SHOOTING RANDOMLY AGAINST A LINE IN EUCLIDEAN AND NON-EUCLIDEAN SPACES

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ABSTRACT. In this paper we study a class of distributions related to the r.v. $C_n(t) = t \tan^{\frac{1}{n}} \Theta$, for different distributions of Θ. The problem is related to the hitting point of a randomly oriented ray and generalize the Cauchy distribution in different directions. We show that the distribution of $C_n(t)$ solves the Laplace equation of order $2n$, possesses even moments of order $2k < 2n-1$, and has bimodal structure when Θ is uniform. We study also a number of distributional properties of functionals of $C_n(t)$, including those related to the arcsine law. Finally we study the same problem in the Poincaré half-plane and this leads to the hyperbolic distribution $\Pr\{\eta \in dw\} = \frac{dw}{\pi \cosh w}$ of which the main properties are explored. In particular we study the distribution of hyperbolic functions of η, the law of sums of i.i.d. r.v.'s η_j and the distribution of the area of random hyperbolic right triangles.

1. INTRODUCTION

In this paper we consider the random variables of the form

$$C_n(t) = \begin{cases} t \tan^{\frac{1}{n}} \Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \\ -t \tan^{\frac{1}{n}} |\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right), \end{cases} \tag{1.1}$$

under different assumptions on the distribution of Θ.

First of all we consider the case where the random angle Θ has distribution

$$q_n(\theta) = \frac{\sin \frac{\pi}{2n}}{\pi} \cot^{\frac{n-1}{n}} |\theta|, \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), n \in \mathbb{N}, \tag{1.2}$$

and we show that in this case $C_n(t)$ has probability density

$$p_n(x,t) = \left(\frac{n \sin \frac{\pi}{2n}}{\pi}\right) \frac{t^{2n-1}}{t^{2n} + x^{2n}}, \qquad x \in \mathbb{R}, t > 0. \tag{1.3}$$

We regard (1.3) as a generalization of the classical symmetric Cauchy law under many viewpoints. First of all because, for $n = 1$, the angle has uniform distribution and the law of $C_1(t)$ becomes

$$p_1(x,t) = \frac{1}{\pi} \frac{t}{t^2 + x^2}, \qquad x \in \mathbb{R}, t > 0. \tag{1.4}$$

Furthermore in this case $C_1(t) = t \tan \Theta$ represents the segment intersected by a ray shooted from the point O against the parallel t units away.

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In the case $n > 1$ we mantain the same interpretation but here the angle has a law which becomes increasingly concentrated around $\theta = 0$ as n increases.

For $x^2 < t^2$ the cumulative distribution of (1.3) has the form

$$\Pr\{C_n(t) < x\} = \frac{1}{2} + \frac{n \sin\frac{\pi}{2n}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2nk+1} \left(\frac{x}{t}\right)^{2nk+1} \tag{1.5}$$

where

$$\mathcal{O}_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2\alpha k+1} z^{2\alpha k+1}, \qquad z^2 < 1, \alpha > 0, \tag{1.6}$$

represents a generalization of the $\arctan z$ function and reduces to it for $\alpha = 1$.

The density (1.3) is a solution to the $2n$-th order Laplace equation

$$\left(\frac{\partial^{2n}}{\partial t^{2n}} + \frac{\partial^{2n}}{\partial x^{2n}}\right) p_n(x,t) = 0. \tag{1.7}$$

However (1.3) differs from the classical Cauchy because even moments of order $2k < 2n-1$ exist and is non longer infinitely divisible as the characteristic function shows.

Some other properties of the Cauchy are lost but considering some other related distributions we are able to give a picture of generalized higher-order Cauchy distributions with interesting interlaced distributional properties. In the case Θ is uniform in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the probability density of

$$\hat{C}_n(t) = \begin{cases} t \tan^{\frac{1}{n}} \Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \\ -t \tan^{\frac{1}{n}} |\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right), \end{cases} \tag{1.8}$$

reads

$$\hat{p}_n(x,t) = \frac{n}{\pi} \left(\frac{|x|}{t}\right)^{n-1} \frac{t^{2n-1}}{t^{2n} + x^{2n}}, \qquad x \in \mathbb{R}, t > 0. \tag{1.9}$$

The distribution has a bimodal structure and thus substantially differs from (1.3). The maxima of (1.9) are located at $x = \pm t \left(\frac{n-1}{n+1}\right)^{\frac{1}{2n}}$. For the r.v. $\hat{C}_n(t)$ the following remarkable property holds

$$\frac{1}{\hat{C}_n\left(\frac{1}{t}\right)} \overset{\text{i.d.}}{=} \hat{C}_n(t). \tag{1.10}$$

In our view it is relevant that the probability law (1.9) shares with the classical Cauchy also the property that

$$\hat{Z}_n(t) = \frac{t}{1 + \left(\frac{\hat{C}_n(t)}{t}\right)^{2n}} \tag{1.11}$$

possesses arcsine distribution, that is

$$\Pr\left\{\hat{Z}_n(t) \in dw\right\} = \frac{dw}{\pi\sqrt{w(t-w)}}, \qquad 0 < w < t. \tag{1.12}$$

Curiously enough the ratio of independent r.v.'s $\hat{W}_n(t) = \frac{\hat{C}_n^1(t)}{\hat{C}_n^2(t)}$ has a distribution which generalizes that of the ratio of independent Cauchy r.v.'s; that is

$$\Pr\left\{\frac{\hat{C}_n^1(t)}{\hat{C}_n^2(t)} \in dw\right\} = \frac{dw}{\pi^2}\frac{nt^n|w|^{n-1}}{(t^{2n}-w^{2n})}\log\left(\frac{t}{w}\right)^{2n}, \qquad w \in \mathbb{R}, t>0. \tag{1.13}$$

However the distribution (1.9) does not satisfy an higher-order Laplace equation as (1.3) does.

The third r.v. considered below is

$$\tilde{C}_n(t) = t\tan\Theta \tag{1.14}$$

where Θ has distribution $q_n(\theta)$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The distribution of (1.14) is unimodal and has analytical form

$$\tilde{p}_n(x,t) = \frac{1}{\pi}\sin\frac{\pi}{2n}\frac{t}{t^2+x^2}\left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}, \qquad x \in \mathbb{R}, t>0. \tag{1.15}$$

We note that for (1.15) the r.v.

$$\tilde{Z}_n(t) = \frac{t}{1+\left(\frac{\tilde{C}_n(t)}{t}\right)^2} \tag{1.16}$$

has Beta distribution with parameters $\left(\frac{1}{2n}, 1-\frac{1}{2n}\right)$.

In the last section of the paper we consider the problem of shooting against a geodesic line in the Poincaré half-plane $\mathbb{H}_2^+$. We shoot from a point O of the x-axis, representing the infinite in $\mathbb{H}_2^+$ against a half-circumference of radius t and center O (see figure 6a below). The hyperbolic distance η between the points P and Q, is given by

$$\eta = \begin{cases} -\log\tan\frac{\theta}{2}, & \theta \in \left(0, \frac{\pi}{2}\right), \\ \log\tan\frac{\theta}{2}, & \theta \in \left(\frac{\pi}{2}, \pi\right), \end{cases} \tag{1.17}$$

because the metric in $\mathbb{H}_2^+$ is

$$ds^2 = \frac{dx^2+dy^2}{y^2}. \tag{1.18}$$

Considering Θ uniformly distributed, the random variable (1.17) has probability density

$$\Pr\{\eta \in dw\} = \frac{4}{\pi}\frac{e^{-w}dw}{1+e^{-2w}} = \frac{2}{\pi}\frac{1}{\cosh w}dw, \qquad w>0. \tag{1.19}$$

The symmetric r.v.

$$\hat{\eta} = -\log\tan\frac{\Theta}{2} \tag{1.20}$$

has density

$$\Pr\{\hat{\eta} \in dw\} = \frac{1}{\pi}\frac{1}{\cosh w}dw, \qquad w \in \mathbb{R}, \tag{1.21}$$

and characteristic function

$$\mathbb{E}e^{i\beta\hat{\eta}} = \frac{1}{\cosh\frac{\beta\pi}{2}}, \qquad \beta \in \mathbb{R}. \tag{1.22}$$

The hyperbolic r.v. $\hat{\eta}$ has the unusual property that its density and characteristic function have the same analytic form. The even-order moments

$$\mathbb{E}\hat{\eta}^{2n} = \left(\frac{\pi}{2}\right)^{2n} |E_{2n}| \tag{1.23}$$

show an interesting relationship with the Euler numbers E_{2n}. We produce a direct derivation of the distribution

$$\Pr\{\hat{\eta}_1 + \hat{\eta}_2 \in dw\} = \frac{2}{\pi} \frac{w}{\sinh w}, \qquad w \in \mathbb{R}, \tag{1.24}$$

by means of the Cauchy residue theorem and we give also the explicit distribution of sums $\hat{\boldsymbol{\eta}}_n = \sum_{j=1}^n \hat{\eta}_j$ for any $n \in \mathbb{N}$. We obtain the distribution of all hyperbolic functions of $\hat{\eta}$ and of other related functionals. For example, we prove that the law of $\sinh \hat{\eta}$ coincides with the standard Cauchy and that

$$Y = \frac{1}{\cosh^2 \hat{\eta}} = \frac{1}{1 + \sinh^2 \hat{\eta}} \tag{1.25}$$

has arcsine distribution. In the last section of the paper we also derive the distribution of the area K of the hyperbolic right triangle (see fig. 6a one side of which has length η defined in (1.17). We show that the distribution of K is

$$\Pr\{K \in dw\} = \frac{2}{\pi} \frac{dw}{1 + \sin w}, \qquad w \in \left(0, \frac{\pi}{2}\right), \tag{1.26}$$

with mean

$$\mathbb{E}K = \frac{2}{\pi} \log 2. \tag{1.27}$$

2. The higher order Cauchy random variables

We consider the angular distribution

$$q_n(\theta) = \frac{\sin \frac{\pi}{2n}}{\pi} \cot^{\frac{n-1}{n}} |\theta| \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{2.1}$$

which for $n = 1$ coincides with the uniform law in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The distribution (2.1) is concentrated around $\theta = 0$ (see figure 1) and its spread around the mean decreases as n increases. One expects that the shots must be concentrated around the target and (2.1) satisfies this requirement. In order to check that (2.1) integrates to unity we perform the following calculation

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q_n(\theta) d\theta &= \frac{2 \sin \frac{\pi}{2n}}{\pi} \int_0^{\frac{\pi}{2}} \cot^{\frac{n-1}{n}} |\theta| \, d\theta \\
&\overset{\sin\theta = \sqrt{y}}{=} \frac{1}{\pi} \sin \frac{\pi}{2n} \int_0^1 y^{\frac{1}{2n} - 1} (1 - y)^{-\frac{1}{2n}} \, dy \\
&= \frac{1}{\pi} \sin\left(\frac{\pi}{2n}\right) \Gamma\left(\frac{1}{2n}\right) \Gamma\left(1 - \frac{1}{2n}\right) = 1,
\end{aligned} \tag{2.2}$$

because $\Gamma\left(\frac{1}{2n}\right)\Gamma\left(1 - \frac{1}{2n}\right) = \frac{\pi}{\sin \frac{\pi}{2n}}$ for the reflection formula of the Gamma integral.

We note that the related random variable $\cos \Theta$ with Θ distributed as (2.1) has even-order moments equal to

$$\mathbb{E} \cos^m \Theta = 2 \int_0^{\frac{\pi}{2}} \cos^m \theta q_n(\theta) \, d\theta$$

$$= \frac{1}{\pi} \sin \frac{\pi}{2n} \frac{\Gamma\left(\frac{1}{2n}\right) \Gamma\left(\frac{m}{2}+1-\frac{1}{2n}\right)}{\Gamma\left(\frac{m}{2}+1\right)}. \tag{2.3}$$

The special case $m = 2$ yields to $\mathbb{E}\cos^2\Theta = 1 - \frac{1}{2n}$.

We now pass to the derivation of the distribution of

$$C_n(t) = \begin{cases} t\tan^{\frac{1}{n}}\Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \\ -t\tan^{\frac{1}{n}}|\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right). \end{cases} \tag{2.4}$$

Theorem 2.1. *The explicit law of $C_n(t)$, where Θ possesses distribution (2.1), reads*

$$\Pr\{C_n(t) \in dx\} = \frac{n\sin\frac{\pi}{2n}}{\pi} \frac{t^{2n-1}}{t^{2n}+x^{2n}}\, dx \qquad x \in \mathbb{R}, t > 0, \tag{2.5}$$

and for $x^2 < t^2$

$$\Pr\{C_n(t) < x\} = \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2nk+1} \left(\frac{x}{t}\right)^{2nk+1}. \tag{2.6}$$

Proof. For $x > 0$, we have that

$$\begin{aligned} \Pr\{C_n(t) < x\} &= \Pr\left\{t\tan^{\frac{1}{n}}\Theta < x\right\} \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi} \int_0^{\arctan\left(\frac{x}{t}\right)^n} \cot^{\frac{n-1}{n}}\theta\, d\theta. \end{aligned} \tag{2.7}$$

On deriving (2.7) with respect to x we readily have the density (2.5). In the same spirit of the previous calculation we obtain the result for $x < 0$. By means of the substitution $\tan\theta = y$ we reduce (2.7) to the form

$$\begin{aligned} \Pr\{C_n(t) < x\} &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi} \int_0^{\left(\frac{x}{t}\right)^n} \frac{1}{y^{\frac{n-1}{n}}} \frac{dy}{1+y^2} \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi} \sum_{k=0}^{\infty} (-1)^k \int_0^{\left(\frac{x}{t}\right)^n} y^{2k-1+\frac{1}{n}}\, dy \\ &= \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{t}\right)^{2nk+1}}{2nk+1}, \qquad x^2 < t^2, \end{aligned} \tag{2.8}$$

which coincides with (2.6). The intermediate step shows why the cumulative function can be written as in (2.6) for $x^2 < t^2$. ∎

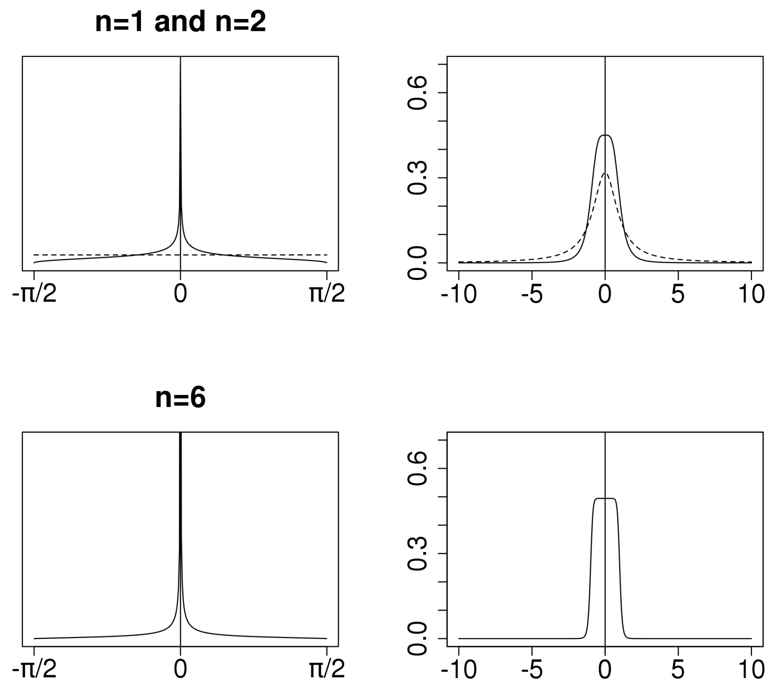
Remark 2.2. The density (2.5) has the alternative form

$$p_n(x,t) = \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi} \int_0^{\infty} e^{-zt} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz, \tag{2.9}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad z \in \mathbb{R}, \alpha > 0, \beta > 0, \tag{2.10}$$

FIGURE 1. The probabilty function (2.1) of the r.v. Θ (left column) and the related distribution of $C_n(t)$ (right column). The dotted lines represent the uniform law and the Cauchy density.



is the Mittag-Leffler function, see for example Haubold, Mathai and Saxena [7]. The representation (2.9) permits us to show that it satisfies the Laplace equation of order $2n$. Since

$$\frac{\partial^{2n}}{\partial x^{2n}} E_{2n,1}\left(-z^{2n}x^{2n}\right) = -z^{2n} E_{2n,1}\left(-z^{2n}x^{2n}\right), \tag{2.11}$$

we have that

$$\begin{aligned} &\frac{\partial^{2n}}{\partial x^{2n}} \int_0^\infty e^{-zt} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz \\ &= -\int_0^\infty e^{-zt} z^{2n} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz \\ &= -\frac{\partial^{2n}}{\partial t^{2n}} \int_0^\infty e^{-zt} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz. \end{aligned} \tag{2.12}$$

The probability density (2.5) is an unimodal function which for $n \to \infty$ converges to the uniform law in $(-t, t)$. For increasing values of n it takes the form of a rectangular wave as figure 1 shows.

Remark 2.3. The distribution function of $C_n(t)$, $t > 0$, can be represented in terms of hypergeometric functions for all $w \in \mathbb{R}$ without the restriction $\left(\frac{w}{t}\right)^2 < 1$.

For $w > 0$ we have that

$$\begin{aligned}
\Pr\{C_n(t) < w\} &= \frac{1}{2} + \frac{n \sin \frac{\pi}{2n}}{\pi} \int_0^w \frac{t^{2n-1}}{t^{2n} + x^{2n}} \\
&\overset{x=ty}{=} \frac{1}{2} + \frac{n \sin \frac{\pi}{2n}}{\pi} \int_0^{\frac{w}{t}} dy \int_0^\infty du \, e^{-u\left(1+y^{2n}\right)} \\
&\overset{y=x^{\frac{1}{2n}}}{=} \frac{1}{2} + \frac{n \sin \frac{\pi}{2n}}{\pi} \int_0^\infty du \, e^{-u} \int_0^{\left(\frac{w}{t}\right)^{2n}} dx \, e^{-ux} x^{\frac{1}{2n}-1} \\
&= \frac{1}{2} + \frac{\sin \frac{\pi}{2n}}{\pi} \int_0^\infty e^{-u} \, u^{-\frac{1}{2n}} \, \gamma\left(\frac{1}{2n}, \, u\left(\frac{w}{t}\right)^{2n}\right) du \\
&= \frac{1}{2} + \frac{\sin \frac{\pi}{2n}}{\pi} \frac{\frac{w}{t}\,\Gamma(1)}{\frac{1}{2n}\left(\left(\frac{w}{t}\right)^{2n}+1\right)} F\left(1, 1; \frac{1}{2n}+1; \frac{\left(\frac{w}{t}\right)^{2n}}{\left(\frac{w}{t}\right)^{2n}+1}\right) \\
&= \frac{1}{2} + \frac{n \sin \frac{\pi}{2n}}{\pi} \frac{w\, t^{2n-1}}{w^{2n}+t^{2n}} F\left(1, 1; \frac{1}{2n}+1; \frac{w^{2n}}{w^{2n}+t^{2n}}\right). \qquad (2.13)
\end{aligned}$$

In the above steps we denoted by

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} \, dt \qquad (2.14)$$

the incomplete Gamma function. By

$$\begin{aligned}
F(a, b; c; z) &= \sum_{k=0}^\infty \frac{(a)_k \, (b)_k}{(c)_k} \frac{z^k}{k!} \\
&= \sum_{k=0}^\infty \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)} \frac{1}{B(a,b)} \frac{z^k}{k!}, \qquad (2.15)
\end{aligned}$$

we denote the hypergeometric function. In the last step we used formula 6.455, page 657, of Gradshteyn and Ryzhik [5], that is

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \, \gamma(\nu, \alpha x) \, dx = \frac{\alpha^\nu \Gamma(\mu+\nu)}{\nu(\alpha+\beta)^{\mu+\nu}} F\left(1, \mu+\nu; \nu+1; \frac{\alpha}{\alpha+\beta}\right), \qquad (2.16)$$

valid for $\Re(\alpha+\beta) > 0$, $\Re(\beta) > 0$, $\Re(\mu+\nu) > 0$. With little changes we can see that (2.13) holds also for $w < 0$. By means of formula (see [5], 9.131, page 1008),

$$F(a, b; c; z) = (1-z)^{c-a-b} \, F(c-a, c-b; c; z) \qquad (2.17)$$

the cumulative function (2.13) can also be written as

$$\Pr\{C_n(t) < w\} = \frac{1}{2} + \frac{n \sin \frac{\pi}{2n}}{\pi} \frac{w}{\left(w^{2n}+t^{2n}\right)^{\frac{1}{2n}}} F\left(\frac{1}{2n}, \frac{1}{2n}; \frac{1}{2n}+1; \frac{w^{2n}}{t^{2n}+z^{2n}}\right). \qquad (2.18)$$

We note that, for $n = 1$, the function (2.18) coincides with the expansion of the arctangent function,

$$\begin{aligned}
\Pr\{C_1(t) < w\} &= \frac{1}{2} + \frac{1}{\pi} \frac{w}{\sqrt{w^2+t^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{w^2}{w^2+t^2}\right) \\
&\overset{\text{see [5], 1.641, pag. 60}}{=} \frac{1}{2} + \frac{1}{\pi} \arctan \frac{w}{t}. \qquad (2.19)
\end{aligned}$$

By applying the following formula

$$F(a,b;c;z) = (1-z)^{-b} F\left(b, c-a; c; \frac{z}{z-1}\right), \qquad \left|\frac{z}{z-1}\right| < 1, \tag{2.20}$$

we can rewrite the distribution function (2.13), for $\frac{w^2}{t^2} < 1$, as

$$\begin{aligned}
\Pr\{C_n(t) < w\} &= \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi}\frac{w}{t} F\left(1, \frac{1}{2n}, \frac{1}{2n}+1, -\frac{w^{2n}}{t^{2n}}\right) \\
&= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)} + \frac{w}{t}\sum_{k=0}^{\infty}(-1)^k \frac{(1)_k\left(\frac{1}{2n}\right)_k}{\left(\frac{1}{2n}+1\right)_k}\frac{1}{k!}\frac{w^{2nk}}{t^{2nk}}\right) \\
&= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)} + \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{2n}\right)_k}{\left(\frac{2n+1}{2n}\right)_k}\frac{w^{2nk+1}}{t^{2nk+1}}\right) \\
&= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\frac{w^{2nk+1}}{t^{2nk+1}}\right).
\end{aligned} \tag{2.21}$$

In (2.21) we retrive the result (2.8) which was obtained without resorting to the hypergeometric functions.

Other useful representations of the cumulative function of $C_n(t)$ can be given in integral form as

$$\begin{aligned}
\Pr\{C_n(t) < w\} &= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\frac{w^{2nk+1}}{t^{2nk+1}}\right) \\
&= \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi}\frac{w}{t}\sum_{k=0}^{\infty}(-1)^k\left(\frac{w}{t}\right)^{2nk}\int_0^\infty du\, e^{-u(2nk+1)} \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\int_0^\infty du\, e^{-u}\frac{w}{t}\sum_{k=0}^{\infty}\left(-\frac{e^{-2nu}w^{2n}}{t^{2n}}\right)^k \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\int_0^\infty du\, e^{-u}\frac{t^{2n-1}}{t^{2n}+w^{2n}e^{-2nu}}.
\end{aligned} \tag{2.22}$$

Formula (2.22) can be also rewritten as

$$\begin{aligned}
&\Pr\{C_n(t) < w\} = \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\int_0^\infty du\, e^{-u}\int_0^\infty dz\, e^{-zt}E_{2n,1}\left(-\left(we^{-u}\right)^{2n}z^{2n}\right) \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\sum_{k=0}^{\infty}\frac{(-1)^k w^{2nk}}{\Gamma(2nk+1)}\int_0^\infty dz\, e^{-zt}z^{2nk}\int_0^\infty du\, e^{-u(1+2nk)} \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\sum_{k=0}^{\infty}\frac{(-1)^k w^{2nk}}{\Gamma(2nk+1)(1+2nk)}\int_0^\infty dz\, e^{-zt}z^{1+2nk-1} \\
&= \frac{1}{2} + \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{w}{t}\right)^{2nk+1}}{2nk+1},
\end{aligned} \tag{2.23}$$

which coincides with (2.7).

Remark 2.4. In force of formula 3.738 pag. 430 of Gradshteyn and Ryzhik [5], we can give a representation of the characteristic function of (2.5) as follows

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\beta x} p_n(x,t)dx &= \frac{2n\sin\frac{\pi}{2n}}{\pi} t^{2n-1}\int_0^{\infty} \frac{\cos\beta x}{x^{2n}+t^{2n}}dx \\ &= \sin\frac{\pi}{2n}\sum_{k=1}^{n} e^{-|\beta| t\sin\frac{(2k-1)\pi}{2n}} \sin\left(\frac{(2k-1)\pi}{2n} + |\beta|\, t\cos\frac{(2k-1)\pi}{2n}\right), \end{aligned} \tag{2.24}$$

which coincides, for $n=1$, with the characteristic function of the Cauchy distribution.

Remark 2.5. Other generalizations of the Cauchy are obtained by considering two different types of r.v.'s. The first one is

$$\hat{C}_n(t) = \begin{cases} t\tan^{\frac{1}{n}}\Theta, & \Theta\in\left(0,\frac{\pi}{2}\right) \\ -t\tan^{\frac{1}{n}}|\Theta| & \Theta\in\left(-\frac{\pi}{2},0\right), \end{cases} \tag{2.25}$$

where Θ has uniform law. The distribution function of (2.25) is

$$\Pr\left\{\hat{C}_n(t)<x\right\} = \begin{cases} \frac{1}{2}+\frac{1}{\pi}\int_0^{\arctan\left(\frac{x}{t}\right)^n} d\theta, & x>0 \\ \frac{1}{\pi}\int_{\arctan\left(-\frac{x}{t}\right)^n}^{\frac{\pi}{2}} d\theta, & x<0, \end{cases} \tag{2.26}$$

and thus the density reads

$$\hat{p}_n(x,t) = \frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+x^{2n}}, \qquad x\in\mathbb{R}, t>0, n\in\mathbb{N}, \tag{2.27}$$

and possesses the following representation

$$\hat{p}_n\left(x,t\right) = \frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\int_0^{\infty} e^{-zt}E_{2n,1}\left(-x^{2n}z^{2n}\right)\,dz. \tag{2.28}$$

In Figure 2a we give a picture of density (2.27) for different values of n. It is interesting to note that the distribution is bimodal with two symmetric maxima at

$$x = \pm t\left(\frac{n-1}{n+1}\right)^{\frac{1}{2n}}, \qquad n>1. \tag{2.29}$$

Furthermore, the characteristic function of the distribution (2.27), in force of formula 3.738 of Gradshteyn and Ryzhik [5] pag 430, reads

$$\int_{-\infty}^{\infty} e^{i\beta x}\hat{p}_n\left(x,t\right)\,dx = \sum_{k=1}^{n} e^{-|\beta| t\sin\frac{(2k-1)\pi}{2n}} \sin\left(\frac{(2k-1)\pi}{2} + |\beta|\, t\cos\frac{(2k-1)\pi}{2n}\right). \tag{2.30}$$

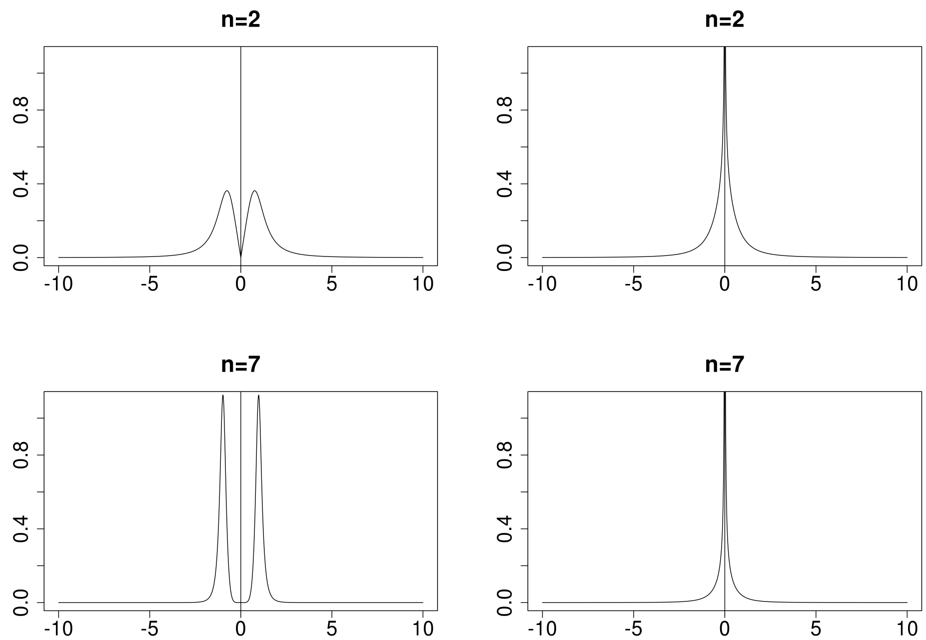
For the r.v.

$$\tilde{C}_n(t) = t\tan\Theta, \tag{2.31}$$

with Θ endowed with the distribution $q_n\left(\theta\right)$ given in (2.1), we have that

$$\begin{aligned}\tilde{p}_n\left(x,t\right) &= \frac{d}{dx}\Pr\left\{t\tan\Theta<x\right\} \\ &= \frac{d}{dx}\left[\frac{1}{2}+\frac{\sin\frac{\pi}{2n}}{\pi}\int_{-\frac{\pi}{2}}^{\arctan\frac{x}{t}} \cot^{\frac{n-1}{n}}|\theta|\,d\theta\right]\end{aligned}$$

$$= \frac{\sin\frac{\pi}{2n}}{\pi} \frac{t}{x^2+t^2} \left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}. \tag{2.32}$$



(A) (B)

FIGURE 2. The probability density function of $\hat{C}_n(t)$, (A), and $\tilde{C}_n(t)$, (B), for different values of n.

Remark 2.6. Since the following identity holds

$$\frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1} \frac{t^{2n-1}}{t^{2n}+x^{2n}} = n|x|^{n-1} \int_0^\infty \frac{e^{-\frac{x^{2n}}{2s}}}{\sqrt{2\pi s}} t^n \frac{e^{-\frac{t^{2n}}{2s}}}{\sqrt{2\pi s^3}}\, ds, \tag{2.33}$$

for the hyperCauchy (2.27) a subordination similar to that of the classical Cauchy law can be established and reads

$$\Pr\left\{\hat{C}_n(t) \in dx\right\} = \int_0^\infty \Pr\left\{\mathcal{B}^{\frac{1}{n}}(s) \in dx\right\} \Pr\left\{T_{t^n} \in ds\right\} ds, \tag{2.34}$$

where

$$\mathcal{B}(s) = \begin{cases} B(s), & B(s) > 0, \\ -B(s), & B(s) < 0. \end{cases} \tag{2.35}$$

With $B(s)$ we denote a standard Brownian motion and T_{t^n} is defined as

$$T_{t^n} = \inf\left\{s > 0 : B(s) = t^n\right\} \tag{2.36}$$

Now we pass to the derivation of the moments of (2.1).

Theorem 2.7. *For $2n > 2k+1$, $k > 0$, we have that*

$$\mathbb{E}C_n^{2k}(t) = \frac{\sin\frac{\pi}{2n}}{\sin\left(\frac{2k+1}{2n}\pi\right)}\, t^{2k}$$

$$= \frac{t^{2k}}{\cos\frac{k\pi}{n} + \cot\frac{\pi}{2n}\sin\frac{k\pi}{n}}. \tag{2.37}$$

Proof.

$$\begin{aligned}
\mathbb{E}C_n^{2k}(t) &= \frac{n}{\pi}\sin\frac{\pi}{2n} t^{2n-1}\int_{-\infty}^{\infty}\frac{x^{2k}}{x^{2n}+t^{2n}}\,dx \\
&= 2\frac{n}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}x^{2k}\int_0^{\infty}e^{-w\left(x^{2n}+t^{2n}\right)}\,dw\,dx \\
&\overset{x=y^{\frac{1}{2n}}}{=} \frac{1}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}e^{-wt^{2n}}\,dw\int_0^{\infty}e^{-wy}\,y^{\frac{2k+1}{2n}-1}\,dy \\
&= \frac{\Gamma\left(\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}e^{-wt^{2n}}\,w^{-\frac{2k+1}{2n}}\,dw \\
&= \frac{\Gamma\left(\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2k}\int_0^{\infty}e^{-y}\,y^{-\frac{2k+1}{2n}+1-1}\,dy \\
&= \frac{\Gamma\left(\frac{2k+1}{2n}\right)\Gamma\left(1-\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2k} \\
&= \frac{\sin\frac{\pi}{2n}}{\sin\left(\frac{2k+1}{2n}\pi\right)}\,t^{2k}. \qquad (2.38)
\end{aligned}$$

□

Remark 2.8. For $k = 1$, formula (2.37) gives the variance of the hyperCauchy

$$\mathbb{E}C_n^2(t) = \operatorname{Var} C_n(t) = \frac{\sin\frac{\pi}{2n}}{\sin\frac{3\pi}{2n}}t^2 = \frac{t^2}{1+2\cos\frac{\pi}{n}}. \tag{2.39}$$

The last expression shows that the variance is a decreasing function of n.

Furthermore we have the following interesting relationships:

$$\begin{aligned}
\mathbb{E}C_n^{2(n-1)}(t) &= t^{2n-2}, \\
\mathbb{E}C_n^{2(n-2)}(t) &= \frac{\sin\frac{\pi}{2n}}{\sin\frac{3\pi}{2n}}t^{2(n-2)} = \operatorname{Var}\left(C_n(t)\right)\, t^{2n-4} = \frac{t^{2n-2}}{1+2\cos\frac{\pi}{n}}, \\
\mathbb{E}C_n^4(t) &= \frac{t^4 \operatorname{Var} C_n(t)}{2t^2\cos\frac{\pi}{n} - \operatorname{Var} C_n(t)}. \qquad (2.40)
\end{aligned}$$

For the distribution (2.32) it is possible to evaluate only the moment $\mathbb{E}\left|\tilde{C}_n(t)\right|$ by performing the following calculation

$$\begin{aligned}
\mathbb{E}\left|\tilde{C}_n(t)\right| &= \frac{\sin\frac{\pi}{2n}}{\pi}\int_{-\infty}^{\infty}|x|\,\frac{t}{x^2+t^2}\left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}\,dx \\
&= 2\frac{\sin\frac{\pi}{2n}}{\pi}\int_0^{\infty}\frac{t^{2-\frac{1}{n}}x^{\frac{1}{n}}}{x^2+t^2}\,dx \\
&\overset{x=ty}{=} \frac{2\sin\frac{\pi}{2n}}{\pi}\,t\int_0^{\infty}\frac{y^{\frac{1}{n}}}{1+y^2}\,dy
\end{aligned}$$

$$\overset{y=x^{\frac{1}{2}}}{=} \frac{\sin\frac{\pi}{2n}}{\pi}\, t \int_0^\infty e^{-u} \int_0^\infty x^{\frac{1}{2}+\frac{1}{2n}-1} e^{-ux} du\, dx$$

$$= \frac{\sin\frac{\pi}{2n}}{\pi}\, \Gamma\left(\frac{1}{2}+\frac{1}{2n}\right) t \int_0^\infty e^{-u}\, u^{1-\frac{1}{2}-\frac{1}{2n}-1}\, du$$

$$= \frac{\sin\frac{\pi}{2n}}{\pi}\, \Gamma\left(\frac{1}{2}+\frac{1}{2n}\right) \Gamma\left(1-\frac{1}{2}-\frac{1}{2n}\right) t$$

$$= \frac{\sin\frac{\pi}{2n}}{\sin\left(\left(\frac{1}{2}+\frac{1}{2n}\right)\pi\right)}\, t \;=\; t\tan\frac{\pi}{2n}. \tag{2.41}$$

3. Distributional properties of the hyperCauchy

In this section we consider a number of r.v.'s related to the hyperCauchy previously introduced. We start by examining the properties of the reciprocal of the hyperCauchy.

3.1. **Distribution of the reciprocal.** It is well known that the symmetrical Cauchy r.v. $C_1(t)$, $t > 0$, has the property that

$$\frac{1}{C_1\left(\frac{1}{t}\right)} \sim C_1(t). \tag{3.1}$$

For the hyperCauchy $C_n(t)$, $\hat{C}_n(t)$ and $\tilde{C}_n(t)$ we have the following theorem

Theorem 3.1. *We have that*

i)

$$\Pr\left\{\frac{1}{C_n\left(\frac{1}{t}\right)} \in dw\right\} = \frac{n\sin\frac{\pi}{2n}}{\pi} \frac{t^{2n-1}}{t^{2n}+w^{2n}} \left(\frac{w}{t}\right)^{2n-2} dw$$

$$= \left(\frac{w}{t}\right)^{2n-2} \Pr\left\{C_n(t) \in dw\right\}, \qquad w \in \mathbb{R}, t > 0, \tag{3.2}$$

ii)

$$\Pr\left\{\frac{1}{\hat{C}_n\left(\frac{1}{t}\right)} \in dw\right\} = \Pr\left\{\hat{C}_n(t) \in dw\right\}, \qquad w \in \mathbb{R}, t > 0, \tag{3.3}$$

iii)

$$\Pr\left\{\frac{1}{\tilde{C}_n(t)} \in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi} \frac{t}{t^2+w^2} \left(\frac{|w|}{t}\right)^{\frac{n-1}{n}} dw$$

$$= \left(\frac{t}{|x|}\right)^{\frac{2n-2}{n}} \Pr\left\{\tilde{C}_n(t) \in dw\right\}, \qquad w \in \mathbb{R}, t > 0. \tag{3.4}$$

Proof. The density of

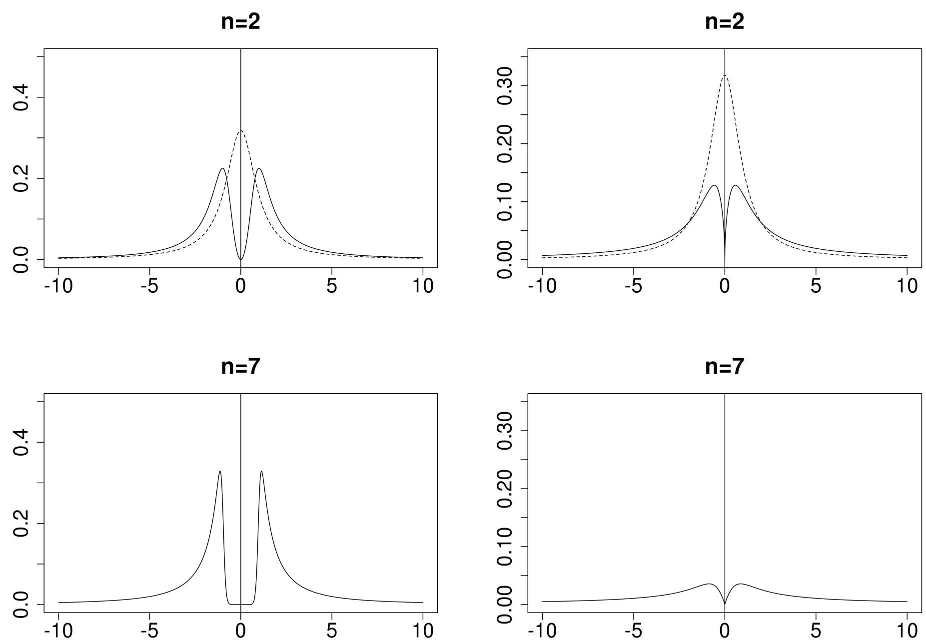
$$V_n(t) = \frac{1}{C_n\left(\frac{1}{t}\right)} \tag{3.5}$$

reads

$$v_n(w,t) = \frac{d}{dw} \frac{n\sin\frac{\pi}{2n}}{\pi} \int_{\frac{1}{w}}^\infty \frac{\left(\frac{1}{t}\right)^{2n-1}}{\frac{1}{t^{2n}}+x^{2n}}\, dx$$

$$= \frac{n \sin \frac{\pi}{2n}}{\pi} \frac{t^{2n-1}}{t^{2n} + w^{2n}} \left(\frac{w}{t} \right)^{2n-2}, \qquad w \in \mathbb{R}, t > 0, \tag{3.6}$$

and for $n = 1$ we retrive the previous result of the classical Cauchy r.v.. The density (3.2) has a bimodal structure (with maxima at $x = \pm t$) as illustrated in figure 3a.



(A) (B)

FIGURE 3. The probability density function (3.6), (A), and (3.10), (B), for different values of n.

Instead, the r.v. $\hat{C}_n(t)$ preserves the fine property of the classical Cauchy distribution because

$$\Pr \left\{ \frac{1}{\hat{C}_n \left(\frac{1}{t} \right)} < w \right\} = \frac{n}{\pi} \int_{\frac{1}{w}}^{\infty} (t|x|)^{n-1} \frac{\left(\frac{1}{t} \right)^{2n-1}}{\frac{1}{t^{2n}} + x^{2n}} dx, \tag{3.7}$$

and so, by taking the derivative with respect to w we get

$$\Pr \left\{ \frac{1}{\hat{C}_n \left(\frac{1}{t} \right)} \in dw \right\} = \frac{n}{\pi} \left(\frac{|w|}{t} \right)^{n-1} \frac{t^{2n-1}}{t^{2n} + w^{2n}} dw, \tag{3.8}$$

which coincides with the law of $\hat{C}_n(t)$. For the law of the r.v. $\tilde{C}_n(t)$ we get that

$$\Pr \left\{ \frac{1}{\tilde{C}_n \left(\frac{1}{t} \right)} < w \right\} = \frac{\sin \frac{\pi}{2n}}{\pi} \int_{\frac{1}{w}}^{\infty} \frac{\frac{1}{t}}{\frac{1}{t^2} + x^2} \left(\frac{\frac{1}{t}}{|x|} \right)^{\frac{n-1}{n}} dx, \tag{3.9}$$

and thus

$$\Pr\left\{\frac{1}{\tilde{C}_n\left(\frac{1}{t}\right)} \in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{t^2+w^2}\left(\frac{|w|}{t}\right)^{\frac{n-1}{n}} dw. \tag{3.10}$$

□

Distributions (3.6) and (3.10) are presented respectively in fig 3a and 3b, for different values of n and the dotted line represents the classical Cauchy density.

3.2. **Distributions of the ratio.** For the ratios of the three types of hyperCauchy distributions dealt with so far we have the following theorem.

Theorem 3.2. *In the following table we have the ratios of the r.v.'s and the corresponding densities*

r.v.	*density for* $w \in \mathbb{R}$
$W_n(t) = t\frac{C_n^1(t)}{C_n^2(t)}$	$\mathfrak{w}_n(w,t) = \frac{n}{2\pi}\tan\frac{\pi}{2n}t\frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}$
$\hat{W}_n(t) = t\frac{\hat{C}_n^1(t)}{\hat{C}_n^2(t)}$	$\hat{\mathfrak{w}}_n(w,t) = \frac{nt^n\|w\|^{n-1}}{\pi^2(t^{2n}-w^{2n})}\log\left(\frac{t}{w}\right)^{2n}$
$\tilde{W}_n(t) = \frac{\tilde{C}_n^1(t)}{\tilde{C}_n^2(t)}$	$\tilde{\mathfrak{w}}_n(w) = \frac{1}{2\pi}\tan\frac{\pi}{2n}\frac{\|w\|^{\frac{1}{n}-1}}{1-w^2}\left(1-w^{2-\frac{2}{n}}\right)$

Proof. We give a hint of the derivation of the densities above. For $w > 0$,

$$\Pr\left\{t\frac{C_n^1(t)}{C_n^2(t)} < \frac{w}{t}\right\} = \frac{1}{2} + 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\int_0^\infty dx\int_0^{\frac{wx}{t}} dy\frac{t^{2n-1}}{t^{2n}+x^{2n}}\frac{t^{2n-1}}{t^{2n}+y^{2n}}. \tag{3.11}$$

The density is therefore

$$\begin{aligned}
\mathfrak{w}_n(w,t) &= 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\int_0^\infty dx\frac{x}{t}t^{4n-2}\frac{1}{t^{2n}+x^{2n}}\frac{2}{t^{2n}+\left(\frac{wx}{t}\right)^{2n}} \\
&= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}-w^{2n}}\frac{t^{4n-2}}{t}\left[\int_0^\infty\frac{x\,dx}{t^{2n}+x^{2n}} - \frac{w^{2n}}{t^{2n}}\int_0^\infty\frac{x\,dx}{t^{2n}+\left(\frac{w^{2n}x^{2n}}{t^{2n}}\right)}\right], \qquad (3.12)
\end{aligned}$$

and with the change of variable $\frac{wx}{t} = y$ in the second integral of (3.12) we obtain

$$\begin{aligned}
\mathfrak{w}_n(w,t) &= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}+w^{2n}}\frac{t^{4n-2}}{t}\left(1-\frac{w^{2n}}{t^{2n}}\frac{t^2}{w^2}\right)\int_0^\infty dx\,\frac{x}{t^{2n}+x^{2n}} \\
&\overset{x=ty}{=} \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}+w^{2n}}t^{4n-3}\left(1-\frac{w^{2n-2}}{t^{2n-2}}\right)\frac{t^2}{t^{2n}}\int_0^\infty dy\,\frac{y}{1+y^{2n}} \\
&= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}+w^{2n}}\left(t^{2n-2}-w^{2n-2}\right)\frac{t^{2n-1}}{t^{2n}}t^2\int_0^\infty dy\,\frac{y}{1+y^{2n}} \\
&= 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}\frac{t}{2n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \\
&= \frac{n}{2\pi}\tan\frac{\pi}{2n}t\frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}. \qquad (3.13)
\end{aligned}$$

For the r.v. $\hat{W}_n(t)$ the density reads

$$\begin{aligned}
\hat{\mathfrak{w}}_n(w,t) &= \frac{2n^2}{\pi^2}\int_0^\infty \frac{x}{t}\frac{t^{2n-1}}{t^{2n}+x^{2n}}\left(\frac{x}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+\left(\frac{wx}{t}\right)^{2n}}\left(\frac{wx}{t^2}\right)^{n-1}dx \\
&= \frac{2n^2t^{3n}w^{n-1}}{\pi^2}\int_0^\infty \frac{x^{2n-1}}{t^{2n}+x^{2n}}\frac{dx}{t^{4n}+(wx)^{2n}} \\
&\overset{\left(\frac{x}{t}\right)^{2n}=y}{=} \frac{2n^2t^{3n}w^{n-1}}{\pi^2}\frac{1}{t^{2n}}\frac{1}{2n}\int_0^\infty \frac{1}{1+y}\frac{dy}{t^{2n}+w^{2n}y}\,dy \\
&= \frac{nt^nw^{n-1}}{\pi^2\left(t^{2n}-w^{2n}\right)}\int_0^\infty\left(\frac{1}{1+y}-\frac{w^{2n}}{t^{2n}+w^{2n}y}\right)dy \\
&= \frac{nt^nw^{n-1}}{\pi^2\left(t^{2n}-w^{2n}\right)}\log\left(\frac{t}{w}\right)^{2n}. \qquad (3.14)
\end{aligned}$$

For the r.v. $\tilde{W}_n(t)$ the density, not depending on t, has a structure different from the previous ones and is obtained by means of the following calculation

$$\begin{aligned}
\tilde{\mathfrak{w}}_n(w) &= 2\left(\frac{\sin\frac{\pi}{2n}}{\pi}\right)^2\int_0^\infty \frac{t}{t^2+x^2}\left(\frac{t}{x}\right)^{\frac{n-1}{n}}\frac{t}{t^2+w^2+x^2}\left(\frac{t}{wx}\right)^{\frac{n-1}{n}}x\,dx \\
&= 2\left(\frac{\sin\frac{\pi}{2n}}{\pi}\right)^2\frac{t^{2+2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}}\int_0^\infty \frac{x}{x^2+t^2}\frac{x^{-2\left(\frac{n-1}{n}\right)}}{t^2+w^2x^2}\,dx \\
&= 2\frac{\left(\sin\frac{\pi}{2n}\right)^2}{\pi^2}\frac{t^{2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}\left(1-w^2\right)}\left(\int_0^\infty \frac{x^{\frac{2}{n}-1}}{x^2+t^2}\,dx - w^2\int_0^\infty \frac{x^{\frac{2}{n}-1}}{t^2+w^2x^2}\,dx\right) \\
&= 2\frac{\left(\sin\frac{\pi}{2n}\right)^2}{\pi^2}\frac{t^{2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}\left(1-w^2\right)}\left(\frac{\pi}{2t^{2-\frac{2}{n}}\sin\frac{\pi}{n}}-\frac{w^{2-\frac{2}{n}}\pi}{2t^{2-\frac{2}{n}}\sin\frac{\pi}{n}}\right) \\
&= \frac{1}{2\pi}\tan\frac{\pi}{2n}\frac{w^{\frac{1}{n}-1}}{\left(1-w^2\right)}\left(1-w^{2-\frac{2}{n}}\right). \qquad (3.15)
\end{aligned}$$

Similar calculation performed for $w<0$ yield the previous distributions for $w\in\mathbb{R}$. □

Remark 3.3. We note that by setting $n=2$ in the law $\mathfrak{w}_n(w,t)$ we retrive the standard Cauchy density. Indeed

$$\begin{aligned}
\mathfrak{w}_2(w,t) &= \frac{1}{\pi}\tan\frac{\pi}{4}t\frac{t^2-w^2}{t^4-w^4} \\
&= \frac{1}{\pi}\frac{t}{t^2+w^2}. \qquad (3.16)
\end{aligned}$$

This means that if $C_2^1(t)$ and $C_2^2(t)$ are two independent random variables with law

$$p_2(w,t) = \frac{1}{\sqrt{2}\pi}\frac{t^3}{t^4+w^4}, \qquad (3.17)$$

the distribution of

$$W_2(t) = \frac{C_2^1(t)}{C_2^2(t)} \qquad (3.18)$$

is a standard Cauchy.

Furthermore we have that the distribution (3.14) coincides with formula (4.6) of D'Ovidio and Orsinger [3] for $n = 1$. We can check that the r.v.

$$\left(\frac{1}{t} \frac{C_1^1(t)}{C_1^2(t)} \right)^{\frac{1}{n}}, \tag{3.19}$$

where C_1^1, C_1^2 are two independent Cauchy r.v.'s, possesses distribution (3.14). In other words we have the following equality in distribution

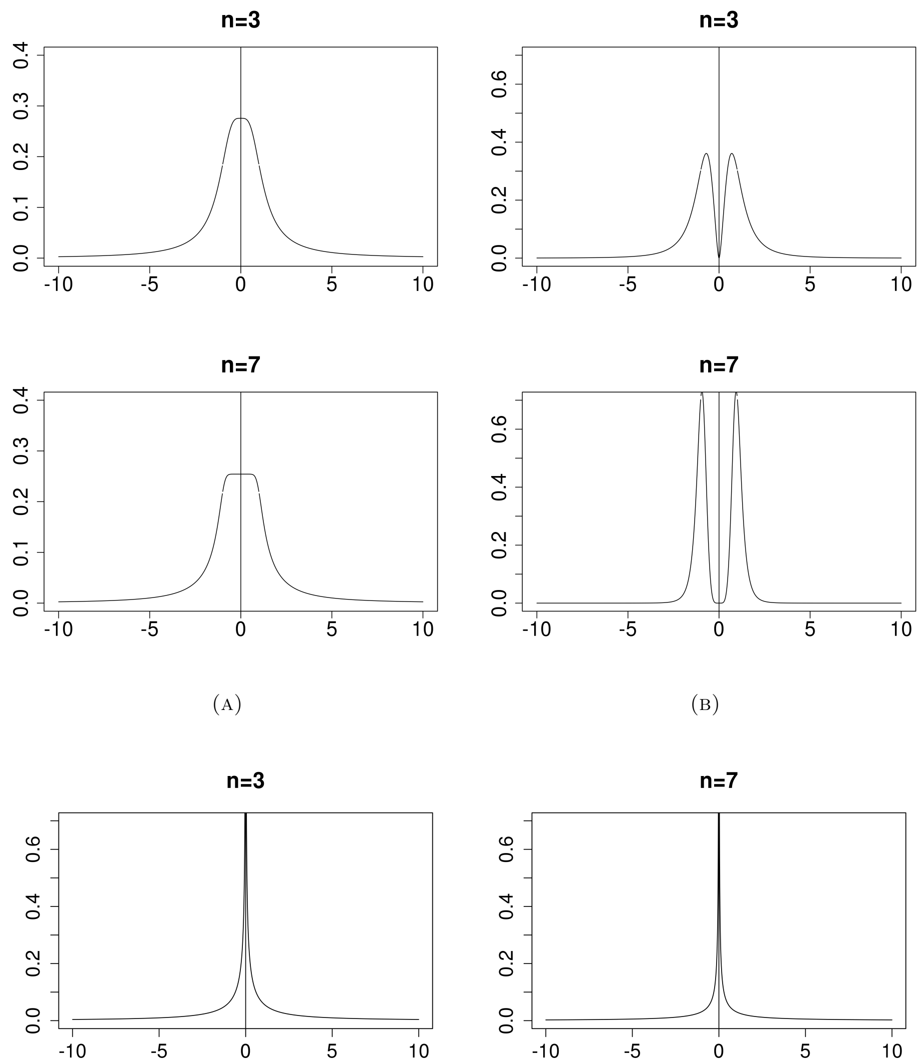
$$t \frac{\hat{C}_n^1(t)}{\hat{C}_n^2(t)} \overset{\text{i.d.}}{=} \left(\frac{1}{t} \frac{C_1^1(t)}{C_1^2(t)} \right)^{\frac{1}{n}}. \tag{3.20}$$

Remark 3.4. In order to check that the density (3.15) integrates to unity we perform the following calculation

$$\begin{aligned}
&\int_{-\infty}^{\infty} \tilde{\mathfrak{w}}_n(w)\, dw = \\
&= \frac{\tan\frac{\pi}{2n}}{2\pi} \int_{-\infty}^{\infty} \frac{|w|^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right) dw}{(1-w^2)} = \frac{\tan\frac{\pi}{2n}}{\pi} \int_0^{\infty} \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right) dw}{(1-w^2)} \\
&= \frac{1}{\pi} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw + \frac{1}{\pi} \tan\frac{\pi}{2n} \int_1^{\infty} \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw.
\end{aligned} \tag{3.21}$$

With the change of variable $y = \frac{1}{w}$ in the second integral of (3.21), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \tilde{\mathfrak{w}}_n(w)\, dw &= \frac{1}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw + \\
&\quad + \frac{1}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{\left(1 - y^{2-\frac{2}{n}}\right) y^{\frac{1}{n}-1}}{1-y^2}\, dy \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2}\, dw \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=0}^{\infty} \int_0^1 \left(w^{\frac{1}{n}-1} - w^{-\frac{1}{n}+1} \right) w^{2k} \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=0}^{\infty} \left[\frac{w^{2k+\frac{1}{n}}}{2k+\frac{1}{n}} - \frac{w^{2k-\frac{1}{n}+2}}{2k-\frac{1}{n}+2} \right]_0^1 \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=0}^{\infty} \left(\frac{1}{2k+\frac{1}{n}} - \frac{1}{2(k+1)-\frac{1}{n}} \right) \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=1}^{\infty} \left(\frac{1}{2k+\frac{1}{n}} - \frac{1}{2k-\frac{1}{n}} + n \right) \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \left(n - \sum_{k=1}^{\infty} \frac{\frac{2}{n}}{(2k)^2 - \frac{1}{n^2}} \right).
\end{aligned} \tag{3.22}$$



(A) (B) (C)

Figure 4. The probability density function $\mathfrak{w}_n(w,t)$, (A), $\hat{\mathfrak{w}}_n(w,t)$, (B), and $\tilde{\mathfrak{w}}_n(w)$, (C), for different values of n.

Considering the relationship (see Smirnov [11] pag 410)

$$z \cot z \,=\, 1 - \sum_{k=1}^{\infty} \frac{2z^2}{k^2\pi^2 - z^2} \tag{3.23}$$

and setting $z = \frac{\pi}{2n}$, we get

$$\frac{\pi}{2n}\cot\frac{\pi}{2n} = 1 - \sum_{k=1}^{\infty} \frac{2\left(\frac{\pi}{2n}\right)^2}{k^2\pi^2 - \left(\frac{\pi}{2n}\right)^2}$$

$$= 1 - \sum_{k=1}^{\infty} \frac{\frac{2}{n^2}}{(2k)^2 - \frac{1}{n^2}}, \tag{3.24}$$

and thus

$$\frac{\pi}{2}\cot\frac{\pi}{2n} = n - \frac{2}{n}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - \frac{1}{n^2}}. \tag{3.25}$$

Considering (3.25) we can rewrite (3.22) as follows

$$\int_{-\infty}^{\infty} \tilde{\mathfrak{w}}_n(w)\, dw = \frac{2}{\pi}\tan\frac{\pi}{2n}\left(n - \sum_{k=1}^{\infty} \frac{\frac{2}{n}}{(2k)^2 - \frac{1}{n^2}}\right)$$

$$= \frac{2}{\pi}\tan\frac{\pi}{2n}\left(\frac{\pi}{2}\cot\frac{\pi}{2n}\right) = 1. \tag{3.26}$$

The previous calculation yields an interesting integral represention of the cotangent function. Indeed, in light of (3.22) and (3.23) we can write

$$\cot z = \frac{1}{z} - \sum_{k=0}^{\infty} \frac{2z}{(2k)^2 - z^2}$$

$$= \int_0^1 \frac{w^{z-1}}{1-w^2}\left(1 - w^{2(1-z)}\right) dw$$

$$= \frac{1}{2}\int_0^{\infty} \frac{w^{z-1}}{1-w^2}\left(1 - w^{2(1-z)}\right) dw. \tag{3.27}$$

For a representation of (3.13), (3.14) and (3.15), see Fig. 4a, 4b and 4c.

3.3. **The higher-order arcsine law.** It is well-known that for the classical Cauchy r.v., $C_1(t)$, holds the following relationship (see Chaumont and Yor [2] pag. 104)

$$Z_1(t) = \frac{t}{1 + (C_1(t))^2} \overset{\text{i.d.}}{=} \frac{1}{\pi}\frac{1}{\sqrt{w\,(t-w)}}, \qquad 0 < w < t. \tag{3.28}$$

which is known as the arcsine law. For the hyperCauchy we get similar relationships.

Theorem 3.5. *We have the following distributions.*

r.v.	*probability density for* $0 < w < t$
$Z_n(t) = \frac{t}{1+\left(\frac{\lvert C_n(t)\rvert}{t}\right)^{2n}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}\, w^{-\frac{1}{2n}}$
$\hat{Z}_n(t) = \frac{t}{1+\left(\frac{\lvert \hat{C}_n(t)\rvert}{t}\right)^{2n}}$	$\frac{1}{\pi\sqrt{(t-w)w}}$
$\tilde{Z}_n(t) = \frac{t}{1+\left(\frac{\lvert \tilde{C}_n(t)\rvert}{t}\right)^{2}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}\, w^{-\frac{1}{2n}}$

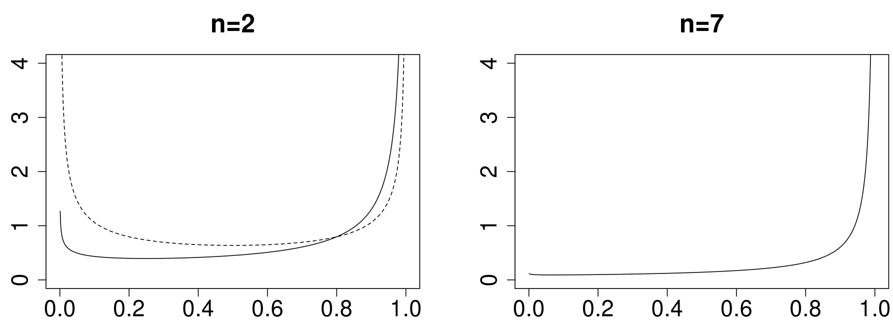
Proof. We get for $0 < w < t$,

$$\Pr\left\{\hat{Z}_n(t) < w\right\} \;=\; 2\frac{n}{\pi}\int_{t\left(\frac{t-w}{w}\right)^{\frac{1}{2n}}}^{\infty} \left(\frac{|x|}{t}\right)^{n-1} \frac{t^{2n-1}}{t^{2n}+x^{2n}}\,dx \tag{3.29}$$

and thus

$$\hat{z}_n(w,t)\,dw \;=\; \Pr\left\{\hat{Z}_n(t) \in dw\right\} \;=\; \frac{1}{\pi}\frac{dw}{\sqrt{w(t-w)}}. \tag{3.30}$$

FIGURE 5



For the r.v. $C_n(t)$ the distribution becomes

$$\Pr\left\{Z_n(t) < w\right\} \; 2\frac{n\sin\frac{\pi}{2n}}{\pi}\int_{t\left(\frac{t-w}{w}\right)^{\frac{1}{2n}}}^{\infty} \frac{t^{2n-1}}{t^{2n}+x^{2n}}dx \tag{3.31}$$

and

$$z_n(w,t)\,dw \;=\; \Pr\left\{Z_n(t) \in dw\right\} \;=\; \frac{\sin\frac{\pi}{2n}}{\pi}w^{-\frac{1}{2n}}\,(t-w)^{\frac{1}{2n}-1}\;dw. \tag{3.32}$$

Similar calculations for $\tilde{Z}_n(t)$ yield

$$\Pr\left\{\tilde{Z}_n(t) \in dw\right\} \;=\; \frac{\sin\frac{\pi}{2n}}{\pi}\,(t-w)^{\frac{1}{2n}-1}\;w^{-\frac{1}{2n}}dw. \tag{3.33}$$

□

The density

$$z_n(w,t) \;=\; \tilde{z}_n(w,t) \;=\; \frac{\sin\frac{\pi}{2n}}{\pi}\,(t-w)^{\frac{1}{2n}-1}\;w^{-\frac{1}{2n}} \tag{3.34}$$

is a Beta with parameters $\left(\frac{1}{2n}-1,-\frac{1}{2n}\right)$ and for increasing values of n becomes more asymmetric, as shown in Fig. 5 for $t = 1$ (the dotted line represents the classical arcsine law).

TABLE 1. In the following table we sum up our results on the hyperCauchy functionals

Variable	Law	Transformation	Law of the transformation
$C_n(t)$	$\frac{n \sin \frac{\pi}{2n}}{\pi} \frac{t^{2n-1}}{t^{2n}+x^{2n}}$	$\frac{1}{C_n\left(\frac{1}{t}\right)}$	$\frac{n \sin \frac{\pi}{2n}}{\pi} \frac{t^{2n-1}}{t^{2n}+w^{2n}} \left(\frac{w}{t}\right)^{2n-2}$
		$t \frac{C_n^1(t)}{C_n^2(t)}$	$\frac{n}{2\pi} \tan \frac{\pi}{2n} \frac{t\left(t^{2n-2}-w^{2n-2}\right)}{t^{2n}-w^{2n}}$
		$\frac{t}{1+\left(\frac{\lvert C_n(t)\rvert}{t}\right)^{2n}}$	$\frac{\sin \frac{\pi}{2n}}{\pi} (t-w)^{\frac{1}{2n}-1} w^{-\frac{1}{2n}}$, for $0 < w < t$
$\hat{C}_n(t)$	$\frac{n}{\pi} \frac{t^{2n-1}}{t^{2n}+x^{2n}} \left(\frac{\lvert x\rvert}{t}\right)^{n-1}$	$\frac{1}{\hat{C}_n\left(\frac{1}{t}\right)}$	$\frac{n}{\pi} \frac{t^{2n-1}}{t^{2n}+w^{2n}} \left(\frac{\lvert w\rvert}{t}\right)^{n-1}$
		$t \frac{\hat{C}_n^1(t)}{\hat{C}_n^2(t)}$	$\frac{n t^n w^{n-1}}{\pi^2(t^{2n}-w^{2n})} \log \left(\frac{t}{w}\right)^{2n}$
		$\frac{t}{1+\left(\frac{\lvert\hat{C}_n(t)\rvert}{t}\right)^{2n}}$	$\frac{1}{\pi} w^{-\frac{1}{2}} (t-w)^{-\frac{1}{2}}$, for $0 < w < t$
$\tilde{C}_n(t)$	$\frac{\sin \frac{\pi}{2n}}{\pi} \frac{t}{t^2+x^2} \left(\frac{t}{\lvert x\rvert}\right)^{\frac{n-1}{n}}$	$\frac{1}{\tilde{C}_n\left(\frac{1}{t}\right)}$	$\frac{\sin \frac{\pi}{2n}}{\pi} \frac{t}{t^2+w^2} \left(\frac{\lvert w\rvert}{t}\right)^{\frac{n-1}{n}}$
		$\frac{\tilde{C}_n^1(t)}{\tilde{C}_n^2(t)}$	$\frac{1}{2\pi} \tan \frac{\pi}{2n} \frac{\lvert w\rvert^{\frac{1}{n}-1}}{(1-w^2)} \left(1 - w^{2-\frac{2}{n}}\right)$
		$\frac{t}{1+\left(\frac{\lvert\tilde{C}_n(t)\rvert}{t}\right)^{2}}$	$\frac{\sin \frac{\pi}{2n}}{\pi} (t-w)^{\frac{1}{2n}-1} w^{-\frac{1}{2n}}$, for $0 < w < t$

4. The Hyperbolic case

Let us consider the Poincaré half-plane $\mathbb{H}_2^+ = \{x, y : x \in \mathbb{R}, y > 0\}$ (see for example Gruet [6]; Lao and Orsingher [8]) endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{4.1}$$

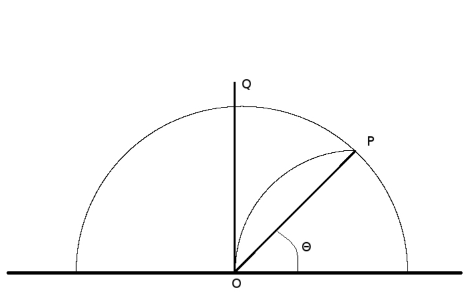
We assume that a particle is shooted from the point $O(0,0)$, see figure 6a, on the x-axis (representing the infinite of $\mathbb{H}_2^+$), and moves along the geodesic line joining O with an arbitrary point P on the half-circle centered at O, denoted by C_O, and with arbitrary radius t. The hyperbolic distance η between P and Q (Q is the intersection of the vertical geodesic line through O and the half-circle C_O), does not depend on t, because the half-circumferences centered at O form a system of horocycles, and will be denoted by η. Thus the hyperbolic distance η is obtained by evaluating the line integral

$$\begin{aligned}\eta &= \int_{\Theta}^{\frac{\pi}{2}} \frac{\sqrt{(x'(s))^2 + (y'(s))^2}}{y(s)}\, ds, \qquad \Theta \in \left(0, \frac{\pi}{2}\right) \\ &= \int_{\Theta}^{\frac{\pi}{2}} \frac{ds}{\sin s} = -\log \tan \frac{\Theta}{2}, \end{aligned} \tag{4.2}$$

where Θ is the random angle formed by OP and the x-line. Formula (4.2) can be rewritten as

$$e^{-\eta} = \tan \frac{\Theta}{2} \tag{4.3}$$

which is the celebrated Lobachevsky law for the angle of parallelism.



(A) (B)

FIGURE 6. The probability density function of $\hat{C}_n(t)$, (A), and $\tilde{C}_n(t)$, (B), for different values of n.

If Θ is uniformly distributed in $(0,\pi)$, the non-negative random variable η (representing the hyperbolic distance of P from Q)

$$\eta = \begin{cases} -\log \tan \frac{\Theta}{2}, & \Theta \in \left(0, \frac{\pi}{2}\right), \\ \log \tan \frac{\Theta}{2}, & \Theta \in \left(\frac{\pi}{2}, \pi\right), \end{cases} \tag{4.4}$$

has distribution function

$$\begin{aligned}\Pr\{\eta < w\} &= 2\Pr\left\{0 < -\log \tan \frac{\theta}{2} < w\right\} = 2\Pr\left\{0 > \log \tan \frac{\theta}{2} > -w\right\} \\ &= 2\Pr\left\{1 > \tan \frac{\theta}{2} > e^{-w}\right\} = 2\Pr\left\{\frac{\pi}{2} > \theta > 2\arctan e^{-w}\right\} \\ &= 2\int_{2\arctan e^{-w}}^{\frac{\pi}{2}} \frac{d\theta}{\pi} = 1 - \frac{4}{\pi}\arctan e^{-w}, \qquad w > 0. \end{aligned} \tag{4.5}$$

The density related to (4.5) reads

$$\Pr\{\eta \in dw\} = \frac{4}{\pi}\frac{e^{-w}}{1+e^{-2w}}\,dw = \frac{2}{\pi}\frac{dw}{\cosh w}, \qquad w > 0. \tag{4.6}$$

If we consider the symmetric r.v. (see fig 7)

$$\hat{\eta} = -\log\tan\frac{\Theta}{2}, \qquad \Theta \in (0,\pi), \tag{4.7}$$

we obtain that

$$\Pr\{\hat{\eta} \in dw\} = \frac{1}{\pi}\frac{dw}{\cosh w}, \qquad w \in \mathbb{R}, \tag{4.8}$$

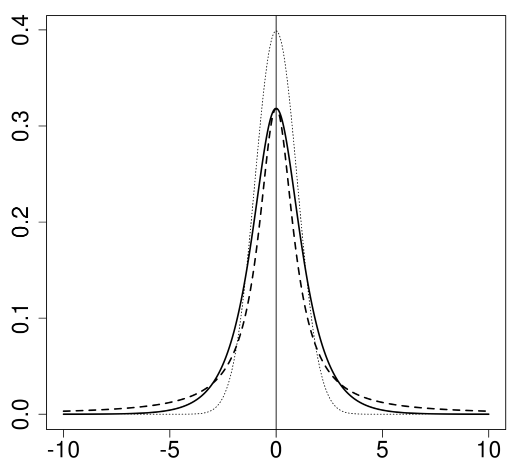
with distribution function

$$\Pr\{\hat{\eta} < w\} = 1 - \frac{2}{\pi}\arctan e^{-w}. \tag{4.9}$$

The distribution (4.8) appears in Feller [4] pag. 503 and emerges in the analysis of the successive overshoots by a Cauchy process in Pitman and Yor [10].

The r.v.'s η and $\hat{\eta}$ can be also viewed on the Poincaré disc, where the shooting point O is on the circumference and η represents the distance between Q and P (see figure 6b).

Figure 7. The density of the hyperbolic r.v. (black line) is compared with the standard normal (which has high concentration of the probability around zero) and the Cauchy law.



We give a derivation of the characteristic function of (4.8) different from the series expansion of Feller [4]. Our approach is based on the residue theorem.

Theorem 4.1. *The characteristic function of* (4.8) *is written as*

$$\mathbb{E}e^{i\beta\hat{\eta}} = \frac{1}{\cosh\frac{\beta\pi}{2}}. \tag{4.10}$$

Proof. The integral (4.10) can be evaluated by means of the residue theorem applied to the function

$$f(z) = \frac{e^{i\beta\pi z}}{\cosh \pi z}, \qquad z \in \mathbb{C}, \tag{4.11}$$

By considering the contour of Fig. 8a we have that

$$\int_{-r}^{r} \frac{e^{i\beta\pi x}dx}{\cosh \pi x} + \int_{0}^{i} \frac{e^{i\beta\pi(r+iy)}dy}{\cosh(r+iy)} + \int_{r}^{-r} \frac{e^{i\beta\pi(x+i)}dx}{\cosh \pi(x+i)} + \int_{i}^{0} \frac{e^{i\beta\pi(-r+iy)}dy}{\cosh \pi(-r+iy)} = \tag{4.12}$$
$$= 2\pi i \,\mathrm{Res} f(z)|_{z=\frac{i}{2}},$$

where $\mathrm{Res} f(z)|_{z=\frac{i}{2}}$ is the residue of the pole at $z = \frac{i}{2}$, the contour of integration is represented in Fig. 8a. By taking the limit for $r \to \infty$ the second and the fourth integral disappear and thus

$$\int_{-\infty}^{\infty} \frac{e^{i\beta\pi x}dx}{\cosh \pi x} + \int_{-\infty}^{\infty} \frac{e^{i\beta\pi x - \beta\pi}dx}{\cosh \pi x} = 2e^{-\frac{\beta\pi}{2}}$$
$$\left(1 + e^{-\beta\pi}\right) \int_{-\infty}^{\infty} \frac{e^{i\beta\pi x}}{\cosh \pi x} = 2e^{-\frac{\beta\pi}{2}}. \tag{4.13}$$

In conclusion we have that

$$\int_{-\infty}^{\infty} \frac{e^{i\beta\pi x}}{\cosh \pi x}dx = \frac{2e^{-\frac{\beta\pi}{2}}}{1+e^{-\beta\pi}} = \frac{1}{\cosh \frac{\beta\pi}{2}}, \tag{4.14}$$

which is the desired result. □

From (4.10) we obtain that

$$\mathrm{Var}\,\hat{\eta} = \left(\frac{\pi}{2}\right)^2. \tag{4.15}$$

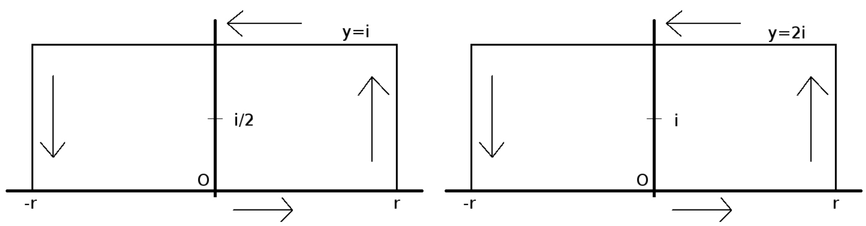
The even-order moments of $\hat{\eta}$ can be expressed in terms of the Euler numbers E_{2n}

$$\mathbb{E}\hat{\eta}^{2n} = \left(\frac{\pi}{2}\right)^{2n} |E_{2n}|, \tag{4.16}$$

in view of formula 3.523 pag 376 of [5]. The Euler numbers have generating function

$$\frac{1}{\cosh t} = \sum_{k=0}^{\infty} E_n \frac{t^n}{n!}, \qquad |t| < \frac{\pi}{2}. \tag{4.17}$$

Formula (4.17) gives, for $|t| < \frac{\pi}{2}$, a possible representation of the density (4.8).



(A) (B)

FIGURE 8. The contours of integration for Theorems 4.1 and 4.2.

4.1. **Distributional Properties of the hyperbolic distribution.**

Theorem 4.2. *Let* η_1 *and* η_2 *be two independent copies of* (4.7)*. Thus the distribution of*

$$\hat{\boldsymbol{\eta}}_{\mathbf{2}} = \hat{\eta}_1 + \hat{\eta}_2, \tag{4.18}$$

is given by

$$\Pr\{\hat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{2x}{\pi^2 \sinh x}\, dx. \tag{4.19}$$

Proof. In view of (4.10) we have

$$\Pr\{\hat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-izx}}{\cosh^2 \frac{z\pi}{2}} dz. \tag{4.20}$$

The inverse Fourier transform appearing in the right-hand side of (4.20) can be evaluated by means of the residue theorem, applied to the function

$$f(z) = \frac{e^{-ixz}}{\cosh^2 \frac{z\pi}{2}}, \qquad z \in \mathbb{C}, \tag{4.21}$$

along the contour of the form in Fig. 8b. In the same spirit of Theorem 4.1 we get

$$\begin{aligned} \frac{1}{2\pi} \Bigg[\int_{-r}^{r} \frac{e^{-ixw}}{\cosh^2 \frac{w\pi}{2}} dw + \int_{r}^{-r} \frac{e^{-ix(w+2i)}}{\cosh^2 \frac{\pi}{2}(w+2i)} dw + \int_0^{2i} \frac{e^{-ix(r+iy)}}{\cosh \frac{\pi(r+iy)}{2}} dy \; + \\ + \int_{2i}^{0} \frac{e^{-ix(-r+iy)}}{\cosh \frac{\pi(-r+iy)}{2}} dy \Bigg] = i \operatorname{Res} f(z)\big|_{z=i} \end{aligned} \tag{4.22}$$

and taking the limit for $r \to \infty$ we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-ixw}}{\cosh^2 \frac{w\pi}{2}} dw = \frac{i \operatorname{Res} f(z)\big|_{z=i}}{1 - e^{2x}} = -\frac{i}{2\sinh x} e^{-x} \operatorname{Res} f(z)\big|_{z=i}. \tag{4.23}$$

The residue in $z = i$ is given by

$$\begin{aligned} \operatorname{Res} f(z)\big|_{z=i} &= \lim_{z\to i} \frac{d}{dz} \left[(z-i)^2 \, \frac{e^{-ixz}}{\cosh^2 \frac{z\pi}{2}} \right] \\ &= \lim_{z\to i} \frac{d}{dz} \left[(z-i)^2 \, \frac{e^{-ixz}}{1+\cosh \pi z} \right] \\ &= \lim_{z\to i} e^{-ixz} \left[\frac{-2(z-i)^2 ix + 4(z-i)}{1+\cosh \pi z} - \frac{2\pi (z-i)^2 \sinh(\pi z)}{(1+\cosh \pi z)^2} \right] \\ &= \frac{2^2 xi}{\pi^2} e^x + \lim_{z\to i} e^{-ixz} \left[\frac{4(z-i)}{1+\cosh \pi z} - \frac{2\pi (z-i)^2 \sinh \pi z}{(1+\cosh z\pi)^2} \right] \\ &\overset{\text{Taylor}}{=} \frac{2^2 xi}{\pi^2} e^x + \lim_{z\to i} e^{-ixz} \left[\frac{4(z-i)}{-\frac{\pi^2}{2}(z-i)^2} + \frac{2\pi^2 (z-i)^3}{\left(-\frac{\pi^2}{2}(z-i)^2\right)^2} \right] \\ &= \frac{2^2 xi}{\pi^2} e^x, \end{aligned} \tag{4.24}$$

where, in the last step, we used the following Taylor's series expansions in a neighborhood of the point $z = i$

$$\begin{aligned} 1 + \cosh \pi z &= -\frac{(z-i)^2}{2}\pi^2 + o\left((z-i)^2\right) \\ \sinh \pi z &= -(z-i)\pi + o(z-i). \end{aligned} \tag{4.25}$$

In conclusion, considering (4.23) and (4.24), we obtain

$$\Pr\{\hat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{2x}{\pi^2 \sinh x}\, dx. \tag{4.26}$$

□

Remark 4.3. In order to check that (4.26) integrates to unity we refer to formula 3.521 pag. 375 of Gradshteyn and Ryzhik [5] obtaining

$$\int_{-\infty}^{\infty} \frac{2x}{\pi^2 \sinh x}\, dx = 1. \tag{4.27}$$

For a picture of distribution (4.19) see Fig. 9.

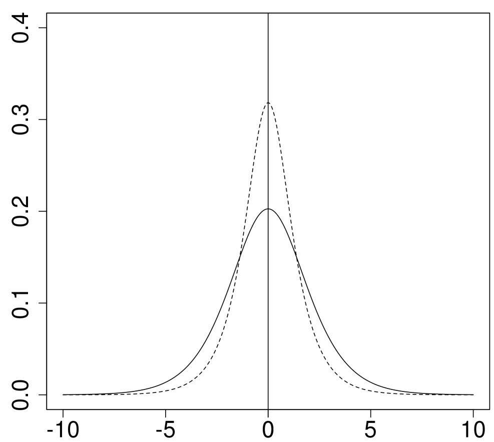


FIGURE 9. The dotted line represents the hyperbolic distribution (4.8) and the bold one represents the density (4.26) of the sum $\eta_1 + \eta_2$.

In general, for

$$\hat{\boldsymbol{\eta}}_{\boldsymbol{n}} = \hat{\eta}_1 + \hat{\eta}_2 + \cdots + \hat{\eta}_n, \qquad n \in \mathbb{N}, \tag{4.28}$$

we have, in force of formula 3.985 pag 512 of [5],

(4.29)

$$\Pr\{\hat{\boldsymbol{\eta}}_{\boldsymbol{n}} \in dw\} = \begin{cases} \frac{4^k w}{2(2k-1)!\pi^2 \sinh w} \prod_{r=1}^{k-1} \left(\frac{w^2}{\pi^2} + r^2\right) dw, & n = 2k, 2 \le k \in \mathbb{N} \\ \frac{2^{2k}}{(2k)!\pi \cosh w} \prod_{r=1}^{k} \left[\frac{w^2}{\pi^2} + \left(\frac{2r-1}{2}\right)^2\right] dw, & n = 2k+1, k \in \mathbb{N}. \end{cases}$$

The proof of (4.29) is based on the evaluation of the integral

$$\int_\Gamma f(z)\, dz, \qquad z \in \mathbb{C}, \tag{4.30}$$

where

$$f(z) = \frac{e^{-ixz}}{\cosh^n \frac{\pi z}{2}}, \qquad z \in \mathbb{C}, \tag{4.31}$$

and the contour Γ is that of figure 8b. The proof follows the same line of Theorem 4.2 and we arrive at

$$\int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh \frac{w\pi}{2}} dw - \frac{e^{2x}}{(-1)^n} \int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh^n \frac{w\pi}{2}} dw = 2\pi i \, \mathrm{Res} f(z)|_{z=i} \tag{4.32}$$

where $\mathrm{Res} f(z)|_{z=i}$ is the residue of $f(z)$ at $z = i$. The inverse Fourier transform is therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh^n \frac{w\pi}{2}} dw = \frac{i}{1 + (-1)^{n+1} e^{2x}} \mathrm{Res} f(z)|_{z=i} \, . \tag{4.33}$$

The evaluation of $\mathrm{Res} f(z)|_{z=i}$ leads to (4.29). For $n = 2$ we clearly retrive the result of Theorem 4.2.

A particle performing a random walk on the geodesic line QP of figure 6a, after n steps occupies the position $\tilde{\eta}_n$ with distribution (4.29) and characteristic function

$$\mathbb{E} e^{i\beta\hat{\eta}} = \frac{1}{\cosh^n \frac{\beta\pi}{2}}. \tag{4.34}$$

We present now some transformation of the hyperbolic distribution of $\hat{\eta}$. We start by showing that $\sinh \hat{\eta}$ has Cauchy distribution. We have for the r.v.

$$O(\eta) = \sinh \eta \tag{4.35}$$

that

$$\begin{aligned} \Pr\{\sinh \eta < y\} &= \Pr\{\eta < \arg\sinh y\} = \Pr\left\{\eta < \log\left(y + \sqrt{y^2+1}\right)\right\} \\ &= \int_{-\infty}^{\log\left(y+\sqrt{1+y^2}\right)} \frac{dx}{\pi \cosh x}. \end{aligned} \tag{4.36}$$

and thus

$$\begin{aligned} \frac{\Pr\{O(\eta) \in dy\}}{dy} &= \frac{1}{\pi} \frac{1}{y + \sqrt{1+y^2}} \left(1 + \frac{y}{\sqrt{1+y^2}}\right) \frac{2}{e^{\log\left(y+\sqrt{1+y^2}\right)} + e^{-\log\left(y+\sqrt{1+y^2}\right)}} \\ &= \frac{1}{\pi} \frac{y + \sqrt{1+y^2}}{y^2 + 1 + y\sqrt{1+y^2}} \frac{1}{\sqrt{1+y^2}} \\ &= \frac{1}{\pi(1+y^2)}. \end{aligned} \tag{4.37}$$

Furthermore, considering the r.v. $\cosh \eta$ we get, for $w > 1$

$$\begin{aligned} \Pr\{1 < \cosh \eta < w\} &= \frac{2}{\pi} \int_0^{\arg\cosh w} \frac{dx}{\cosh x} \\ &= \frac{2}{\pi} \int_0^{\log\left(w+\sqrt{w^2-1}\right)} \frac{dx}{\cosh x}, \end{aligned} \tag{4.38}$$

and thus the density reads

$$\Pr\{\cosh\eta \in dw\} = \frac{2}{\pi}\frac{dw}{w\sqrt{w^2-1}}, \qquad w>1. \tag{4.39}$$

The distribution (4.39) integrates to unity since

$$\frac{2}{\pi}\int_1^\infty \frac{dw}{w\sqrt{w^2-1}} \overset{\frac{1}{w^2}=y}{=} \frac{1}{\pi}\int_0^1 \frac{dy}{\sqrt{y(1-y)}} = 1. \tag{4.40}$$

The last step suggests a relationship between the r.v. $\cosh\eta$ and the arcsine law. The r.v.

$$Y = \frac{1}{\cosh^2\eta} \tag{4.41}$$

possesses arcsine distribution, as the following detailed calculation shows

$$\begin{aligned} \Pr\{Y<w\} &= \Pr\left\{\eta > \arg\cosh\frac{1}{\sqrt{w}}\right\} \\ &= \frac{2}{\pi}\int_{\log\left(\frac{1}{\sqrt{w}}+\frac{1}{\sqrt{w}}\sqrt{1-w}\right)}^\infty \frac{dx}{\cosh x} \\ &= \frac{2}{\pi}\int_{-\frac{1}{2}\log w+\log\left(1+\sqrt{1-w}\right)}^\infty \frac{dx}{\cosh x}, \end{aligned} \tag{4.42}$$

and thus

$$\begin{aligned} \frac{\Pr\{Y\in dw\}}{dw} &= \frac{1}{\pi}\left[\frac{1}{w}+\frac{1}{\sqrt{1-w}\left[1+\sqrt{1-w}\right]}\right]\frac{2}{\frac{1}{\sqrt{w}}\left(1+\sqrt{1-w}\right)+\sqrt{w}\frac{1}{1+\sqrt{1-w}}} \\ &= \frac{2}{\sqrt{\pi}}\frac{\sqrt{1-w}(1+\sqrt{1-w}+w}{w\sqrt{1-w}(1+\sqrt{1-w})}\frac{\sqrt{w}(1+\sqrt{1-w})}{(1+\sqrt{1-w})^2+w} \\ &= \frac{1}{\pi}\frac{1+\sqrt{1-w}}{\sqrt{w}\sqrt{1-w}}\frac{1}{\left(\sqrt{1-w}+1\right)} = \frac{1}{\pi}\frac{1}{\sqrt{w}\sqrt{1-w}}, \qquad 0<w<1. \end{aligned} \tag{4.43}$$

Remark 4.4. Result (4.43) can be also obtained observing that

$$Y = \frac{1}{\cosh^2\eta} = \frac{1}{1+\sinh^2\eta} = \frac{1}{1+O(\eta)^2}, \tag{4.44}$$

and we have shown that O possesses Cauchy distribution. The transformation (4.44) is the classical way to obtain the arcsine law from the Cauchy distribution.

Remark 4.5. Let us recall the hyperbolic version of the Pythagorean theorem which reads

$$\cosh a\cosh b = \cosh c, \tag{4.45}$$

where c is the hypotenuse of the right triangle with sides a and b. Considering a and b distributed as (4.8) their hyperbolic cosine has law (4.39). The random length of the hypotenuse is therefore written as

$$\begin{aligned} \Pr\{\cosh\eta_1\cosh\eta_2\in dw\} &= dw\left(\frac{2}{\pi}\right)^2\frac{1}{w}\int_1^w \frac{dx}{\sqrt{x^2-1}\sqrt{w^2-x^2}}dx \\ &\overset{x=\cosh y}{=} dw\int_0^{\log\left(w+\sqrt{w^2-1}\right)}\frac{1}{\sqrt{w^2-\cosh^2 y}}dy. \end{aligned} \tag{4.46}$$

Table 2. For the hyperbolic r.v. $\hat{\eta}$ we have the following table of distributional relationships for the related hyperbolic function.

Variable	$\sinh\hat{\eta}$	$\cosh\hat{\eta}$	$\tanh\hat{\eta}$	$\tanh^2\hat{\eta}$
Density	$\frac{1}{\pi(1+z^2)}$ $z\in\mathbb{R}$	$\frac{2}{\pi z\sqrt{z^2-1}}$ $z>1$	$\frac{1}{\pi\sqrt{1-z^2}}$ $-1<z<1$	$\frac{1}{\pi\sqrt{z(1-z)}}$ $0<z<1$
Reciprocal	$\frac{1}{\sinh\hat{\eta}}$	$\frac{1}{\cosh\hat{\eta}}$	$\coth\hat{\eta}$	$\coth^2\hat{\eta}$
Density	$\frac{1}{\pi(1+z^2)}$ $z\in\mathbb{R}$	$\frac{2}{\pi\sqrt{1-z^2}}$ $0<z<1$	$\frac{1}{\pi\lvert z\rvert\sqrt{z^2-1}}$ $z\in\mathbb{R}\backslash[-1,1]$	$\frac{1}{\pi z\sqrt{z-1}}$ $z>1$

Remark 4.6. Considering the r.v.

$$\tilde{\eta} = -\log\tan^{\alpha}\frac{\Theta}{2}, \qquad \alpha>0, \tag{4.47}$$

with Θ uniformly distributed in $(0,\pi)$ we get

$$\Pr\{\tilde{\eta}\in dw\} = \frac{2}{\alpha\pi}\frac{e^{-\frac{w}{\alpha}}dw}{1+e^{-\frac{2w}{\alpha}}} = \frac{1}{\pi\alpha}\frac{dw}{\cosh\frac{w}{\alpha}}, \qquad w\in\mathbb{R}. \tag{4.48}$$

The density (4.48) is a generalization with paramater α of (4.8).

4.2. **The area of hyperbolic random triangles.** It is well known that the area A of an hyperbolic triangle is given by

$$A = \pi-(\alpha+\beta+\gamma) \tag{4.49}$$

where α, β and γ are the angles pertaining to vertices not lying on the (x-axis). A triangle which has three vertices on the x-axis has area $A=\pi$.

Let us consider the triangle with vertices O, P, and Q in Fig. 6a or 6b, thus the area K is given by $K=\frac{\pi}{2}-\alpha$ where α is the angle of the vertex $O\hat{P}Q$, formally we have $K\in\left(0,\frac{\pi}{2}\right)$.

Theorem 4.7. *For the random area K of the hyperbolic triangle OPQ where PQ has length η with distribution (4.8), we have that*

$$\Pr\{K\in dw\} = \frac{2}{\pi}\frac{dw}{1+\sin w}, \qquad w\in\left(0,\frac{\pi}{2}\right). \tag{4.50}$$

Proof. In view of formula

$$\tan\frac{A}{2} = \tanh\frac{a}{2}\tanh\frac{b}{2} \tag{4.51}$$

where a, b are the sides of an hyperbolic right triangle of area A, we have

$$\tan\frac{K}{2} = \tanh\frac{\eta}{2}. \tag{4.52}$$

For $w>0$

$$\Pr\{K<w\} = \Pr\left\{\eta<2\,\mathrm{arctanh}\tan\frac{w}{2}\right\}$$

$$= \frac{2}{\pi} \int_0^{\log \frac{1+\sin w}{\cos w}} \frac{1}{\cosh x} \, dx$$

$$= 1 - \frac{4}{\pi} \arctan \frac{\cos w}{1+\sin w}, \tag{4.53}$$

and thus

$$\Pr\{K \in dw\} = \frac{2}{\pi} \frac{dw}{1+\sin w}. \tag{4.54}$$

□

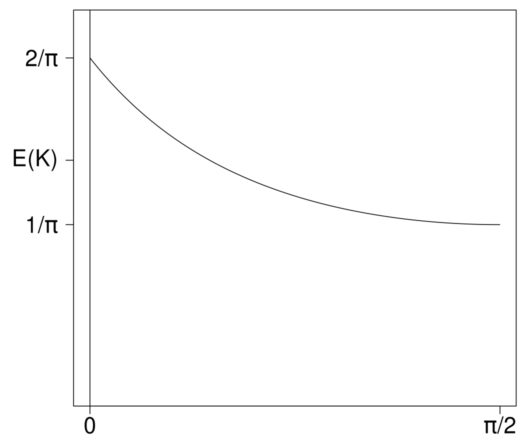


FIGURE 10. The distribution (4.50) of the random area K.

In view of formula 3.791 pag. 448 of [5] we have

$$\mathbb{E}K = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{x}{1+\sin x} \, dx = \frac{2}{\pi} \log 2. \tag{4.55}$$

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