2} = -\lambda\left(1 - \sqrt{1 - \frac{\|\underline{x}_2^F\|^2}{c^2}}\right) = -I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2^F).$$

Thus the proof of the upper bound is complete noting that $I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2^F) = \inf_{\underline{x}_2 \in F} I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2)$ because $I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2) \geq (\text{resp.} =) I_{\lambda,c}^{\underline{X}_2}(\underline{y}_2)$ if and only if $\|\underline{x}_2\| \geq (\text{resp.} =) \|\underline{y}_2\|$. $\square$

Now we prove the LDP for the non-conditional distributions.

Proposition 4.2. *If $w_t \to w \in (0,\infty)$, then $\left\{P\left(\frac{\underline{X}_2(t)}{t} \in \cdot \middle| \frac{N(t)}{t} = w_t\right) : t > 0\right\}$ satisfies the LDP with good rate function $I_{\lambda,c}^{\underline{X}_2|N}(\cdot; w)$ defined by*

$$I_{\lambda,c}^{\underline{X}_2|N}(\underline{x}_2; w) = \begin{cases} w\log\left(\frac{c}{\sqrt{c^2 - \|\underline{x}_2\|^2}}\right) & \textit{if } \|\underline{x}_2\| < c \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. In this proof we consider the conditional law of $\underline{X}_2(t)$ given $\{N(t) = h\}$ for $t > 0$ and $h \geq 1$ (see e.g. Theorem 1 in [14]):

$$P(\underline{X}_2(t) \in E|N(t) = h) = \int_{E\cap B_{ct}(\underline{0}_2)} \frac{h}{2\pi(ct)^h}(c^2t^2 - \|\underline{y}_2\|^2)^{\frac{h}{2}-1}dy_1dy_2$$

for any measurable set E. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t}\log P\left(\frac{\underline{X}_2(t)}{t} \in G\Big|\frac{N(t)}{t} = w_t\right) \geq -w\log\left(\frac{c}{\sqrt{c^2 - \|\underline{x}_2\|^2}}\right)$$

for all $\underline{x}_2 \in \mathbb{R}^2$ such that $\|\underline{x}_2\| < c$ and for all open sets G such that $\underline{x}_2 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{x}_2) \subset G \cap B_c(\underline{0}_2)$. Then we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_2(t)}{t} \in G\Big|\frac{N(t)}{t} = w_t\right) \geq P\left(\frac{\underline{X}_2(t)}{t} \in B_\varepsilon(\underline{x}_2)\Big|\frac{N(t)}{t} = w_t\right)\\
&= \int_{B_{\varepsilon t}(\underline{x}_2 t)} \frac{tw_t}{2\pi(ct)^{tw_t}}(c^2t^2 - \|\underline{y}_2\|^2)^{\frac{tw_t}{2}-1}dy_1dy_2\\
&= \int_{B_\varepsilon(\underline{x}_2)} \frac{tw_t}{2\pi(ct)^{tw_t}}t^{tw_t-2}(c^2 - \|\underline{y}_2\|^2)^{\frac{tw_t}{2}-1}t^2dy_1dy_2\\
&\geq \frac{tw_t}{2\pi c^{tw_t}}(c^2 - \sup\{\|\underline{y}_2\|^2 : \underline{y}_2 \in B_\varepsilon(\underline{x}_2)\})^{\frac{tw_t}{2}-1}\pi\varepsilon^2,
\end{aligned}$$

and we conclude following the lines of the first part of the proof of Proposition 4.1.
2) Proof of the upper bound for open sets. We have to check

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{\underline{X}_2(t)}{t} \in F\Big|\frac{N(t)}{t} = w_t\right) \leq -\inf_{\underline{x}_2\in F} I_{\lambda,c}^{\underline{X}_2|N}(\underline{x}_2; w) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_2 \in F$ and if $F \cap B_c(\underline{0}_2) = \emptyset$. Thus, from now on, we assume that $\underline{0}_2 \notin F$ and $F \cap B_c(\underline{0}_2) \neq \emptyset$. We can find $\underline{x}_2^F \in F$ such that $\|\underline{x}_2^F\| = \inf\{\|\underline{x}_2\| : \underline{x}_2 \in F \cap B_c(\underline{0}_2)\}$; note that $r_F := \|\underline{x}_2^F\| \in (0, c)$. Then, since $F \subset (B_{r_F}(\underline{0}_2))^c$, we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_2(t)}{t} \in F\Big|\frac{N(t)}{t} = w_t\right) \leq P\left(\frac{\underline{X}_2(t)}{t} \in (B_{r_F}(\underline{0}_2))^c\Big|\frac{N(t)}{t} = w_t\right)\\
&= \int_{r_F t}^{ct} \frac{tw_t}{2\pi(ct)^{tw_t}}(c^2t^2 - \rho^2)^{\frac{tw_t}{2}-1}2\pi\rho d\rho\\
&= \frac{1}{(ct)^{tw_t}}\left[-(c^2t^2 - \rho^2)^{\frac{tw_t}{2}}\right]_{\rho=r_F t}^{\rho=ct} = \left(\frac{c^2 - r_F^2}{c^2}\right)^{\frac{tw_t}{2}},
\end{aligned}$$

and we conclude following the lines of the second part of the proof of Proposition 4.1. □

We conclude with some remarks; in particular, in the spirit of Remark 3.5 for the one-dimensional case, we compare the LDPs of non-conditional distributions and of conditional distributions (Remark 4.3(ii)).

Remark 4.3. *(i) We can check that, for all $\underline{x}_2 \in \mathbb{R}^2$, we have $I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2) = I_{\lambda,\lambda,c,c}^{X}(\|\underline{x}_2\|)$ and $I_{\lambda,c}^{\underline{X}_2|N}(\underline{x}_2; w) = I_{\lambda,\lambda,c,c}^{X|N}(\|\underline{x}_2\|; w)$ (for all $w > 0$). These equalities are not surprising because the planar random flight is an extension of the telegraph process in $\mathbb{R}^2$. Indeed, for each fixed $t > 0$, the density of the absolutely continuous part of the law of $\underline{X}_2(t)$ satisfies the two-dimensional telegraph*

equation (see e.g. equations (2)-(3) in [14]).
(ii) We can obtain inequalities between $I_{\lambda,c}^{X_2|N}(\cdot;w)$ *and* $I_{\lambda,c}^{X_2}$ *by Proposition 3.4 (with* $c_1 = c_2 = c$*). Then, by taking into account Remark 2.1, we can say that the convergence of the conditional distributions is faster than the convergence of the non-conditional distributions if and only if* $w \geq \lambda$*; this inequality has the same interpretation given in Remark 3.5.*

4.2 The case $d = 4$

We start with the LDP for the non-conditional distributions. It is interesting to note that the rate function $I_{\lambda,c}^{X_4}$ can be seen as the restriction on $\overline{B_c(\underline{0}_4)}$ (the set where $I_{\lambda,c}^{X_4}$ is finite) of a large deviation rate function for centered Gaussian distributions; this is not surprising if we look at the expression of the density of $\underline{X}_4(t)$ provided by Theorem 3.2 in [28].

Proposition 4.4. *The family* $\left\{P\left(\frac{\underline{X}_4(t)}{t} \in \cdot\right) : t > 0\right\}$ *satisfies the LDP with good rate function* $I_{\lambda,c}^{X_4}$ *defined by*

$$I_{\lambda,c}^{X_4}(\underline{x}_4) = \begin{cases} \frac{\lambda}{c^2}\|\underline{x}_4\|^2 & \text{if } \|\underline{x}_4\| \leq c \\ \infty & \text{otherwise.} \end{cases}$$

Proof. In this proof we consider the law of $\underline{X}_4(t)$ for any fixed $t > 0$ (see e.g. Theorem 3.2 in [28]): we have an absolutely continuous part given by

$$\frac{\lambda}{c^4t^3\pi^2}e^{-\frac{\lambda}{c^2t}\|\underline{y}_4\|^2}\left(2+\frac{\lambda}{c^2t}\left(c^2t^2-\|\underline{y}_4\|^2\right)\right)1_{B_{ct}(\underline{0}_4)}(\underline{y}_4)dy_1dy_2dy_3dy_4,$$

and a singular part uniformly distributed on the boundary of $B_{ct}(\underline{0}_4)$ with weight $e^{-\lambda t}$. The proof is divided in two parts.
1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{\underline{X}_4(t)}{t}\in G\right) \geq -\frac{\lambda}{c^2}\|\underline{x}_4\|^2$$

for all $\underline{x}_4 \in \mathbb{R}^4$ such that $\|\underline{x}_4\| \leq c$ and for all open sets G such that $\underline{x}_4 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{x}_4) \subset G$; moreover, if $\underline{x}_4 \in B_c(\underline{0}_4)$, we also require that $B_\varepsilon(\underline{x}_4) \subset B_c(\underline{0}_4)$. Then we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_4(t)}{t}\in G\right) \geq P\left(\frac{\underline{X}_4(t)}{t}\in B_\varepsilon(\underline{x}_4)\right)\\
&\geq \int_{B_{\varepsilon t}(\underline{x}_4 t)\cap B_{ct}(\underline{0}_4)} \frac{\lambda}{c^4t^3\pi^2}e^{-\frac{\lambda}{c^2t}\|\underline{y}_4\|^2}\left(2+\frac{\lambda}{c^2t}\left(c^2t^2-\|\underline{y}_4\|^2\right)\right)dy_1dy_2dy_3dy_4\\
&= \int_{B_\varepsilon(\underline{x}_4)\cap B_c(\underline{0}_4)} \frac{\lambda}{c^4t^3\pi^2}e^{-\frac{\lambda t}{c^2}\|\underline{y}_4\|^2}\left(2+\frac{\lambda t}{c^2}\left(c^2-\|\underline{y}_4\|^2\right)\right)t^4dy_1dy_2dy_3dy_4\\
&\geq \frac{\lambda t}{c^4\pi^2}e^{-\frac{\lambda t}{c^2}\sup\{\|\underline{y}_4\|^2:\underline{y}_4\in B_\varepsilon(\underline{x}_4)\cap B_c(\underline{0}_4)\}}\\
&\quad\cdot\left(2+\frac{\lambda t}{c^2}\left(c^2-\sup\{\|\underline{y}_4\|^2:\underline{y}_4\in B_\varepsilon(\underline{x}_4)\cap B_c(\underline{0}_4)\}\right)\right)\underbrace{\text{measure}(B_\varepsilon(\underline{x}_4)\cap B_c(\underline{0}_4))}_{>0},
\end{aligned}$$

and we conclude following the lines of the first part of the proof of Proposition 4.1.
2) Proof of the upper bound for open sets. We have to check

$$\limsup_{t\to\infty}\frac{1}{t}\log P\left(\frac{\underline{X}_4(t)}{t}\in F\right) \leq -\inf_{\underline{x}_4\in F} I_{\lambda,c}^{X_4}(\underline{x}_4) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_4 \in F$ and if $F \cap \overline{B_c(\underline{0}_4)} = \emptyset$. Thus, from now on, we assume that $\underline{0}_4 \notin F$ and $F \cap \overline{B_c(\underline{0}_4)} \neq \emptyset$. We can find $\underline{x}_4^F \in F$ such that $\|\underline{x}_4^F\| = \inf\{\|\underline{x}_4\| : \underline{x}_4 \in F \cap \overline{B_c(\underline{0}_4)}\}$; note that $r_F := \|\underline{x}_4^F\| \in (0, c]$. Then, since $F \subset (B_{r_F}(\underline{0}_4))^c$, we have

$$\begin{aligned} &P\left(\frac{\underline{X}_4(t)}{t} \in F\right) \leq P\left(\frac{\underline{X}_4(t)}{t} \in (B_{r_F}(\underline{0}_4))^c\right) \\ &= \int_{r_F t}^{ct} \frac{\lambda}{c^4 t^3 \pi^2} e^{-\frac{\lambda}{c^2 t}\rho^2} \left(2 + \frac{\lambda}{c^2 t}(c^2t^2 - \rho^2)\right) 2\pi^2 \rho^3 d\rho + e^{-\lambda t}; \end{aligned}$$

moreover we note that, if $\rho \in [r_F t, ct]$, we have

$$\begin{cases} 0 \leq \frac{\lambda}{c^2 t}(c^2t^2 - \rho^2) = \lambda t \left(1 - \left(\frac{\rho}{ct}\right)^2\right) \leq \lambda t \\ 0 \leq \frac{2\lambda\rho^3}{c^4t^3} = 2\left(\frac{\rho}{ct}\right)^2 \frac{\lambda\rho}{c^2 t} \leq \frac{2\lambda\rho}{c^2 t}, \end{cases}$$

whence we obtain

$$\begin{aligned} &P\left(\frac{\underline{X}_4(t)}{t} \in F\right) \leq (2 + \lambda t) \int_{r_F t}^{ct} \frac{2\lambda\rho}{c^2 t} e^{-\frac{\lambda}{c^2 t}\rho^2} d\rho + e^{-\lambda t} \\ &= (2 + \lambda t) \left[-e^{-\frac{\lambda}{c^2 t}\rho^2}\right]_{\rho = r_F t}^{\rho = ct} + e^{-\lambda t} = (2 + \lambda t) e^{-\lambda \frac{r_F^2}{c^2} t} \left(1 - e^{-\lambda t\left(1 - \frac{r_F^2}{c^2}\right)}\right) + e^{-\lambda t}; \end{aligned}$$

actually, if $r_F = c$, we have $P\left(\frac{\underline{X}_4(t)}{t} \in (B_{r_F}(\underline{0}_4))^c\right) = e^{-\lambda t}$ because $\left\{\frac{\underline{X}_4(t)}{t} \in (B_{r_F}(\underline{0}_4))^c\right\} = \{N(t) = 0\}$. In conclusion we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(\frac{\underline{X}_4(t)}{t} \in F\right) \leq -\frac{\lambda}{c^2} r_F^2 = -I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4^F),$$

(if $r_F \in (0, c)$, this is a consequence of Lemma 1.2.15 in [6]). Thus the proof of the upper bound is complete noting that $I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4^F) = \inf_{\underline{x}_4 \in F} I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4)$ because $I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) \geq$ (resp. $=$)$I_{\lambda,c}^{\underline{X}_4}(\underline{y}_4)$ if and only if $\|\underline{x}_4\| \geq$ (resp. $=$)$\|\underline{y}_4\|$. □

Now we prove the LDP for the conditional distributions.

Proposition 4.5. *If $w_t \to w \in (0, \infty)$, then $\left\{P\left(\frac{\underline{X}_4(t)}{t} \in \cdot \middle| \frac{N(t)}{t} = w_t\right) : t > 0\right\}$ satisfies the LDP with good rate function $I_{\lambda,c}^{\underline{X}_4|N}(\cdot; w)$ defined by*

$$I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) = \begin{cases} 2w \log\left(\frac{c}{\sqrt{c^2 - \|\underline{x}_4\|^2}}\right) & \textit{if } \|\underline{x}_4\| < c \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. In this proof we consider the conditional law of $\underline{X}_4(t)$ given $\{N(t) = h\}$ for $t > 0$ and $h \geq 1$ (see e.g. Theorem 3.1 in [28]):

$$P(\underline{X}_4(t) \in E | N(t) = h) = \int_{E \cap B_{ct}(\underline{0}_4)} \frac{h(h+1)}{\pi^2 (ct)^{2h+2}} (c^2t^2 - \|\underline{y}_4\|^2)^{h-1} dy_1 dy_2 dy_3 dy_4$$

for any measurable set E. The proof is divided in two parts.

1) Proof of the lower bound for open sets. We want to check the equivalent condition (1), i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log P\left(\frac{\underline{X}_4(t)}{t} \in G \middle| \frac{N(t)}{t} = w_t\right) \geq -2w \log\left(\frac{c}{\sqrt{c^2 - \|\underline{x}_4\|^2}}\right)$$

for all $\underline{x}_4 \in \mathbb{R}^4$ such that $\|\underline{x}_4\| < c$ and for all open sets G such that $\underline{x}_4 \in G$. Firstly we can take $\varepsilon > 0$ small enough to have $B_\varepsilon(\underline{x}_4) \subset G \cap B_c(\underline{0}_4)$. Then we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_4(t)}{t} \in G \Big| \frac{N(t)}{t} = w_t\right) \geq P\left(\frac{\underline{X}_4(t)}{t} \in B_\varepsilon(\underline{x}_4) \Big| \frac{N(t)}{t} = w_t\right) \\
&= \int_{B_{\varepsilon t}(\underline{x}_4 t)} \frac{tw_t(tw_t+1)}{\pi^2(ct)^{2tw_t+2}}(c^2t^2 - \|\underline{y}_4\|^2)^{tw_t-1} dy_1dy_2dy_3dy_4 \\
&= \int_{B_\varepsilon(\underline{x}_4)} \frac{tw_t(tw_t+1)}{\pi^2(ct)^{2tw_t+2}} t^{2(tw_t-1)}(c^2 - \|\underline{y}_4\|^2)^{tw_t-1} t^4 dy_1dy_2dy_3dy_4 \\
&\geq \frac{tw_t(tw_t+1)}{\pi^2 c^{2tw_t+2}}(c^2 - \sup\{\|\underline{y}_4\|^2 : \underline{y}_4 \in B_\varepsilon(\underline{x}_4)\})^{tw_t-1} \underbrace{\text{measure}(B_\varepsilon(\underline{x}_4))}_{>0},
\end{aligned}$$

and we conclude following the lines of the first part of the proof of Proposition 4.1.
2) Proof of the upper bound for open sets. We have to check

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(\frac{\underline{X}_4(t)}{t} \in F \Big| \frac{N(t)}{t} = w_t\right) \leq -\inf_{\underline{x}_4 \in F} I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) \quad \text{for all closed sets } F.$$

Firstly note that this condition trivially holds if $\underline{0}_4 \in F$ and if $F \cap B_c(\underline{0}_4) = \emptyset$. Thus, from now on, we assume that $\underline{0}_4 \notin F$ and $F \cap B_c(\underline{0}_4) \neq \emptyset$. We can find $\underline{x}_4^F \in F$ such that $\|\underline{x}_4^F\| = \inf\{\|\underline{x}_4\| : \underline{x}_4 \in F \cap B_c(\underline{0}_4)\}$; note that $r_F := \|\underline{x}_4^F\| \in (0, c)$. Then, since $F \subset (B_{r_F}(\underline{0}_4))^c$, we have

$$\begin{aligned}
&P\left(\frac{\underline{X}_4(t)}{t} \in F \Big| \frac{N(t)}{t} = w_t\right) \leq P\left(\frac{\underline{X}_4(t)}{t} \in (B_{r_F}(\underline{0}_4))^c \Big| \frac{N(t)}{t} = w_t\right) \\
&= \int_{r_F t}^{ct} \frac{tw_t(tw_t+1)}{\pi^2(ct)^{2tw_t+2}}(c^2t^2 - \rho^2)^{tw_t-1} 2\pi^2\rho^3 d\rho \leq \frac{tw_t(tw_t+1)}{(ct)^{2tw_t+2}}(c^2t^2 - r_F^2t^2)^{tw_t-1} 2\int_{r_F t}^{ct} \rho^3 d\rho \\
&= \frac{tw_t(tw_t+1)}{(ct)^{2tw_t+2}}(c^2t^2 - r_F^2t^2)^{tw_t-1} \frac{c^4t^4 - r_F^4t^4}{2} = \frac{tw_t(tw_t+1)}{c^{2tw_t+2}}(c^2 - r_F^2)^{tw_t-1} \frac{c^4 - r_F^4}{2}
\end{aligned}$$

and we conclude following the lines of the second part of the proof of Proposition 4.1. □

In Remark 4.7 at the end of this subsection we compare the LDPs of non-conditional distributions and of conditional distributions in the spirit of Remarks 3.5 and 4.3(ii); in view of this we refer to the inequalities between the rate functions $I_{\lambda,c}^{\underline{X}_4}$ and $I_{\lambda,c}^{\underline{X}_4|N}(\cdot; w)$ proved in the next proposition (see Figure 2 for the case $\lambda = c = 1$).

Proposition 4.6. *We have two cases. (i) For $w \geq \lambda$, we have $I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) \leq I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w)$ for all $\underline{x}_4 \in \mathbb{R}^4$; moreover the inequality is strict for $\underline{x}_4 \in \overline{B_c(\underline{0}_4)} \setminus \{\underline{0}_4\}$. (ii) For $w \in (0, \lambda)$, there exists $\gamma \in (\xi, 1)$ where $\xi = \sqrt{1 - \frac{w}{\lambda}}$ such that: $I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) > I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4)$ for $\|\underline{x}_4\| \in (\gamma c, c]$, $I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) < I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4)$ for $\|\underline{x}_4\| \in (0, \gamma c)$ and $I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) = I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4)$ otherwise.*

Proof. Firstly note that, for all $w > 0$, we have: $I_{\lambda,c}^{\underline{X}_4|N}(\underline{0}_4; w) = I_{\lambda,c}^{\underline{X}_4}(\underline{0}_4) = 0$; $I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) = I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) = \infty$ for all $\|\underline{x}_4\| > c$; $\lambda = I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) < I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) = \infty$ for $\|\underline{x}_4\| = c$. Moreover let us consider the difference function (for $r = \|\underline{x}_4\| \in [0, c)$) defined by

$$\Delta(r) := I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) - I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) = 2w \log\left(\frac{c}{\sqrt{c^2 - r^2}}\right) - \frac{\lambda}{c^2} r^2$$

(the definition is well-posed). Then the derivative is $\Delta'(r) = \frac{2r}{c^2} \frac{(w-\lambda)c^2 + \lambda r^2}{c^2 - r^2}$, and we complete the proof as follows.

Statement (i) can be proved noting that, for $w \geq \lambda$, the function Δ is increasing in the interval $[0, c)$ and its global minimum - equal to zero - is uniquely attained at $r = 0$.

Statement (ii) can be proved noting that, for $w \in (0, \lambda)$ and $\xi = \sqrt{1 - \frac{w}{\lambda}}$, we have what follows: the function Δ is decreasing in the interval $[0, \xi c)$ and is increasing in the interval $(\xi c, c)$; a local maximum of Δ - equal to zero - is attained at $r = 0$; the minimum of Δ is uniquely attained at $r = \xi c$ and is equal to $-\left[w \log\left(\frac{w}{\lambda}\right) - w + \lambda\right] < 0$; $\Delta(r) \uparrow \infty$ as $r \uparrow c$. $\square$

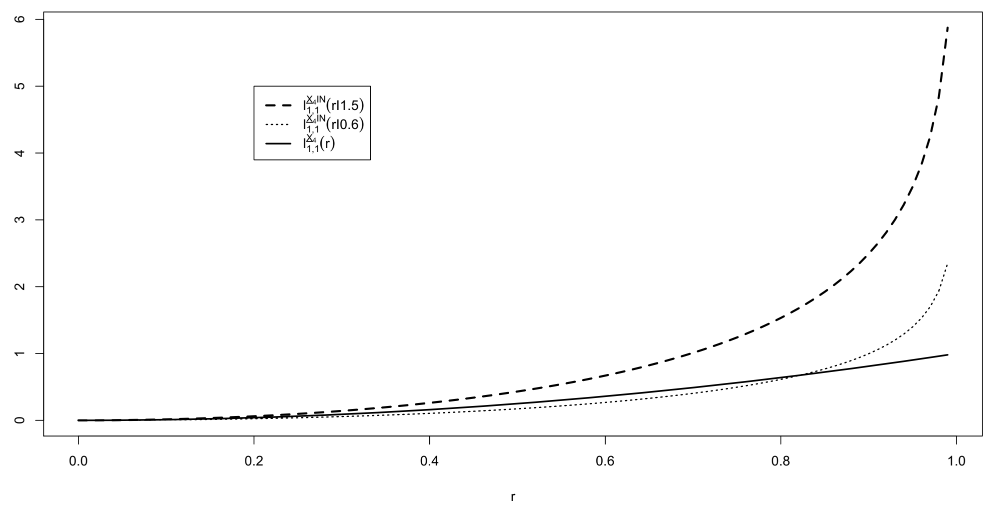

Figure 2: The functions $I_{1,1}^{\underline{X}_4}(\underline{x}_4)$ and $I_{1,1}^{\underline{X}_4|N}(\underline{x}_4; w)$, where $r = \|\underline{x}_4\| \in [0, 1)$. Two choices for w: $w = 1.5 \geq 1$; $w = 0.6 \in (0, 1)$.

Remark 4.7. *By taking into account Remark 2.1 and the inequalities between* $I_{\lambda,c}^{\underline{X}_4|N}(\cdot; w)$ *and* $I_{\lambda,c,}^{\underline{X}_4}$ *proved in Proposition 4.6, we can say that the convergence of the conditional distributions is always faster than the convergence of the non-conditional distributions if and only if* $w \geq \lambda$*; this inequality has the same interpretation given in Remarks 3.5 and 4.3(ii).*

4.3 A discussion on the differences between the cases $d = 2$ and $d = 4$

We already pointed out in Remark 4.3(i) that, for all $\underline{x}_2 \in \mathbb{R}^2$, we have:

$$I_{\lambda,c}^{\underline{X}_2}(\underline{x}_2) = I_{\lambda,\lambda,c,c}^{X}(\|\underline{x}_2\|); \quad I_{\lambda,c}^{\underline{X}_2|N}(\underline{x}_2; w) = I_{\lambda,\lambda,c,c}^{X|N}(\|\underline{x}_2\|; w) \text{ for all } w > 0.$$

These equalities illustrate the strict connection between the case $d = 2$ and the one-dimensional case studied in subsection 3.1 with $\lambda_1 = \lambda_2 = \lambda$ and $c_1 = c_2 = c$. On the contrary we do not have the same situation for the case $d = 4$. For all $\underline{x}_4 \in \mathbb{R}^4$ we have:

$$I_{\lambda,c}^{\underline{X}_4}(\underline{x}_4) = \begin{cases} \frac{\lambda}{c^2}\|\underline{x}_4\|^2 & \text{if } \|\underline{x}_4\| \leq c \\ \infty & \text{otherwise;} \end{cases} \quad I_{\lambda,\lambda,c,c}^{X}(\|\underline{x}_4\|) = \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{\|\underline{x}_4\|^2}{c^2}}\right) & \text{if } \|\underline{x}_4\| \leq c \\ \infty & \text{otherwise;} \end{cases}$$

$$I_{\lambda,c}^{\underline{X}_4|N}(\underline{x}_4; w) = 2I_{\lambda,\lambda,c,c}^{X|N}(\|\underline{x}_4\|; w) \text{ for all } w > 0.$$

Thus the main difference concerns the non-conditional distributions; on the contrary, for the conditional distributions, the difference consists of a multiplicative factor 2.

We have the same differences between the cases $d = 2$ and $d = 4$ even if we consider some LDPs obtained from the ones in subsections 4.1 and 4.2 and an application of the contraction principle. For instance, by taking the inspiration from section 4 in [28] for the case $d = 4$, we present LDPs for the (normalized) random flights in the lower spaces with dimension $s < d$ and the respective rate functions: $\left\{P\left(\frac{\underline{X}_s(t)}{t} \in \cdot\right) : t > 0\right\}$ satisfies the LDP with good rate function $I_{\lambda,c;d}^{\underline{X}_s}$ is defined by

$$I_{\lambda,c;d}^{\underline{X}_s}(\underline{x}_s) := \inf\left\{I_{\lambda,c}^{\underline{X}_d}(\underline{x}_d) : (x_{s+1}, \ldots, x_d) \in \mathbb{R}^{d-s}\right\};$$

if $w_t \to w \in (0, \infty)$, $\left\{P\left(\frac{\underline{X}_s(t)}{t} \in \cdot \middle| \frac{N(t)}{t} = w_t\right) : t > 0\right\}$ satisfies the LDP with good rate function $I_{\lambda,c;d}^{\underline{X}_s|N}(\cdot; w)$ is defined by

$$I_{\lambda,c;d}^{\underline{X}_s|N}(\underline{x}_s; w) := \inf\left\{I_{\lambda,c}^{\underline{X}_d|N}(\underline{x}_s; w) : (x_{s+1}, \ldots, x_d) \in \mathbb{R}^{d-s}\right\}.$$

Firstly, if we specialize the above formulas for $d = 2$, we have:

$$I_{\lambda,c;2}^{X_1} = I_{\lambda,\lambda,c,c}^{X}; \quad I_{\lambda,c;2}^{X_1|N}(\cdot; w) = I_{\lambda,\lambda,c,c}^{X|N}(\cdot; w) \text{ for all } w > 0.$$

Secondly, for $d = 4$ and $s \in \{1, 2, 3\}$, for all $\underline{x}_s \in \mathbb{R}^s$ we have:

$$I_{\lambda,c;4}^{\underline{X}_s}(\underline{x}_s) = \begin{cases} \frac{\lambda}{c^2}\|\underline{x}_s\|^2 & \text{if } \|\underline{x}_s\| \leq c \\ \infty & \text{otherwise;} \end{cases} \quad I_{\lambda,\lambda,c,c}^{X}(\|\underline{x}_s\|) = \begin{cases} \lambda\left(1 - \sqrt{1 - \frac{\|\underline{x}_s\|^2}{c^2}}\right) & \text{if } \|\underline{x}_s\| \leq c \\ \infty & \text{otherwise;} \end{cases}$$

$$I_{\lambda,c;4}^{\underline{X}_s|N}(\underline{x}_s; w) = 2I_{\lambda,\lambda,c,c}^{X|N}(\|\underline{x}_s\|; w) \text{ for all } w > 0.$$

We conclude with a direct comparison between the LDPs concerning the cases $d = 2$ and $d = 4$; more precisely, with an abuse of notation, we write $I_{\lambda,c}^{\underline{X}_d}(r)$ and $I_{\lambda,c}^{\underline{X}_d|N}(r; w)$ where $r = \|\underline{x}_d\|$; note that, in order to avoid the trivial case where both the rate functions are equal to infinity, $r \in [0, c]$ when we deal with non-conditional distributions and $r \in [0, c)$ when we deal with conditional distributions. In the spirit of Remark 2.1 we can say that, for both non-conditional distributions and conditional distributions, the convergence to the origin in $\mathbb{R}^4$ is faster than the analogous convergence in $\mathbb{R}^2$. In some sense this is not surprising because one can expect a faster convergence of a normalized random flight in a higher space. In detail we have: $I_{\lambda,c}^{\underline{X}_4}(r) \geq I_{\lambda,c}^{\underline{X}_2}(r)$ and the inequality turns into an equality for $r \in \{0, \lambda\}$ (see Figure 3 for the case $\lambda = c = 1$; note that $r = 0$ is the limit point and in such a case both the rate functions are equal to zero; moreover $r = \lambda$ concerns the case without changes of directions); $I_{\lambda,c}^{\underline{X}_4|N}(r; w) = 2I_{\lambda,c}^{\underline{X}_2|N}(r; w) \geq I_{\lambda,c}^{\underline{X}_2|N}(r; w)$ and the inequality turns into an equality for $r = 0$ (for the same reasons illustrated for the non-conditional distributions).

Appendix: the extension of equations (2.17)-(2.18) in [5]

In this Appendix we consider the telegraph process $\{X(t) : t \geq 0\}$, with $\lambda_1 = \lambda_2 = \lambda$. The aim is to study the conditional distribution of $X(t)$ given $N(t) = n$, for $t > 0$ and $n \geq 1$. In this case the changes of direction are governed by a homogenous Poisson process $\{N(t) : t \geq 0\}$. If $N(t) = n$ for $n \geq 1$, then the displacements of the telegraph process are of two types, i.e. $c_1 t(T_k - T_{k-1})$ or $-c_2 t(T_k - T_{k-1})$, where tT_k is the instant in which the k-th Poisson event happens. In other words, T_k is the k-th order statistics from the uniform law in $[0, 1]$. By means of the exchangeability of $T_k - T_{k-1}$, we can put together the n_1 forward steps as well as the $n + 1 - n_1$ backward ones.

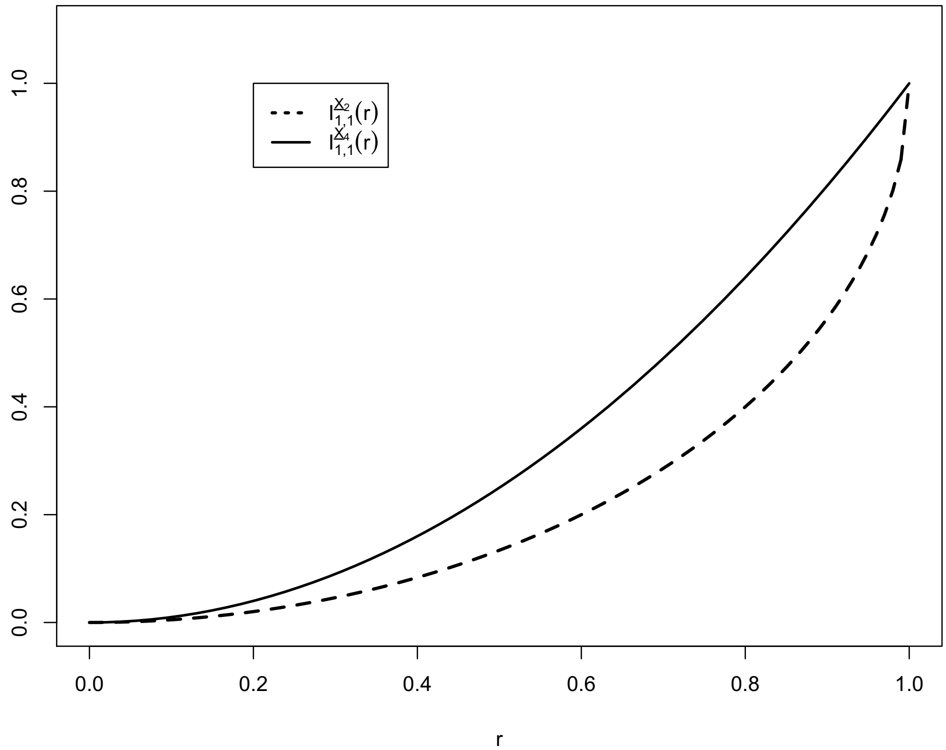

Figure 3: The function $I_{1,1}^{X_d}(r)$ for $d = 2$ and $d = 4$, where $r = \|\underline{x}_d\| \in [0, 1]$.

Then, when n changes of direction happen, the random variables $X(t)$ and $t[c_1 T_{n_1} - c_2(1 - T_{n_1})]$ (for $t > 0$) are equally distributed; thus we obtain

$$P(X(t) \leq x | N(t) = n, J(0) = \{1, 2\}) = P\left(T_{n_1} \leq \frac{c_2 t + x}{(c_1 + c_2)t} \Big| N(t) = n, J(0) = \{1, 2\}\right).$$

Moreover, being the order statistic T_{n_1} a Beta$(n_1, n + 1 - n_1)$ distributed random variable, we have

$$P(X(t) \in dx | N(t) = n, J(0) = \{1, 2\}) = dx \frac{n!}{(n_1 - 1)!(n - n_1)!} \frac{(c_2 t + x)^{n_1 - 1}(c_1 t - x)^{n - n_1}}{((c_1 + c_2)t)^n}. \quad (5)$$

Clearly there is a strict relationship between n and n_1, indeed we have:

$$\begin{cases} \text{if } n = 2k + 1 \text{ (for some } k \geq 0), & \text{then } n_1 = k + 1; \\ \text{if } n = 2k \text{ (for some } k \geq 1) \text{ and } J(0) = 1, & \text{then } n_1 = k + 1; \\ \text{if } n = 2k \text{ (for some } k \geq 1) \text{ and } J(0) = 2, & \text{then } n_1 = k. \end{cases} \quad (6)$$

In conclusion, from (5) and (6), we can provide the following formulas: for all $k \geq 0$,

$$\begin{aligned} P(X(t) \in dx | N(t) = 2k + 1, J(0) = \{1, 2\}) &= P(X(t) \in dx | N(t) = 2k + 1) \\ &= dx \frac{(2k + 1)!}{((c_1 + c_2)t)^{2k+1}(k!)^2} [(c_2 t + x)(c_1 t - x)]^k; \end{aligned}$$

for all $k \geq 1$,

$$\begin{cases} P(X(t) \in dx | N(t) = 2k, J(0) = 1) = dx \frac{(2k)!}{((c_1+c_2)t)^{2k}(k-1)!k!}(c_2 t + x)^k (c_1 t - x)^{k-1} \\ P(X(t) \in dx | N(t) = 2k, J(0) = 2) = dx \frac{(2k)!}{((c_1+c_2)t)^{2k}(k-1)!k!}(c_2 t + x)^{k-1} (c_1 t - x)^{k}, \end{cases}$$

whence we obtain

$$P(X(t) \in dx | N(t) = 2k) = \sum_{j=1}^{2} P(X(t) \in dx | N(t) = 2k, J(0) = j) P(J(0) = j)$$

$$= dx \frac{(2k)!}{k!(k-1)!((c_1 + c_2)t)^{2k}} (c_2 t + x)^{k-1} (c_1 t - x)^{k-1} \{(c_2 t + x)p + (c_1 t - x)(1 - p)\}.$$

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