Calibration of an extended generalized Markov model by using Particle Filtering

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Introduction

The *Solvency II* [6] regulatory regime provides the calculation of a capital requirement, the *Solvency Capital Requirement* (SCR), for the insurance and reinsurance companies. This requirement, based on market-consistent evaluation of the balance sheet, should reflect a level of eligible own funds that enables insurance and reinsurance undertakings to absorb significant losses and that gives reasonable assurance of the company solvency to policy holders and beneficiaries. In fact, the *Solvency Capital Requirement* should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases (probability of at least 99,5%). In practice, the SCR shall correspond to the *Value-at-Risk* of the basic own funds of an insurance or reinsurance undertaking with a confidence level of 99,5% over a one-year time horizon.
European Insurance and Occupational Pensions Authority (EIOPA) proposes three methods to evaluate the SCR: the *Standard Formula*, a modular approach that provides predetermined model and parameters, the *Undertaking Specific Parameters* approach that provides a predetermined model but parameters calibrated by the undertaking and the *Internal Model*, a more sophisticated model developed by the undertaking to evaluate as accurately as possible its risk profile and the consequent requirement.
Therefore the SCR calculation requires the *Basic Own Funds* (BOF) probability distribution forecast over a one-year time horizon and, to ensure the market consistency, must contemplate the use of both *real-world*[1] probability distributions of risk factors to evolving their values over the one-year time horizon required by the directive and risk-neutral

[1] The real-world probability measure is used primarily for risk management purpose and SCR assessment; instead the *risk-neutral* one is used primarily for market-consistent valuations, e.g. for pricing of financial and insurance products.

probability distributions to evaluate the BOF. This framework brings out the issue of using calibration techniques based on market prices time series, which allow to calibrate jointly the parameters under both probability measures. Specifically, this work uses the calibration technique known as *Particle Filtering*, widely used in the financial field and discussed in literature from a theoretical and practical point of view (refer to [15], [14], [4] and [5]).

This paper focuses on the spread risk only, one of the market risks considered by the regulations. Particularly, the spread risk is considered as defined by the Directive: "*the sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure.*" [6].
The model chosen to evaluate the spread risk is an extension of the model proposed by Lando [13] and that proposed by Jarrow, Lando e Turnbull [9]. This extension is also used by Gambaro et al.[7] and models the credit rating transition and the default process using an extension of a time-homogeneous Markov chain; this model explicitly considers the credit rating transitions, defining the price of a zero coupon bond (ZCB) with maturity T and rating i at time t as:

$$v^i(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}(1 - P^i_{DEF}(t,T))|\mathcal{F}_t\right]$$

where $r(t)$ is the risk-free spot rate process and $P^i_{DEF}(t,T)$ is the default probability for an i-rated issuer over the time horizon $(t,T]$.
The paper is structured as follows. In section 1 the Markov chains and the Cox processes are briefly described to introduce the model for the spread risk. In section 2 the model for the spread risk is presented, starting from the Lando model framework. In section 3 the calibration technique, i.e. the *Particle Filtering*, is presented and its application in this work is described. In section 4 the case study is presented. Finally, in the section 5 the results of the calibration procedure are presented focusing on the goodness of fitting.

1 On the Markov chains and the Cox processes

This section presents the main theoretical elements underlying the model used in the paper. Specifically, in the paragraph 1.1 the Markov chains are presented, both in the discrete-time and in the continuous-time case. Instead, in the paragraph 1.2 the Cox processes are presented in the Lando's paper [13] framework.

1.1 The Markov chains

A Markov process is a stochastic process for which the Markov property holds, that is a stochastic process whose future behavior can be determined only by the current state of the process and it is independent of the past. If the state space is finite or countable, a Markov process is called a Markov chain.
Markov chains play a fundamental role in the context of *rating-based* credit risk models, as the Jarrow, Lando e Turnbull model [9] and Lando model [13], which model the transitions between the different rating classes through a Markov chain defined on finite state space, $S = \{1, \cdots, K\}$, which contains all the rating classes and the absorbing default state K.
In general, changes in an issuer's or a security's rating are governed by transition probabilities that can be collected in a transition matrix such as the following:

Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Ca/C	WR	Default
Aaa	0.8771	0.07943	0.0058	0.00072	0.00023	3e-05	0	0	0.03668	0
Aa	0.00818	0.85154	0.08514	0.00424	0.00062	0.00035	0.00017	1e-05	0.04954	2e-04
A	0.00052	0.02464	0.86784	0.05369	0.00484	0.00106	4e-04	5e-05	0.04643	0.00052
Baa	0.00033	0.00143	0.0412	0.85715	0.03787	0.00687	0.00154	2e-04	0.05173	0.00167
Ba	6e-05	0.00041	0.00422	0.06116	0.76321	0.07172	0.00706	0.00111	0.08222	0.00883
B	7e-05	0.00029	0.0014	0.00448	0.04778	0.73486	0.06615	0.00521	0.10704	0.03272
Caa	0	9e-05	0.00022	0.00084	0.00344	0.06512	0.67874	0.02852	0.14348	0.07955
CaC	0	0	0.00049	0	0.00558	0.02289	0.08943	0.39387	0.22116	0.26658

Table 1: Example of transition frequencies matrix - Source: *Moody's*.

The discrete-time Markov chain
Let $\{X_{t_k}, k = 0, 1, \cdots, n\}$ be a discrete-time stochastic process with state space is $\mathcal{N} = \mathbb{Z} \equiv \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. In this work $\mathcal{N}$ is made up of the different rating classes and default state K, i.e. $\mathcal{N} = \{1, 2, \cdots, 7, 8, K\}$ = {Aaa, Aa, A, Baa, Ba, B, Caa, Ca/C, Default}. This process is a discrete-time Markov chain if the distribution of X_{t_n+1} depends only on the current state X_{t_n}, not on the whole history $\{X_{t_0}, \cdots, X_{t_n}\}$, that is if for each time t_k and every state $i_0, \cdots, i_n$:

$$P\left[X_{t_n+1} = j | X_{t_0} = i_0, \cdots, X_{t_n} = i_n\right] = P\left[X_{t_n+1} = j | X_{t_n} = i_n\right]. \quad (1)$$

Markov chains are governed by transition probabilities matrix, one-period or multi-period, that is the probabilities of passing from a generic state i to a generic state j within one or more periods. Specifically, these are the following conditional probabilities:

- $p_{ij} = P\left[X_{t_k+1} = j | X_{t_k} = i\right]$ represents the one-period probability of passing from the state i to the state j;

- $p_{ij}^{(n)} = P\left[X_{t_k+n} = j | X_{t_k} = i\right]$ represents the n-period probability of passing from the state i to the state j.

These probabilities are generally collected in the so-called *transition probabilities matrix* or, more simply, *transition matrix.* According to the state space $\mathcal{N}$ defined above for this work, it can be represented as follows:

$$\mathbf{P} = \begin{bmatrix} p_{11} & \cdots & p_{1K} \\ \vdots & \ddots & \vdots \\ p_{i1} & \cdots & p_{iK} \\ \vdots & \ddots & \vdots \\ p_{K1} & \cdots & p_{KK} \end{bmatrix}$$

with

- $\sum_{j=1}^{K} p_{ij} = 1$ for $i = 1, \cdots, K$

- $p_{ij} \geq 0 \ \forall i, j = 1, \cdots, K.$

Generically, the n-period transition matrix can be achieved multiplying the one-period transition matrices of different periods:

$$\mathbf{P}(t_k, t_{k+n}) = \mathbf{P}(t_k, t_{k+1}) \times \mathbf{P}(t_{k+1}, t_{k+2}) \times \cdots \times \mathbf{P}(t_{k+n-1}, t_{k+n}).$$

If the time-homogeneity property holds, the calculation of the multi-period transition matrix is simplified: indeed, a Markov chain is time-homogeneous if the probability $P\left[X_{t_n+1} = j | X_{t_n} = i\right]$ doesn't depend on n, that is if the one-period transition matrix is constant over time. The consequent n-period transition matrix is given by:

$$\mathbf{P}(t_k, t_{k+n}) = \mathbf{P}^n \tag{2}$$

where $\mathbf{P}$ is the one-period transition matrix. The generic element of $\mathbf{P}(t_k, t_{k+n})$ is given by:

$$p_{ij}^{(n)} = P\left[X_{t_k+n} = j | X_{t_k} = i\right] = (\mathbf{P}^n)_{ij}. \tag{3}$$

The continuous-time Markov chain

The definitions given for the discrete-time case can be extended to the continuous-time one.

A continuous-time stochastic process $\{X_t, t \geq 0\}$ with state space $\mathcal{N} = \mathbb{Z}$ is a time-continuous Markov chain if satisfies the following property for each $t, s \geq 0$ and for each state j:

$$P\left[X_{t+s} = j | X_t = i, \{X_u : 0 \leq u < t\}\right] = P\left[X_{t+s} = j | X_t = i\right]. \quad (4)$$

The time homogeneity property becomes:

$$P\left[X_{t+s} = j | X_t = i\right] = P\left[X_{t+s+k} = j | X_{t+k} = i\right] \quad (5)$$

In the continuous-time case one needs to consider the probability distribution of the holding time S_i, that is the time that an issuer spends in rating grade i. Because of the Markov property this distribution is exponential:

$$S_i \sim \exp(\lambda_i \cdot t).$$

with λ_i a positive constant. Since S_i follows an exponential distribution[2], the probability that one transition occurs during a short interval Δt is given by:

$$P\left[X_{t+\Delta t} \neq i | X_t = i\right] = \lambda_i \cdot \Delta t + o(\Delta t) \quad (6)$$

and is possible to define the transition rate from the state i to the state j as:

$$\lambda_{ij} = \lim_{\Delta_t \to 0} \frac{P\left[X_{\Delta t} = j | X_0 = i\right]}{\Delta t}, \quad (7)$$

that by definition must be non-negative, and the holding rate in the state i as:

$$\lambda_i = \sum_{j=1, j \neq i}^{K} \lambda_{ij}. \quad (8)$$

A time-homogeneous continuous-time Markov chain can be defined by the $K \times K$ generator matrix:

$$\mathbf{\Lambda} = \begin{bmatrix} -\lambda_1 & \cdots & \lambda_{1K} \\ \vdots & \ddots & \vdots \\ \lambda_{i1} & \cdots & \lambda_{iK} \\ \vdots & \ddots & \vdots \\ \lambda_{K1} & \cdots & -\lambda_K \end{bmatrix}$$

which satisfies the following properties:

- $\sum_{j=1}^{K} \lambda_{ij} = 0$, for $1 \leq i \leq K$

[2]Then the number of jump events follows a Poisson distribution.

- $0 \leq -\lambda_{ii} = \lambda_i,$ for $1 \leq i \leq K$
- $\lambda_{ij} \geq 0,$ for $1 \leq i, j \leq K$ with $i \neq j$.

Then the relation between the generator matrix and the transition matrix is:

$$\mathbf{P}(t,s) = \exp\left((s-t)\mathbf{\Lambda}\right). \tag{9}$$

For further details, refer to [8] and [11].

1.2 The Cox processes

A Cox process is a generalization of the Poisson process in which the intensity can be random on condition that, given a particular realization of the intensity, $l(s,\omega)$, the jump process is an inhomogeneous Poisson process[3] with intensity $l(s,\omega)$.
In [13] Lando proposes a random intensity on the form of a function of the current level of the state variables $\mathbf{X}$, that is a $\mathbb{R}^d$-valued stochastic process:

$$l(s,\omega) = \lambda(\mathbf{X}_s)$$

with $\lambda : \mathbb{R}^d \to [0,+\infty)$.
Formally, let $(\Omega, \mathcal{F}, P)$ a probability space large enough to support an $\mathbb{R}^d$-valued stochastic process $\mathbf{X} = \{\mathbf{X}_t : 0 \leq t \leq T_f\}$ and a unit exponential random variable E_1 which is independent of $\mathbf{X}$. Let's assume that $\lambda : \mathbb{R}^d \to \mathbb{R}$ is non-negative and continuous. Then the default time τ is defined as

$$\tau = \inf\left\{t : \int_0^t \lambda(\mathbf{X}_s)\, ds \geq E_1\right\} \tag{10}$$

and can be considered the first jump time of a Cox process with intensity process $\lambda(\mathbf{X}_s)$.
From the previous definitions derive the following probabilities:

$$P\left[\tau > t | \{\mathbf{X}_s\}_{o \leq s \leq t}\right] = e^{-\int_0^t \lambda(\mathbf{X}_s)\, ds} \quad t \in [0, T_f] \tag{11}$$

$$P\left[\tau > t\right] = \mathbf{E}\left[e^{-\int_0^t \lambda(\mathbf{X}_s)\, ds}\right] \quad t \in [0, T_f] \tag{12}$$

[3] An inhomogeneous Poisson process N with non-negative intensity function $l(\cdot)$ satisfies:

$$P\left[N_t - N_s = k\right] = \frac{\left(\int_s^t l(u)\, du\right)^k}{k!}\, e^{-\int_s^t l(u)\, du}$$

and, assuming $N_0 = 0$,

$$P\left[N_t = 0\right] = e^{-\int_s^t l(u)\, du}.$$

To simulate the first jump time τ of N is possible to define

$$\tau = \inf\left\{t : \int_0^t l(u)\, du \geq E_1\right\}$$

with E_1 that is a unit exponential random variable.

To extend this framework in order to model a Cox process past the first jump, essential to consider multiple rating transitions, it's necessary to consider a probability space $(\Omega, \mathcal{F}, P)$ large enough to support a standard unit rate Poisson process N with $N_0 = 0$ and a non-negative stochastic process $\lambda(t)$ which is independent of N and assumed to be right-continuous and integrable on finite intervals, i.e.

$$\Lambda(t) := \int_0^t \lambda(s)\, ds < \infty \quad t \in [0, T].$$

The new process $\tilde{N}_t := N(\Lambda(t))$ is a Cox process with intensity Λ.

2 The model for the spread risk

This section presents first, in the paragraph 2.1, the variables relevant for pricing defaultable bonds, such as the risk-free and risky yield to maturity and the yield spread. Then, in the paragraph 2.2 the model extension proposed in [7] and used in this paper is presented.

2.1 Relevant variables for pricing defaultable bonds

Let τ be the random time at which default occurs and $\mathcal{F}_t$ be the filtration that includes the information at time t about the market state variables and the default. In more detail, $\mathcal{F}_t$ contains the information about the evolution of the market state variables up to time t and about whether default has occurred up to time t.
This filtration can be disassembled in two sub-filtrations:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

with: $\mathcal{G}_t = \sigma\{\mathbf{X}_s : 0 \leq s \leq t\}$, the market state variables filtration, and $\mathcal{H}_t = \sigma\{\mathbf{1}_{\{\tau \leq s\}} : 0 \leq s \leq t\}$, the default filtration.
Let:

- $v(t, T)$ be the price of a unitary risk-free ZCB with maturity T[4] at time t;
- $v^i(t, T)$ be the price of a unitary risky ZCB with maturity T[5] issued by a firm with credit rating i at time t;

Under the model assumptions[6], $v^i(t, T)$ is defined by [9] as:

$$v^i(t, T) = \mathbf{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(\mathbf{X}_u)\, du} \left(\delta \mathbf{1}_{\{\tau \leq T\}} + \mathbf{1}_{\{\tau > T\}} \right) | \mathcal{F}_t \right] \tag{13}$$

[4] The ZCB that pays surely a unit of currency at time T.

[5] The ZCB that pays a unit of currency at time T, if the default of the issuer occurs after the maturity T. If the default occurs before the maturity, the bond pays only $\delta < 1$ units of currency.

[6] The markets for the risk-free and risky bond are complete and arbitrage-free. The recovery rate δ is an exogenous constant.

where $\mathbf{X}_u$ is the level of the state variables at the time u, δ is the recovery rate, i.e. the amount by the ZCB if the default occurs, and $\mathbf{1}_{\{\tau \leq T\}}$ is the indicator function of the event $\{\tau \leq T\}$.
Under the further assumption that the default process is independent of the risk-free spot rate, the equation (13) becomes:

$$\begin{aligned} v^i(t,T) &= \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{X}_u)\,du}|\mathcal{G}_t\right]\mathbf{E}^{\mathbb{Q}}\left[\left(\delta\mathbf{1}_{\{\tau\leq T\}}+\mathbf{1}_{\{\tau>T\}}\right)|\mathcal{H}_t\right] \\ &= v(t,T)\left(\delta+(1-\delta)P_t^{\mathbb{Q}}\left[\tau>t\right]\right), \end{aligned} \tag{14}$$

Starting from $v(t,T)$ and $v^i(t,T)$ definitions, the yields to maturity risk-free and risky for the maturity s are defined respectively as:

$$h(t,s) = -\frac{1}{s-t}\log v(t,s), \tag{15}$$

$$h^i(t,s) = -\frac{1}{s-t}\log v^i(t,s). \tag{16}$$

From the equations (15) and (16) is possible to define the credit spread in terms of yield to maturity:

$$\sigma^i(t,s) = h^i(t,s) - h(t,s). \tag{17}$$

This variable plays a fundamental role in this work, as the calibration procedure works on the credit spreads in terms of yield to maturity.

2.2 The model for the spread risk

The model used in this paper is an extension of the models described in [9] and [13]. The model incorporates state dependence in transition rates and risk premia so that stochastic variations in credit spreads are possible even staying in the same rating class.
The credit rating transition and the default process are modelled with an extension of the classical time-homogeneous Markov chain: the dynamic of a credit rating of a bond $Y(t)$ is a Markov process on a finite state space $S = \{1, 2, \cdots, K\}$ governed by the transition matrix

$$\mathbf{P_X}(s,t) = e^{\mathbf{A}(\tau(t)-\tau(s))}, \tag{18}$$

where K is the absorbing state of default, $\mathbf{A}$ is the generator[7] matrix of a time-homogeneous Markov chain and $\tau(t)$ is a stochastic time. This process conditionally on the evolution of the state variables, $\tau(t)$, is a inhomogeneous Markov chain and the transition probabilities satisfy the Kolmogorov's backward equation:

$$\frac{\partial \mathbf{P_X}(s,t)}{\partial s} = -\mathbf{A_X}(s)\mathbf{P_X}(s,t). \tag{19}$$

[7] $\mathbf{A}$ is a matrix with a non-negative off-diagonal elements and zero row sums.

In order to obtain greater clarity, from now on the dependence on state variables in formulas is omitted.
Under the assumptions that the time-dependent generator matrix $\mathbf{A}$ is diagonalizable and then has the representation

$$\mathbf{A} = \mathbf{B}\,\mathbf{D}\,\mathbf{B}^{-1} \tag{20}$$

where $\mathbf{B}$ is a $K \times K$ matrix whose columns are the K eigenvectors of the generator matrix $\mathbf{A}$ and $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ as the diagonal elements[8], the transition matrix can be written as

$$\mathbf{P}(s,t) = \mathbf{B}\, e^{\mathbf{D}(\tau(s)-\tau(t))}\mathbf{B}^{-1} \tag{21}$$

and continues to satisfy the Kolmogorov's backward equation.

In order for the model to be consistent, the process τ is a stochastic time:

- τ is a real positive and increasing right continuous process with left limits;
- for every $t \geq 0$, $\tau(t)$ is a stopping time;
- for every $t \geq 0$, $\tau(t)$ is finite almost surely;
- $\tau(0) = 0$;
- $\lim_{t\to+\infty} \tau(t) = \infty$.

$\tau(t)$ has stationary non-negative independent increments then is a subordinator of the subordinated process Y, that is unconditionally a Markov chain.
Gambaro et al. in [7] define $\tau(t)$ as an integral of a positive stochastic intensity $\lambda(t)$:

$$\tau(t) = \int_0^t \lambda(s)\, ds, \tag{22}$$

then the equation (19) becomes

$$\frac{\partial \mathbf{P}(s,t)}{\partial s} = -\mathbf{A}\lambda(s)\mathbf{P}(s,t). \tag{23}$$

and the generator matrix of Y is $\mathbf{A}_\lambda(t) = \mathbf{A}\lambda(t)$.
For further analytical details, refer to [13] and [7].

The process λ is modelled as a CIR model [3], i.e. the real-world dynamic is:

$$d\lambda(t) = \alpha(\gamma - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dZ^{\mathbb{P}}(t),\ \lambda(0) = \lambda_0 \tag{24}$$

[8] $\mathbf{D} = \mathrm{diag}(d_1, \cdots, d_{K-1}, 0)$.

where $dZ^{\mathbb{P}}(t)$ is a standard Brownian motion under the real-world measure while under the risk-neutral[9] one become:

$$d\lambda(t) = \hat{\alpha}(\hat{\gamma} - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dZ^{\mathbb{Q}}(t),\ \lambda(0) = \lambda_0, \tag{25}$$

where $dZ^{\mathbb{Q}}(t)$ is a standard Brownian motion under the risk-neutral measure.

Under the CIR model assumption for $\lambda(t)$, the real-world and risk-neutral transition probabilities follow a non-central Chi-squared distribution:

$$p(\lambda_{t+\Delta t}|\lambda_t) \sim \chi^2_{nc}\big(\lambda_{t+\Delta t}; 2\nu, \mu(\lambda_t)\big), \tag{26}$$

where

$$\nu = \frac{2\alpha\gamma}{\sigma^2}$$

$$\mu(\lambda_t) = \lambda_t \frac{4\alpha e^{-\alpha\Delta t}}{(1 - e^{-\alpha\Delta t}\sigma^2)}$$

and the parameters are respectively real-world and risk-neutral.

The non-centrality parameter, as can been seen, depends on the λ_t value then the process is not-stationary.

Then the price at time t of a ZCB with maturity T, issued by a firm with rating i and zero recovery rate is:

$$\begin{aligned} v^i(t,T) &= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\,(1 - \mathbf{P}(t,T)_{i,K})\right] \\ &= \sum_{j=1}^{K-1} -b_{ij}b^{-1}_{jK}\mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\,e^{d_j\int_t^T \lambda(s)\,ds}\right], \end{aligned} \tag{27}$$

where $r(t)$ is the risk-free spot rate, $\mathbf{P}(t,T)_{i,K}$ is the transition probability from a credit rating i at time t to a default state at time T, b_{ij} and b^{-1}_{ij} are respectively the (i,j) elements of the matrices $\mathbf{B}$ and $\mathbf{B}^{-1}$ and d_j is the j-th diagonal element of the matrix $\mathbf{D}$.

Under the assumption that the process r and λ are independent, the equation (27) becomes:

$$v^i(t,T) = v(t,T)\sum_{j=1}^{K-1} -b_{ij}b^{-1}_{jK}\mathbf{E}_t^{\mathbb{Q}}\left[e^{d_j\int_t^T \lambda(s)\,ds}\right], \tag{28}$$

where $v(t,T)$ is the price of the risk-free bond with maturity T.

Under the CIR model hypothesis, the expectation in equation (28) has an analytical expression given by [10]:

$$\mathbf{E}_t^{\mathbb{Q}}\left[e^{d_j\int_t^T \lambda(s)\,ds}\right] = A(u)e^{-B(u)(-d_j)\lambda_t},\ \ u = T - t \tag{29}$$

[9]Gambaro et al. in [7] use a CIR++ model.

where:

$$A(x) = \left[\frac{2\, b\, e^{\frac{\hat{\alpha}+b}{2} x}}{(\hat{\alpha}+b)(e^{b\,t}-1)+2b} \right]^{\nu},$$

$$B(x) = \frac{2\,(e^{b\,T}-1)}{(\hat{\alpha}+b)(e^{b\,x}-1)+2b},$$

$$b = \sqrt{\hat{\alpha}^2 + 2(-d_j)\sigma^2}.$$

As proposed in [7], a more flexible dependence structure among the credit ratings can be obtained adding a rating-specific liquidity spread to the ZCB price formula (28):

$$v^i(t,T) = v(t,T)\mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T l^i(s)\, ds}\right] \sum_{j=1}^{K-1} -b_{ij} b_{jK}^{-1} \mathbf{E}_t^{\mathbb{Q}}\left[e^{d_j \int_t^T \lambda(s)\, ds}\right], \quad (30)$$

where rating-specific liquidity spread intensity $l^i(t)$ can be modelled in several ways, but this additional term represents an extension of the model in (28) and its study is deferred to future works.

3 The *Particle Filter*

This section presents the *Particle Filter* calibration technique. First, from a theoretical point of view (paragraph 3.1), i.e. the Bayes' theorem-based algorithm to calculate the likelihood value and to estimate the optimal parameters vector. Then, the *Particle Filter* is presented applied in the framework of this paper (paragraph 3.2).

3.1 Theoretical aspects

The state space Hidden Markov Model (HMM) are compatible with this structure:

- a multidimensional χ-valued discrete-time Markov process $\{\mathbf{x}_n\}_{n\geq 0}$, unobservable, called *hidden* or *state variable*;
- a multidimensional $\mathcal{Y}$-valued process $\{\mathbf{y}_n\}_{n\geq 0}$, observable, whose observations are conditionally independent, given $\mathbf{x}_n$ value ($\{\mathbf{y}_n|\mathbf{x}_n\}$). This process is called *observation variable*.

It is assumed that:

1. in $n = 0$ (initial time), the *state variable* has an initial probability density $p_\theta(\mathbf{x}_0)$; this can be considered the Bayesian prior distribution.
2. for $n \geq 1$, the *state variable* evolves according to the the transition probability density $p_\theta(\mathbf{x}_n|\mathbf{x}_{n-1})$: this is called *state equation*.

3. for $n \geq 1$, the *observation variable* has marginal probability density $p_\theta(\mathbf{y}_n|\mathbf{x}_n)$: this is called *measure equation* and provides the observation likelihood.

The assumptions 1-2 define the prior distribution of the *state process* $\{\mathbf{x}_n\}_{n\geq 0}$, that is:

$$p_\theta(\mathbf{x}_{1:n}) = p_\theta(\mathbf{x}_0) \prod_{k=1}^{n} p_\theta(\mathbf{x}_k|\mathbf{x}_{k-1}). \tag{31}$$

The third assumption, instead, defines the likelihood function:

$$p_\theta(\mathbf{y}_{1:n}|\mathbf{x}_{1:n}) = \prod_{k=0}^{n} p_\theta(\mathbf{y}_k|\mathbf{x}_k). \tag{32}$$

The problem of filtering consists of characterising the distribution and the full trajectory of the *state variable* of the HMM at the present time, given the observations up to the present time. *Particle Filtering* is proposed as one of the techniques used for this purpose.

The posterior distribution $p_\theta(\mathbf{x}_{1:n}|\mathbf{y}_{1:n})$ satisfies the following recursive relation:

$$p_\theta(\mathbf{x}_{1:n}|\mathbf{y}_{1:n}) = \frac{p_\theta(\mathbf{x}_{1:n-1}|\mathbf{y}_{1:n-1})\, p_\theta(\mathbf{x}_n|\mathbf{x}_{n-1})\, p_\theta(\mathbf{y}_n|\mathbf{x}_n)}{p_\theta(\mathbf{y}_n|\mathbf{y}_{1:n-1})}. \tag{33}$$

In the literature, the recursive relation for the marginal posterior distribution $p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n})$[10] is presented as:

$$p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n}) = \frac{p_\theta(\mathbf{y}_n|\mathbf{x}_n)\, p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n-1})}{p_\theta(\mathbf{y}_n|\mathbf{y}_{1:n-1})} \tag{34}$$

where $p_\theta(\mathbf{y}_n|\mathbf{x}_n)$ is the likelihood and it's defined by assumption 3, whereas

$$p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n-1}) = \int_{\Omega_x} p_\theta(\mathbf{x}_n|\mathbf{x}_{n-1})\, p_\theta(\mathbf{x}_{n-1}|\mathbf{y}_{1:n-1})\; d\mathbf{x}_{n-1} \tag{35}$$

is the conditional distribution of the *state variable* $\mathbf{x}_n$, given the information up to $n-1$, and

$$p_\theta(\mathbf{y}_n|\mathbf{y}_{1:n-1}) = \int_{\Omega_x} p_\theta(\mathbf{y}_n|\mathbf{x}_n)\, p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n-1})\; d\mathbf{x}_n \tag{36}$$

is the constant that allows to compute the marginal likelihood, i.e.:

$$\mathcal{L}(\theta) = p_\theta(\mathbf{y}_1) \prod_{k=2}^{n} p_\theta(\mathbf{y}_k|\mathbf{y}_{1:k-1}) \tag{37}$$

Starting from $p_\theta(\mathbf{x}_0)$ and using recursively equation (35), known as the *prediction step*, and equation (34), known as *updating step*, is possible to compute $\{p_\theta(\mathbf{x}_n|\mathbf{y}_{1:n})\}$ sequentially[11] so to obtain the marginal like-

[10] Obtained integrating out $\mathbf{x}_{1:n-1}$ in equation (33).

[11] Starting from $p_\theta(\mathbf{x}_{n-1}|\mathbf{y}_{1:n-1})$.

lihood $\mathcal{L}(\theta)$.

Specifically, the *Particle Filter* is a particular method belonging to the class of filtering methods in which the integrals are solved with numerical methods. Alternatively, if the distributions are Gaussian and the relation between the *state variable* and the *observation variable* is linear, then analytical solutions to recursive equations are provided and the method is known as *Kalman Filter*.

3.2 The *Particle Filter* applied to the model

In this work, two calibration procedures are proposed, or rather the same calibration procedure is applied to different data (*single-rating* and *multi-rating*[12] calibration). In both cases the *Particle Filter* is used:

1. the *state variable* is the subordinator intensity $\lambda(t)$ and the *state equation* follows a non-central Chi-squared distribution.

2. the *observation variable* is the term structure of credit spread in terms of yield to maturity $\sigma^i(t,s)$, defined in equation (17), either way and the *measure equation* is assumed Gaussian:

$$p(\mathbf{y}_t|\mathbf{x}_t) = p(\sigma_t^i(\mathbf{s})|\mathbf{x}_t) \sim N(\sigma_t^i(\mathbf{s}), \mathbf{\Sigma}), \tag{38}$$

   where $\mathbf{\Sigma}$ is a diagonal variance-covariance matrix and each non-zero element is the constant parameter ω^2.

For the modelling assumptions made, the integrals in equations (35) and (36) have no closed formula solution and therefore must be solved using numerical methods, in particular the Gauss-Legendre quadrature with $n = 1024$ nodes in the range [0,10].
Then the filter recursive equations become:

$$\hat{p}_\theta\left(x_t^j|\sigma_{1:t-1}^i(\mathbf{s_t})\right) = \sum_{k=1}^{n} p_\theta\left(x_t^j|x_{t-1}^k\right)\tilde{p}_\theta\left(x_{t-1}^k|\sigma_{1:t-1}^i(\mathbf{s_t})\right)w^k \tag{39}$$

and

$$\tilde{p}_\theta\left(x_t^j|\sigma_{1:t}^i(\mathbf{s_t})\right) = \frac{p_\theta\left(\sigma_t^i(\mathbf{s_t})|x_t^j\right)\ \hat{p}_\theta\left(x_t^j|\sigma_{1:t-1}^i(\mathbf{s_t})\right)}{\sum_{k=1}^n w^k\ p_\theta\left(\sigma_t^i(\mathbf{s_t})|x_t^k\right)\ \hat{p}_\theta\left(x_{t-1}^k|\sigma_{1:t-1}^i(\mathbf{s_t})\right)}. \tag{40}$$

where θ is the model parameters vector, $w^{(k)}$ are the quadrature weights and x_t^k are the quadrature nodes. Then the marginal likelihood in equation (37) becomes:

$$\mathcal{L}(\theta) = \max_{\theta\in\Theta}\prod_{t=1}^{T} p_\theta(\sigma_t^i(\mathbf{s})|\sigma_{1:t-1}^i(\mathbf{s})). \tag{41}$$

[12]Further details about these two calibration procedure are provided in section 4.

The calibration procedure is implemented using the software `R` and the programming language `C++`; specifically the library `nlopt` and the optimization algorithm `cobyla`[13] are used.

4 Case study

This section presents the data and the metrics involved in the calibration technique applied to the model. Specifically, in the paragraph 4.1 the two calibration procedures are described; in the paragraph 4.2 market data used for the calibration procedure and for the assessment of model goodness of fit are presented; in the paragraph 4.3, instead, the metrics used to assess the model goodness of fit to those data are described.

4.1 Calibration procedures

The two calibration procedures, respectively named *single-rating* and *multi-rating* calibration, provide for the estimation of parameters vector of the subordinator intensity λ and the ω parameter.
Given that we reasonably assuming a unique transition matrix therefore the λ process must also be unique, two paths are followed:

1. *single-rating* calibration: the parameters are calibrated on the historical series of spread of a single rating class. Then this set of parameters is used, jointly with the historical series of λ_t reconstructed by the filter, is tested also on the other rating classes for which calibration has not been carried out.
   This calibration is performed for each available rating and is configured with a maximum number of iterations of 250.

2. *multi-rating* calibration: the parameters are calibrated on the historical series of the spreads of all ratings[14] at the same time.
   This calibration, on the other hand, is configured with a maximum number of iterations of 1000 due to the larger amount of data it has to process.

4.2 Reference database

In this section the input data to the estimation process, i.e. the spreads time series (subparagraph 4.2.1) and the transition matrix (subparagraph 4.2.2), are analysed.

[13] *Constrained Optimization BY Linear Approximations* proposed by Powell in [17] and [18].
[14] All available ratings.

4.2.1 Spreads time series

The spreads time series used in the calibration process are provided by *Bloomberg*. Specifically, the time series of yield spreads related to currency *Euro*, economic sector *Finance* and ratings *Aa*, *A* and *Baa* (according to *Moody's Investors Service* ratings system) are used. They are identified by the following tickers respectively: `IGEEFD` Index, `IGEEFA` Index and `IGEEFB` Index.
The yield curves are constructed daily with bonds that have *BVAL*[15] prices at the market close. The *BVAL* curves are populated with EUR denominated senior unsecured fixed rates bond issued by European Financial companies with a BBG composite rating respectively of Aa+, Aa or Aa- (`IGEEFD` Index), A+, A or A- (`IGEEFA` Index) and Baa+, Baa or Baa- (`IGEEFB` Index)[1].

The valuation date is 30/06/2018 and the historical depth of the time series is 9 years, the maximum available on *Bloomberg*[16]. The residual maturities considered are $\boldsymbol{s} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Therefore the total number of input data is 67474: 23220 for `IGEEFD` Index, 23220 for `IGEEFA` Index and 21034 for `IGEEFB` Index.
The figures 1 represent the market spreads surfaces. Summary statistics (means, standard deviations and 99.5% quantiles) calculated over the full time span are reported in table 2, instead the same statistics calculated over 1-year time span are reported in tables 3, 4 e 5.
From the analysis of the time series statistics it can be seen that the means have an increasing trend in the first years and then drastically collapsed in 2011 for `IGEEFD` Index[17] and in 2013 for the other two indices. As for volatilities, on the other hand, they are much more fluctuating but they too record a sharp drop starting from 2012 for `IGEEFD` Index and from 2013 for the other indices. This highlights the non-stationary nature of the spreads over time.

4.2.2 The transition matrix

The *Average One-Year Letter Rating Migration Rates, 1970-2017* [16] was chosen as the transition matrix, reported in table 6.
This matrix is adjusted by removing the *Without Rating* column and by normalizing the matrix again. The normalization is carried out by dividing the element of the original matrix by the one's complement of

[15]BVAL is the standard for pricing transparency and quality in the fixed income valuation market.
[16]More specifically, starting from 01/07/2009 for `IGEEFD` Index and `IGEEFA` Index and from 02/12/2010 for `IGEEFB` Index.
[17]For `IGEEFD` Index the mean even displays negative value for the 1-year maturity.

the probability of transition to the class *Without Rating*:

$$p_{i,j}^{\text{adj}} = \frac{p_{i,j}}{\sum_{j \neq WR} p_{i,j}} \tag{42}$$

where $p_{i,j}$ is the generic element of the original transition matrix. The resulting adjusted matrix is reported in table 7.
The corresponding generator matrix is obtained through the *Quasi-Optimization* algorithm [12] using the `R` package `ctmcd` and it's reported in table 8. This matrix satisfies the generator properties defined in paragraph 1.1.

4.3 The quality metrics

The calibration quality is assessed through several quality metrics. For the *in-sample* performance, in order to evaluate the goodness of fit, model time series plots superimposed on market time series plots are used for some representative maturities {3, 5, 8, 10} years and for all three rating classes used in the calibration. In addition, model and market means and standard deviations are compared and histograms of residuals, defined as the difference between market and model values, are analysed.
Several synthetic indicators are then calculated:

- the Root Mean Square Error (*RMSE*), that is an absolute risk metric;
- the Coefficient of Determination (R^2), that it's a relative risk metric;
- the Coefficient of Determination on the standardised data (R^2_{std})[18];
- the Coefficient of Determination for the single rating classes (R^2_i)[19];
- the ω parameter which is used to capture market imperfections due, for example, to liquidity effects.

[18] Only for *multi-rating* calibration
[19] Only for *multi-rating* calibration

5 Results

This section presents the results of the calibration procedure for both the *single-rating* and the *multi-rating* case.

5.1 *Single-rating* calibration

Section 7.3 shows tables and figures about the results of the *single-rating* calibration procedure in all its possible configurations. The estimated parameters are difficult to compare with available market data, as they concern the subordinator intensity λ.

Otherwise, all three configurations behave similarly. For the actually calibrated rating class, the parameters are adequate and provide a good fit: R^2 values (reported in tables 10, 12 and 14) are respectively 0.84, 0.85 and 0.84 and the graphical analysis (figures 2, 6 and 10) shows a good adaptation of the model values to the market values, while for the remaining two ratings, the calibration is inadequate (R^2 values much lower: 0.50 or below).

The ω values, instead, are low in all three cases.

These results prompt us to consider a calibration procedure that considers data from all available rating classes to try to obtain an appropriate estimate for all three rating classes.

5.2 *Multi-rating* calibration

Section 7.4 shows tables and figures about the results of the *multi-rating* calibration procedure.

The convenience of switching to a *multi-rating* calibration is confirmed by the statistics in table 16: the overall R^2 is good (0.85) and it's in line with that of *single-rating* calibrations relative to the calibrated rating classes, the standardised R^2 is still good (0.75) and especially the *single-rating* R^2 are significantly higher than those of *single-rating* calibrations (Aa: 0.68, A: 0.68, Baa: 0.84).

In addition, the *RMSE* and the ω parameter have low values.

Turning to the graphical analysis of the comparison between model and market data (figures 11, 12 and 13), it can be seen that model time series, reconstructed by the filter, can adequately replicate market time series, especially for Baa rating class and for middle maturities.

This behaviour is also confirmed by the graphical analysis of the model statistics (figures 14, 15 and 16), the model means are substantially in line with the market ones, as are the volatilities, except for the shorter maturities. This can be justified by the fact the three rating classes have different historical volatilities, especially for shorter maturities, as shown in table 2.

The analysis of the residuals (figures 17, 18 and 19) shows that the normality hypothesis is not respected, however they have a mean not significantly different from 0.

6 Conclusion

This paper aimed to present the application of the *Particle Filtering* calibration technique to the Markovian model for the spread risk presented in [7] with two different approaches, focusing on the *in-sample* analysis. These analyses show that the *Particle Filter* returns acceptable estimates, specially for the *multi-rating* calibration that manages to replicate quite well the market data. However, model volatilities for shorter maturities are too low. Therefore in order to improve this aspect, as well as the goodness of fit further, the model can be extended by introducing a rating-specific liquidity component (formula 28), thus increasing the number of available parameters.
As was to be expected, given the volume of data processed and the complexity of the numerical resolution of the integrals provided by the *Particle Filter*, calibration times were much longer than those for *cross-section* calibrations but still acceptable.

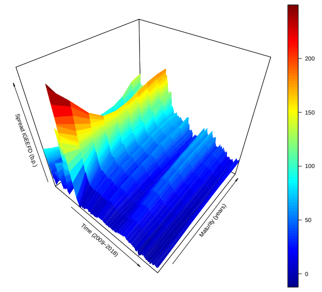
Further analysis referred mostly to the *out-of-sample* performances, the assessment of the Solvency Capital Requirement and the extension of the model are deferred to future work.

7 Plots and tables

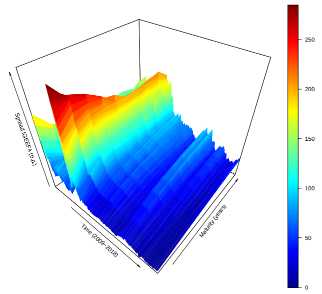
This section contains the graphs and tables of the previous sections.

7.1 Market data plots and statistics

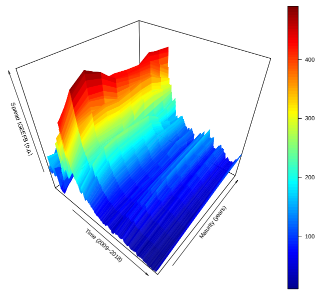
7.1.1 Market spreads surfaces



(a) `IGEEFD` Index time series.



(b) `IGEEFA` Index time series.



(c) `IGEEFB` Index time series.

Figure 1: Spread structures trend - 01/07/2009 to 30/06/2018.

7.1.2 Market spreads statistics

	IGEEFDO Index			IGEEFA Index			IGEEFB Index		
Maturity	Mean	Std Dev.	Quantile 99.5%	Mean	Std Dev.	Quantile 99.5%	Mean	Std Dev.	Quantile 99.5%
1	17.82	39.68	197.44	36.30	42.56	271.45	85.65	75.43	368.06
2	24.16	28.73	157.19	49.37	42.15	261.46	114.01	89.39	425.33
3	29.98	25.61	138.76	57.91	42.70	238.76	129.46	94.40	444.05
4	36.52	26.69	137.45	63.90	42.12	227.05	138.76	92.80	434.60
5	42.56	28.05	137.90	67.32	39.44	207.52	144.87	86.90	417.01
6	48.45	29.36	141.76	69.76	36.44	184.17	144.82	77.25	413.18
7	55.75	32.03	147.44	73.55	34.82	177.12	146.07	72.49	416.86
8	62.10	34.57	157.22	78.83	36.97	180.61	133.62	64.35	418.94
9	67.59	36.85	166.19	84.50	40.04	185.04	129.70	63.03	391.71
10	69.19	35.20	166.36	86.35	38.66	183.40	136.96	69.69	420.94

Table 2: Means, standard deviations and 99.5% quantiles of observed data (b.p.).

		1Y			3Y			5Y			7Y			10Y		
Start Date	End Date	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile
2009-07-01	2009-12-31	44.56	26.58	97.81	40.02	13.82	77.23	54.34	16.96	95.37	64.81	12.38	96.99	97.04	15.20	144.43
2010-01-01	2010-12-31	27.93	13.99	77.86	40.93	7.29	55.84	55.67	8.08	68.20	80.42	12.4	104.34	97.62	11.79	126.27
2011-01-03	2011-12-30	99.25	66.77	270.14	72.92	38.67	181.87	85.46	28.34	149.23	103.31	24.19	160.57	110.67	26.47	178.75
2012-01-02	2012-12-31	-5.52	5.48	6.45	45.40	15.21	93.82	69.00	23.25	131.44	87.13	24.96	146.61	110.65	28.65	167.85
2013-01-01	2013-12-31	-1.37	3.30	8.29	28.63	4.63	38.39	42.06	6.03	54.74	56.38	5.27	71.05	66.16	4.44	79.16
2014-01-02	2014-12-31	3.83	2.03	8.04	19.08	8.57	32.16	28.29	9.54	44.43	38.21	11.59	57.72	47.89	17.4	81.48
2015-01-02	2015-12-31	6.42	3.71	12.80	19.04	8.37	31.98	31.26	11.45	51.17	39.54	12.97	62.82	49.19	13.61	73.43
2016-01-04	2016-12-30	7.76	4.94	17.74	18.38	7.38	32.52	30.47	10.78	51.14	39.50	12.37	63.75	49.10	11.99	73.92
2017-01-02	2017-12-29	-1.83	4.80	10.20	2.26	5.22	14.46	8.13	5.96	20.62	16.17	6.70	29.29	28.9	8.33	44.35
2018-01-02	2018-06-29	0.47	1.99	4.43	3.04	5.32	12.04	6.90	8.50	20.71	12.43	8.37	24.95	22.16	5.68	30.90

Table 3: Means, standard deviations and 99.5% quantiles of observed data - `IGEEFD` Index.

		1Y			3Y			5Y			7Y			10Y		
Start Date	End Date	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile
2009-07-01	2009-12-31	109.77	32.54	206.00	94.37	24.88	158.79	92.83	16.12	140.71	99.36	14.81	140.74	109.51	6.57	123.51
2010-01-01	2010-12-31	65.39	10.91	86.03	84.58	10.28	103.73	91.08	9.25	109.57	93.70	8.89	125.91	116.22	14.06	165.85
2011-01-03	2011-12-30	79.29	73.82	303.40	117.56	48.88	252.52	122.4	39.46	216.08	117.59	26.51	177.46	135.43	23.89	184.91
2012-01-02	2012-12-31	52.74	35.19	171.13	101.33	37.03	211.59	110.51	33.90	208.26	116.12	27.38	205.12	137.24	31.22	192.33
2013-01-01	2013-12-31	18.83	3.16	29.97	49.66	5.03	59.84	64.66	6.31	79.62	76.27	5.74	90.61	78.76	7.71	99.91
2014-01-02	2014-12-31	15.88	2.04	20.89	37.77	7.34	49.87	49.99	8.70	63.32	56.00	10.15	71.55	64.13	10.39	78.44
2015-01-02	2015-12-31	13.84	4.41	21.07	32.60	8.19	47.24	46.52	11.02	69.13	54.83	13.64	82.91	64.43	15.15	96.58
2016-01-04	2016-12-30	15.13	4.45	24.63	29.54	8.76	48.38	41.63	12.14	68.88	50.87	14.04	83.90	59.79	15.67	98.54
2017-01-02	2017-12-29	5.59	4.48	18.05	12.55	5.36	24.81	21.15	7.44	34.85	31.11	9.96	47.01	43.67	12.71	65.32
2018-01-02	2018-06-29	4.55	2.15	8.74	10.26	5.11	19.83	16.85	7.85	31.43	25.8	8.63	41.29	39.13	8.46	54.53

Table 4: Means, standard deviations and 99.5% quantiles of observed data - `IGEEFA` Index.

		1Y			3Y			5Y			7Y			10Y		
Start Date	End Date	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile	Mean	Std Dev.	Quantile
2010-02-12	2010-12-31	106.63	38.48	179.13	153.59	15.78	224.07	182.94	15.03	226.14	207.60	29.37	257.61	76.75	6.19	99.05
2011-01-03	2011-12-30	132.82	80.70	287.71	199.41	90.07	374.12	211.38	69.90	399.07	172.32	27.69	228.43			
2012-01-02	2012-12-31	222.87	63.60	392.06	296.57	71.33	501.26	294.95	63.50	434.50	254.77	86.79	431.04	288.51	56.58	434.93
2013-01-01	2013-12-31	110.65	21.31	156.73	176.99	32.7	249.89	189.31	28.60	259.07	195.48	20.68	250.43	180.02	33.39	241.52
2014-01-02	2014-12-31	52.53	8.51	71.71	83.12	16.34	119.56	98.59	15.23	134.97	114.02	21.96	164.14	129.49	14.33	163.84
2015-01-02	2015-12-31	40.33	6.82	53.96	73.20	9.74	95.37	89.94	13.37	122.9	98.32	14.89	135.55	104.44	18.84	148.75
2016-01-04	2016-12-30	31.95	6.89	48.63	62.59	16.78	101.09	84.91	21.86	135.36	100.37	22.52	151.83	116.19	22.63	168.48
2017-01-02	2017-12-29	19.33	6.10	36.80	35.74	9.71	56.35	53.84	13.58	78.48	69.35	16.32	97.75	88.51	19.32	123.04
2018-01-02	2018-06-29	18.18	9.99	41.44	32.45	13.83	64.63	49.52	17.06	86.63	66.22	18.51	106.24	87.51	18.43	127.70

Table 5: Means, standard deviations and 99.5% quantiles of observed data - `IGEEFB` Index.

7.2 Rating transition matrix

Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Ca/C	WR	Default
Aaa	0.8771	0.07943	0.0058	0.00072	0.00023	3e-05	0	0	0.03668	0
Aa	0.00818	0.85154	0.08514	0.00424	0.00062	0.00035	0.00017	1e-05	0.04954	2e-04
A	0.00052	0.02464	0.86784	0.05369	0.00484	0.00106	4e-04	5e-05	0.04643	0.00052
Baa	0.00033	0.00143	0.0412	0.85715	0.03787	0.00687	0.00154	2e-04	0.05173	0.00167
Ba	6e-05	0.00041	0.00422	0.06116	0.76321	0.07172	0.00706	0.00111	0.08222	0.00883
B	7e-05	0.00029	0.0014	0.00448	0.04778	0.73486	0.06615	0.00521	0.10704	0.03272
Caa	0	9e-05	0.00022	0.00084	0.00344	0.06512	0.67874	0.02852	0.14348	0.07955
CaC	0	0	0.00049	0	0.00558	0.02289	0.08943	0.39387	0.22116	0.26658

Table 6: Average One-Year Letter Rating Migration Rates, 1970-2017 - Moody's

Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Ca/C	Default
Aaa	0.91051	0.08246	0.00602	0.00075	0.00024	3e-05	0	0	0
Aa	0.00861	0.89593	0.08958	0.00446	0.00065	0.00037	0.00018	1e-05	0.00021
A	0.00055	0.02584	0.91011	0.0563	0.00508	0.00111	0.00042	5e-05	0.00055
Baa	0.00035	0.00151	0.04345	0.90392	0.03994	0.00724	0.00162	0.00021	0.00176
Ba	7e-05	0.00045	0.0046	0.06664	0.83158	0.07815	0.00769	0.00121	0.00962
B	8e-05	0.00032	0.00157	0.00502	0.05351	0.82295	0.07408	0.00583	0.03664
Caa	0	0.00011	0.00026	0.00098	0.00402	0.07603	0.79244	0.0333	0.09288
Ca/C	0	0	0.00063	0	0.00716	0.02939	0.11482	0.50571	0.34228
Default	0	0	0	0	0	0	0	0	1

Table 7: Adjusted transition matrix.

Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Ca/C	Default
Aaa	-0.09419	0.09131	0.00209	0.00057	0.00022	0	0	0	0
Aa	0.00951	-0.11175	0.09927	0.00185	0.00043	0.00034	0.00017	0	0.00018
A	0.00046	0.02861	-0.09709	0.06199	0.00437	0.00083	0.00038	0	0.00046
Baa	0.00039	0.00099	0.04787	-0.10421	0.04586	0.00621	0.00146	0	0.00143
Ba	0	4e-04	0.00334	0.07673	-0.1893	0.09426	0.005	0.0013	0.00826
B	0	0.00034	0.00157	0.0033	0.0647	-0.20229	0.09145	0.00641	0.03453
Caa	0	1e-04	0.00017	0.00087	0.00167	0.09363	-0.24104	0.05203	0.09256
Ca/C	0	0	0.00075	0	0.00925	0.03572	0.17864	-0.68752	0.46315
Default	0	0	0	0	0	0	0	0	0

Table 8: Generator matrix.

7.3 *Single-rating* calibration results

IGEEFD Index (Rating class: Aa)

	IGEEFD Index			
	Par0	**Plow**	**Pupp**	**Parfit**
α	0.005	0.001	5	0.010021
γ	2	0	3.2	3.116302
σ	2	0.01	3	0.249897
$\widehat{\alpha}$	0.15	0.001	3	0.122024
$\widehat{\gamma}$	0.5	0	4	0.911254
λ_0	5	0	10	7.883151
ω	0.01	1e-04	0.1	0.001488

Table 9: Starting values (*Par0*), lower bounds (*Plow*) , upper bounds *Pupp* e calibrated values (*Parfit*) of model parameters - IGEEFD Index (Rating class: Aa).

	IGEEFD Index	IGEEFA Index	IGEEFB Index
Synthetic indicator	Value	Value	Value
R^2	0.84	0.45	-0.26
ω	14.88 (p.b.)	14.88 (p.b.)	14.88 (p.b.)

Table 10: Means, standard deviations and 99.5% quantiles of observed data (b.p.) - IGEEFD Index (Rating class: Aa).

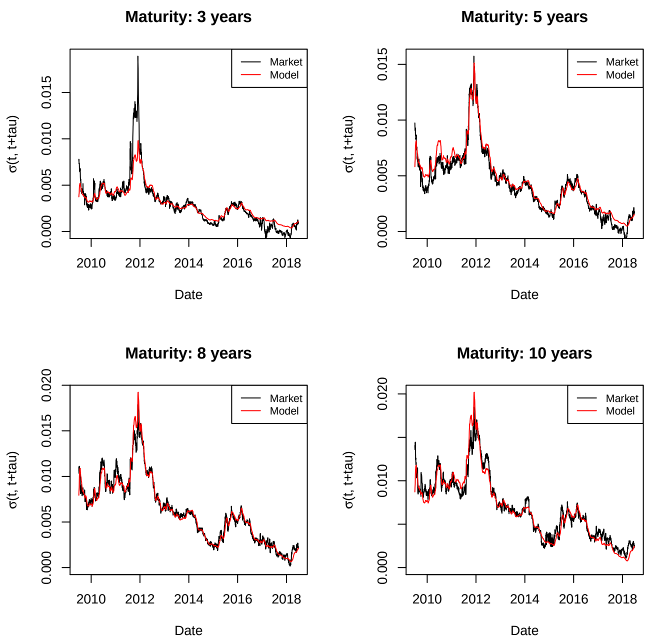


Figure 2: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - IGEEFD Index (Rating class: Aa).

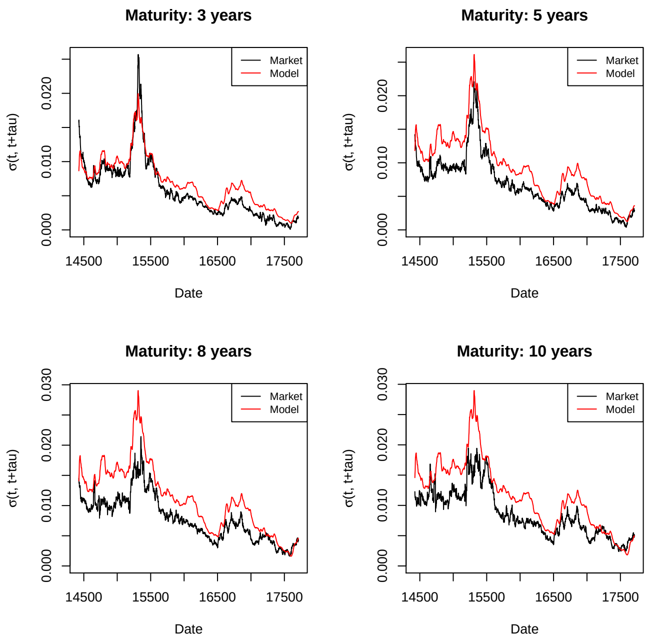


Figure 3: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFA` Index (Rating class: A).

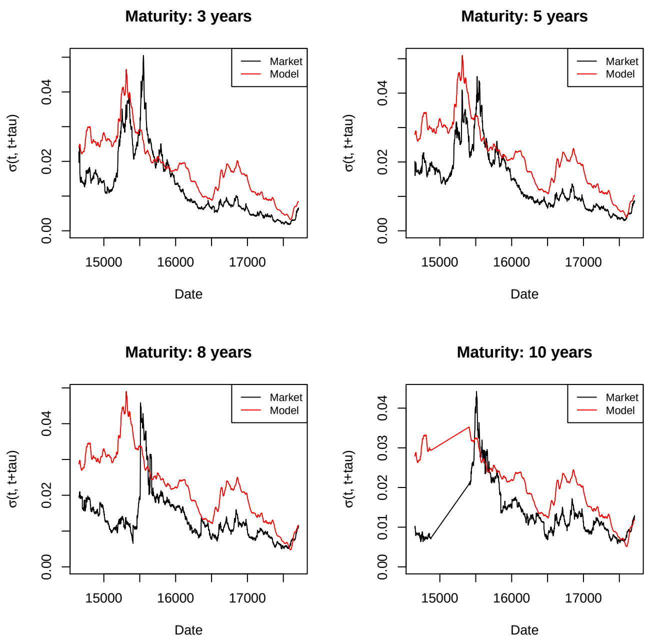


Figure 4: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFB` Index (Rating class: Baa).

`IGEEFA` Index (Rating class: A)

	`IGEEFA` Index			
	Par0	**Plow**	**Pupp**	**Parfit**
α	0.005	0.001	5	0.009194
γ	2	0	3.2	2.769328
σ	2	0.01	3	0.225632
$\widehat{\alpha}$	0.15	0.001	3	0.21007
$\widehat{\gamma}$	0.5	0	4	0.80557
λ_0	5	0	10	9.910955
ω	0.01	1e-04	0.1	0.00146

Table 11: Starting values (*Par0*), lower bounds (*Plow*), upper bounds *Pupp* e calibrated values (*Parfit*) of model parameters - `IGEEFA` Index (Rating class: A).

	`IGEEFD` Index	`IGEEFA` Index	`IGEEFB` Index
Synthetic indicator	Value	Value	Value
R^2	0.51	0.85	0.46
ω	14.60 (p.b.)	14.60 (p.b.)	14.60 (p.b.)

Table 12: Means, standard deviations and 99.5% quantiles of observed data (b.p.) - `IGEEFA` Index (Rating class: A).

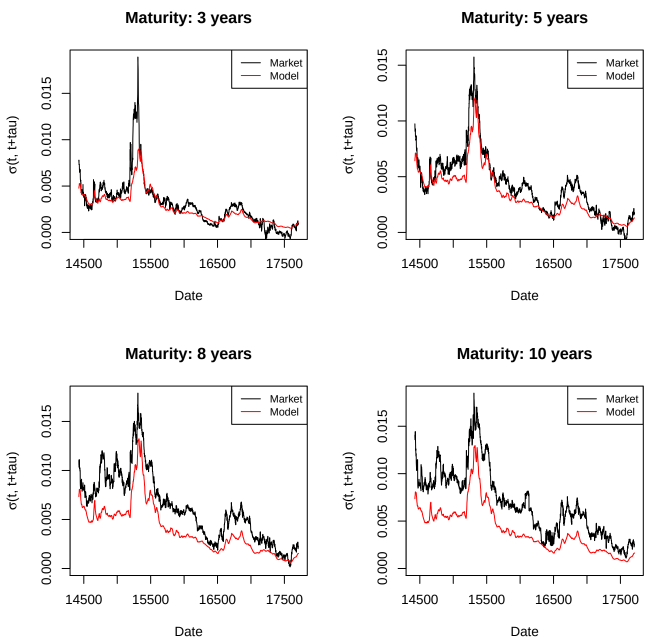


Figure 5: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFD` Index (Rating class: Aa).

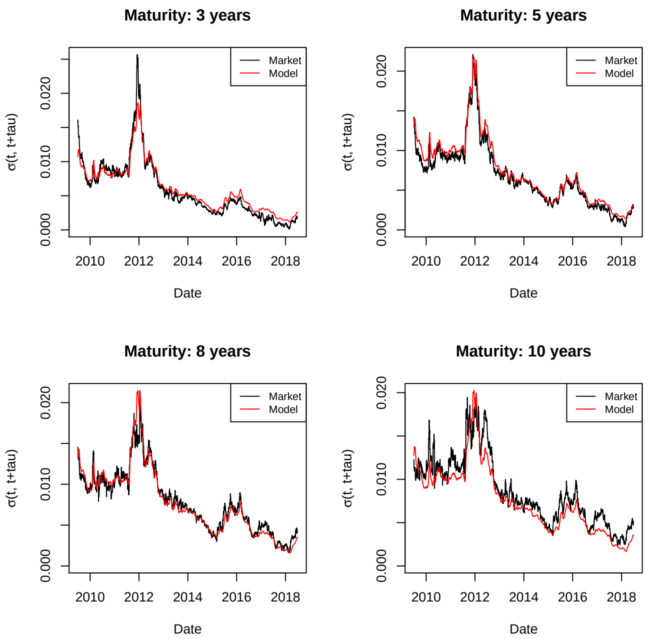


Figure 6: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFA` Index (Rating class: A).

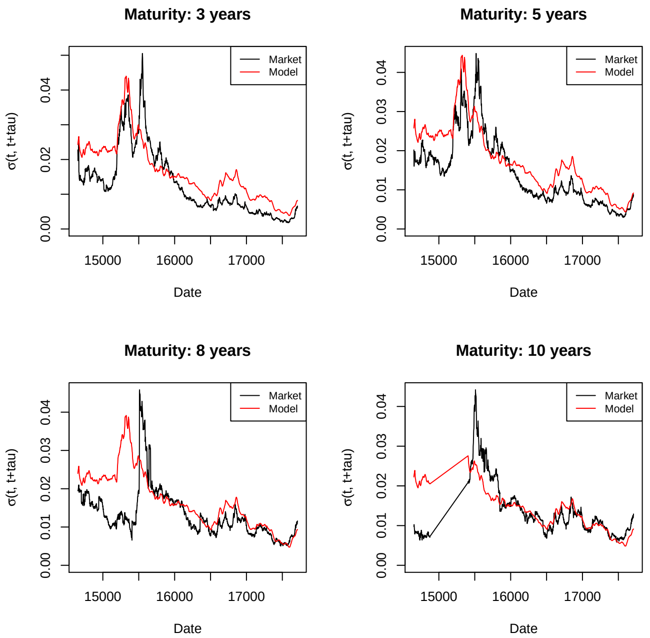


Figure 7: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFB` Index (Rating class: Baa).

`IGEEFB` **Index (Rating class: Baa)**

	`IGEEFB` **Index**			
	Par0	**Plow**	**Pupp**	**Parfit**
α	0.005	0.001	5	0.008384
γ	2	0	3.2	2.565675
σ	2	0.01	3	0.207389
$\widehat{\alpha}$	0.15	0.001	3	0.229926
$\widehat{\gamma}$	0.5	0	4	0.811948
λ_0	5	0	10	6.4365
ω	0.01	1e-04	0.1	0.002755

Table 13: Starting values (*Par0*), lower bounds (*Plow*), upper bounds *Pupp* e calibrated values (*Parfit*) of model parameters - `IGEEFB` Index (Rating class: Baa).

	`IGEEFD` Index	`IGEEFA` Index	`IGEEFB` Index
Synthetic indicator	Value	Value	Value
R^2	0.47	0.17	0.84
ω	27.54 (p.b.)	27.54 (p.b.)	27.54 (p.b.)

Table 14: Means, standard deviations and 99.5% quantiles of observed data (b.p.) - `IGEEFB` Index (Rating class: Baa).

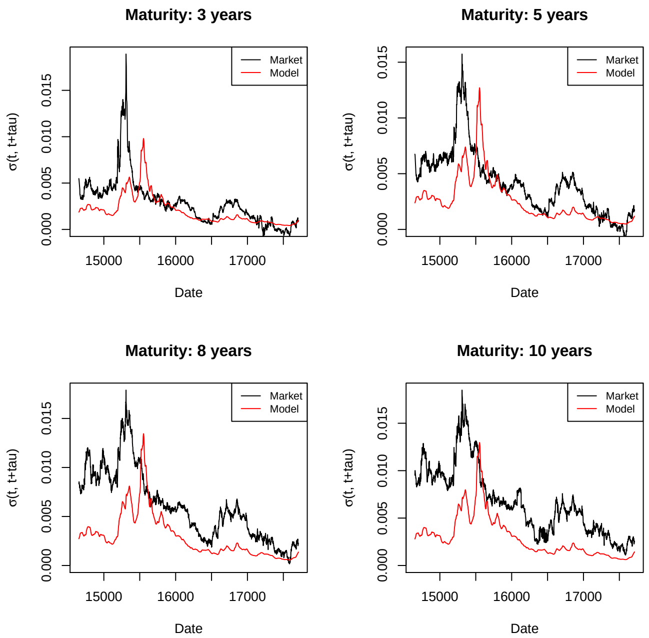


Figure 8: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFD` Index (Rating class: Aa).

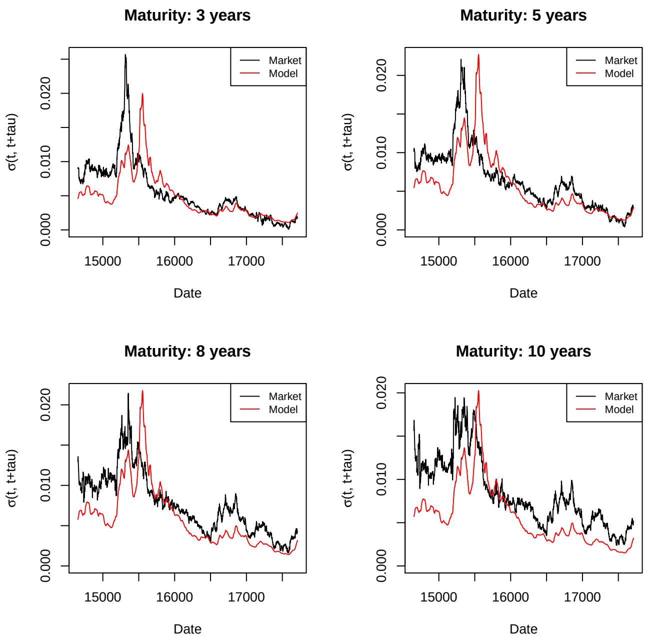


Figure 9: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFA` Index (Rating class: A).

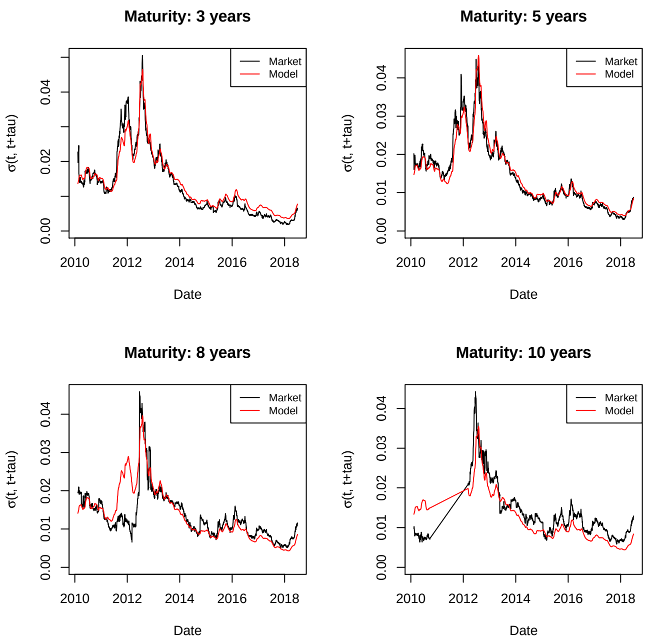


Figure 10: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFB` Index (Rating class: Baa).

7.4 *Multi-rating* calibration results

	Multi-rating			
	Par0	**Plow**	**Pupp**	**Parfit**
α	0.15	0.001	5	0.010700
γ	2	0	3.2	3.186956
σ	2.5	0.001	3	1.997080
$\widehat{\alpha}$	0.2	0.001	3	0.106476
$\widehat{\gamma}$	3.5	0	4	1.983650
λ_0	9.5	1e-04	0.1	9.184817
ω	0.01	0	10	0.002749

Table 15: Starting values (*Par0*), lower bounds (*Plow*), upper bounds *Pupp* e calibrated values (*Parfit*) of model parameters - *Multi-rating*.

Synthetic indicator	Value
R^2	0.85
R^2_{std}	0.75
R^2_{Aa}	0.68
R^2_{A}	0.68
R^2_{Baa}	0.84
$RMSE$	34.17 (p.b.)
ω	27.49 (p.b.)

Table 16: Means, standard deviations and 99.5% quantiles of observed data (b.p.) - *Multi-rating*.

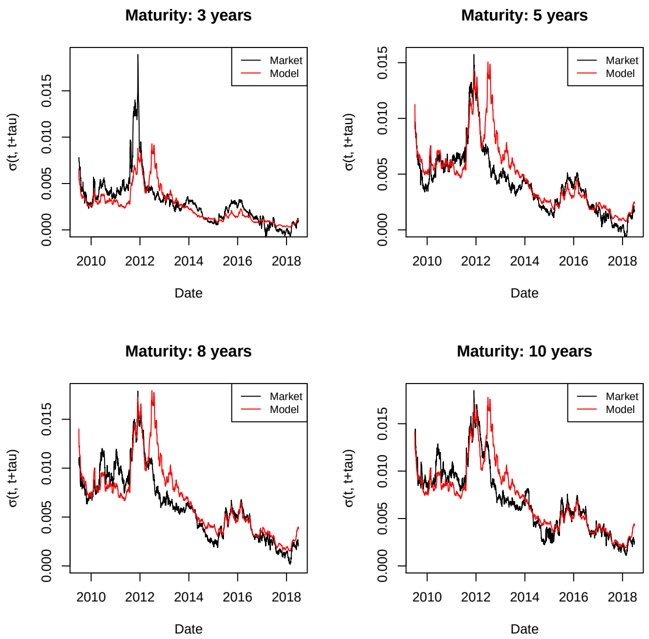


Figure 11: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFD` Index (Rating class: Aa).

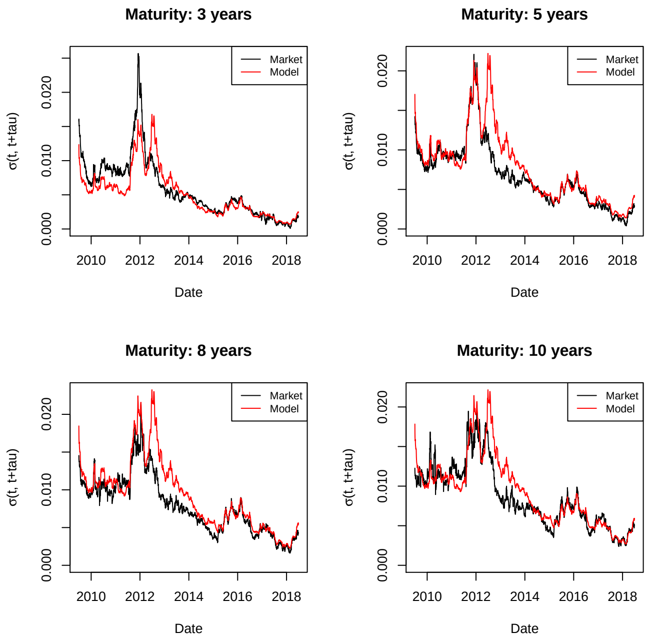


Figure 12: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFA` Index (Rating class: A).

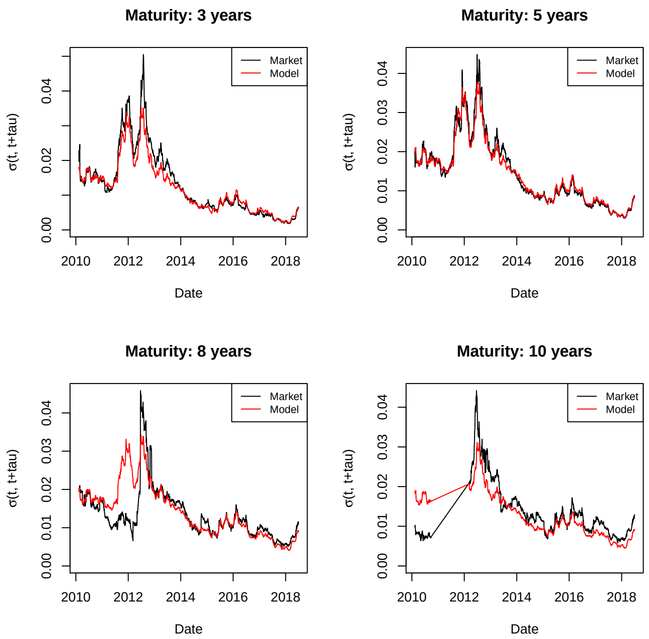


Figure 13: Comparison of model time series and market time series for maturities 3, 5, 8, 10 years - `IGEEFB` Index (Rating class: Baa).

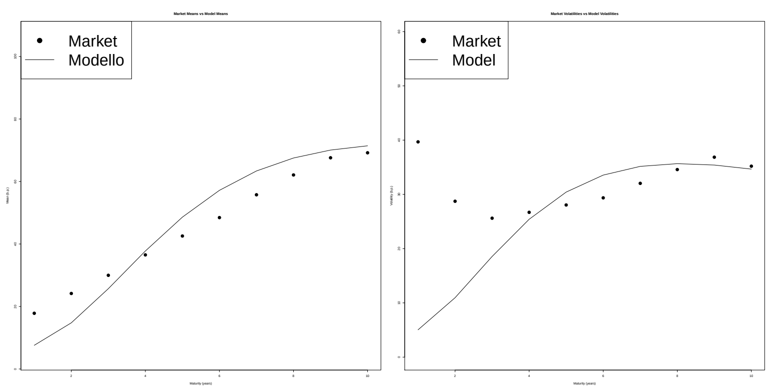


Figure 14: Comparison of model means and standard deviations and market means and standard deviations - `IGEEFD` Index (Rating class: Aa).

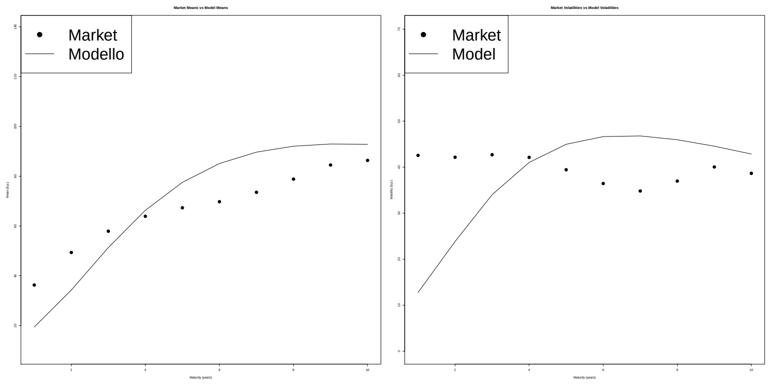


Figure 15: Comparison of model means and standard deviations and market means and standard deviations - `IGEEFA` Index (Rating class: A).

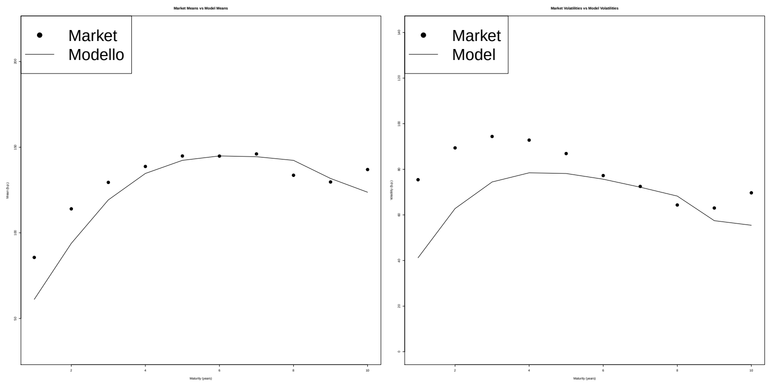


Figure 16: Comparison of model means and standard deviations and market means and standard deviations - `IGEEFB` Index (Rating class: Baa).

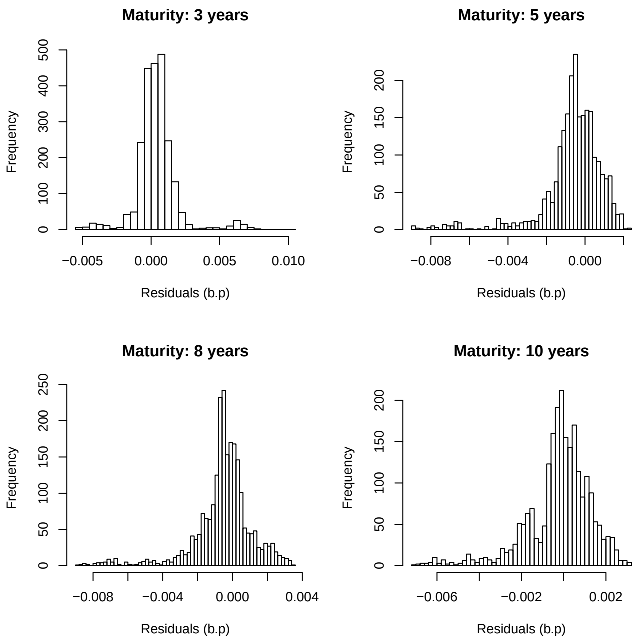


Figure 17: Residuals distribution for maturities 3, 5, 8, 10 years - `IGEEFD` Index (Rating class: Aa).

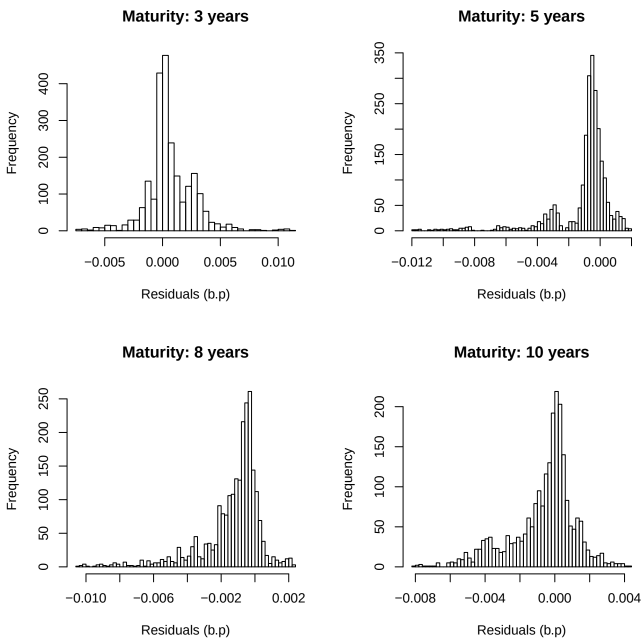


Figure 18: Residuals distribution for maturities 3, 5, 8, 10 years - `IGEEFA` Index (Rating class: A).

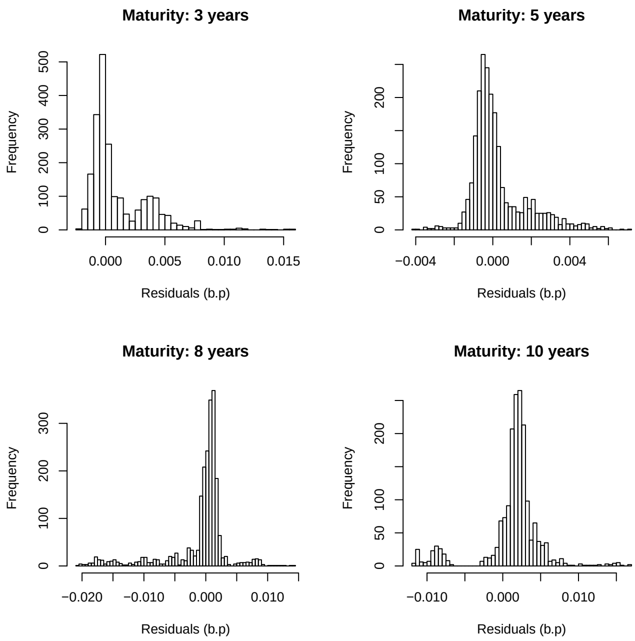


Figure 19: Residuals distribution for maturities 3, 5, 8, 10 years - `IGEEFB` Index (Rating class: Baa).

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