Systemic sovereign risk in the valuation of solvency capital requirements

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Abstract

Motivated by the possibility of a systemic crisis in the quality of sovereign credit standings, we investigate the effects of a systemic component in default correlations for typical Italian life insurance segregated fund portfolios, comparing with a more traditional approach where such component is absent. The systemic effects are modelled with a multidimensional Marshall-Olkin model, that – in addition – allows to describe possible segmentation effects of the market. In particular, we compare the valuation of the Solvency Capital Requirement in the Solvency II framework under the classical CreditRisk$^+$ approach and the Marshall-Olkin model.

1 The problem

We investigate the problem of party default risk measurement for Italian life insurance segregated fund portfolios, by introducing a systemic component in default correlations.

In the Solvency II framework, the measurement of counterparty default risk "should reflect possible losses due to unexpected default, or deterioration in the credit standings, of the counterparties and debtors of undertakings over the forthcoming twelve months "(see [4]). In particular, our investigation is motivated by the peculiarity of Italian life insurance segregated fund portfolios, which are strongly dominated by fixed interest rate assets and, in particular, by Italian sovereign bonds. The possibility of a systemic crisis in the quality of sovereign credit standings can hence be considered quite relevant. It involves either tactical asset allocation issues or strategical solvency, profitability and rating issues for the company as a whole.

The existence of a systemic sovereign risk has been recently analysed by financial regulators. As stated by Bank of Italy, "for several [european] countries ... in the most recent period the sovereign spread vis-à-vis the German Bund has risen well above the value consistent with country-specific fiscal and macroeconomic fundamentals ... For Italian government bonds, most estimates of the 10-year spread fall around 200 basis points, as opposed to a market value of almost 450 points (at end-August 2012). These results are likely due to the fact that the models used so far *do not take into account the new risks* which have recently emerged in euro-area sovereign debt markets. In fact, several reasons suggest that euro-area sovereign spreads are increasingly affected by investors' concerns of a *break-up of the Economic and Monetary Union*" (see [3]).

In the words of the European Central Bank, studying sovereign bond spreads of the G7 countries over the last two decades, "several risk factors have not been priced in the years preceding the financial crisis. This pattern is particularly pronounced for the determinants of the Italian-German and the French-German spreads, i.e. for spreads of the euro area member countries, where macro fundamentals, general risk aversion and liquidity risks used to be priced in the uprun to monetary union and following the outbreak of the financial crisis, but not in the first years of monetary union. ... These findings support the belief that swings in risk appetite have led to an underpricing of risk prior to the global financial crisis, and either an over-pricing of risk during the European sovereign debt crisis or the pricing of *catastrophic events* like a break-up of the euro area and a risk that government bonds of some euro area countries might get re-denominated in other currencies than the euro" (see [2]).

In our investigation, we have modelled the systemic sovereign risk with a multidimensional Marshall-Olkin model, that is particularly suited to describe systemic market shocks. In addition, it allows to describe possible segmentation effects of the market. In particular, we compare the valuation of the Solvency Capital Requirement in the Solvency II framework under the classical CreditRisk$^+$ approach and the Marshall-Olkin model.

2 Portfolio Credit Risk modelling

Since in this work we address the question of default it is quite natural to refer to a standard model such as CreditRisk$^+$ ([1], [7]) as a benchmark. As we shall discuss, CreditRisk$^+$, by construction, has a limited capacity of incorporating default correlations for obligors of high credit standings. Therefore it is interesting to investigate possible modelling alternatives that are more sensitive to systemic crises.

2.1 CreditRisk$^+$

CreditRisk$^+$ is a *conditionally independent* factor model that reduces to the well-known actuarial Poisson-gamma model in absence of correlations. The structure of dependence of defaults is equivalent (see, *e.g* [11]) to a multivariate Clayton copula where the K independent risk factors are gamma-distributed. Conditional default probabilities are a linear combination of risk factors and an idiosyncratic risk component:

$$p_i(\mathbf{x}) = \bar{p}_i \left(\omega_{i0} + \sum_{i=1}^{K} \omega_{ik} \frac{x_k}{\mathbf{E}[x_k]} \right), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}_+^K$ is the vector of risk factors, $x_k \sim \Gamma(\alpha_k, \beta_k)$, $p_i(\mathbf{x})$ (resp. $\bar{p}$) is the conditional (resp. unconditional) default probability of obligor i over the time horizon $[0, T]$, $\omega_{i0} \in [0, 1]$ is the idiosyncratic loading factor and $\omega_{ik} \in [0, 1]$ are the economy risk factors, satisfying $\sum_{k=0}^{K} \omega_{ik} = 1$ for any obligor i. Independence is achieved when the relative weight of the idiosyncratic component is 100% ($\forall i$, $\omega_{i0} = 1$), while maximal correlation is achieved when the idiosyncratic component is absent ($\forall i$, $\omega_{i0} = 0$), and the economy is described by a single ($K = 1$) risk factor.

Since the probability generating function (PGF) of the loss distribution can be computed analytically, the loss distribution can be easily computed in a semi-analytic way using the Panjer recursion or the fast Fourier transform techniques. Moreover, numerical precision can be kept under control by computing the loss distribution moments directly from the PGF.

Default correlations are indirectly related to model parameters. The linear correlation coefficient between default indicators $\mathbb{1}_{\{\tau < T\}}$ (where τ is the time of default) of obligors i and j is known only in the limit of small default probabilities:

$$\rho \left(\mathbb{1}_{\{\tau_i \leq T\}}, \mathbb{1}_{\{\tau_j \leq T\}} \right) \simeq \sqrt{p_i p_j} \sum_{k=1}^{K} \omega_{ik} \omega_{jk} \frac{\mathbf{Var}[x_k]}{\mathbf{E}[x_k]^2}. \tag{2}$$

Thus, the maximal attainable correlation depends on the absolute level of default probabilities and the coefficients of variation of the gamma distributions. Since, typically, default probabilities are small, and the variation

coefficients are of order 1, the model has a limited capacity of incorporating default correlations, in particular for sovereign obligors that have very small default probabilities. In fact, we will show that, for portfolios where the sovereign component is high, the effects of correlations on the risk capital is numerically negligible.

2.2 The Marshall-Olkin model

The Marshall-Olkin model is originally conceived [9] to account for simultaneous defaults in a very general context. In the recent financial literature the Marshall-Olkin model has been considered mainly in the valuation of credit derivatives such as CDO's where its rich default dependence structure allows to reproduce accurately market values [5]. It has also been used to discuss the effects of default dependence structure in the framework of government guarantees [12].

In the following we introduce firstly the bivariate case and discuss some qualitative features. We then introduce the multivariate case and finally the particular version proposed in [13], that, *inter alia*, is computationally faster.

2.2.1 The bivariate Marshall-Olkin model

In the bivariate Marshall-Olkin model [9] there are two obligors subject to default and three independent Poisson processes N_t^1, N_t^2 and N_t^{12} with intensities $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_{12} \geq 0$.

The default of obligor i ($i = 1, 2$) can arrive either at the first jump time $X_i = \inf\{t \geq 0 : N_t^i > 0\}$ of the Poisson process N_t^i, or at the first jump time $X_{12} = \inf\{t \geq 0 : N_t^{12} > 0\}$ of the Poisson process N_t^{12}, which triggers the simultaneous default of both obligors. Therefore, letting τ_i ($i = 1, 2$) the default time of the i-th obligor, then

$$\tau_i = \min\{X_i, X_{12}\} \quad i = 1, 2.$$

The joint distribution of default times (τ_1, τ_2) is called the Marshall-Olkin bivariate exponential distribution with parameters $(\lambda_1, \lambda_2, \lambda_{12})$. The corresponding survival copula is:

$$C_{MO}(u_1, u_2) = u_1 u_2 \min\{u_1^{-\alpha_1}, u_2^{-\alpha_2}\}, \tag{3}$$

where $\alpha_i = \lambda_{12}/(\lambda_i + \lambda_{12})$ ($i = 1, 2$). Marginal distributions are exponential with intensities:

$$g_1 = \lambda_1 + \lambda_{12}, \quad g_2 = \lambda_2 + \lambda_{12}. \tag{4}$$

The intensities g_i are related to default probabilities p_i ($i = 1, 2$):

$$g_i = -\log(1 - p_i). \tag{5}$$

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Eq. (4) shows that (a) the marginal default intensities receive a contribution from a common "systemic" factor, and (b) for given default probabilities p_i, the common systemic intensity λ_{12} cannot exceed the smallest marginal intensity $\min\{g_1, g_2\} = \min\{\log(1 - p_1)^{-1}, \log(1 - p_2)^{-1}\}$.

In the Marshall-Olkin bivariate model both the linear correlation between default times and default indicators are available in closed form:

$$\mathbf{Cor}[\tau_1, \tau_2] = \frac{g_1 + g_2 - g_{12}}{g_{12}} = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}, \tag{6}$$

$$\mathbf{Cor}[\mathbb{1}_{\{\tau_1 < t\}}, \mathbb{1}_{\{\tau_2 < t\}}] = \sqrt{\frac{e^{-(g_1 + g_2)t}}{(1 - e^{-g_1 t})(1 - e^{-g_2 t})}} \left(e^{\lambda_{12} t} - 1\right), \tag{7}$$

where $g_{12} = \lambda_1 + \lambda_2 + \lambda_{12}$. From eq. (6) it is clear that independence is obtained for $\lambda_{12} = 0$ and maximal dependence for $(\lambda_1 + \lambda_2) \to 0$.

However, since $\lambda_{12} \leq \min\{g_1, g_2\}$, the maximal attainable correlation between default times is:

$$\max_{g_1, g_2, \lambda_{12}} \mathbf{Cor}[\tau_1, \tau_2] = \frac{\min\{g_1, g_2\}}{\max\{g_1, g_2\}}, \tag{8}$$

while the maximal attainable correlation between default indicators is:

$$\max_{g_1, g_2, \lambda_{12}} \mathbf{Cor}[\mathbb{1}_{\{\tau_1 < t\}}, \mathbb{1}_{\{\tau_2 < t\}}] = \sqrt{\frac{P_d (1 - P_D)}{P_D (1 - P_d)}}, \tag{9}$$

where $P_d = \min\{p_1, p_2\}$ and $P_D = \max\{p_1, p_2\}$. Notice that it is possible to attain full correlation ($\mathbf{Cor}[\cdot] = 1$) only if $p_1 = p_2$, $i.e.$ if default probabilities are equal, and that a strong asymmetry in default probabilities ($P_d << P_D$) implies low correlation. For a general discussion on attainable correlations, see $e.g.$ [11].

The asymmetry between P_d and P_D is relevant not only for the maximal attainable correlation, but also for the level of idiosyncratic risk. To illustrate this point we compare two set of default intensities, the first in which $(g_1, g_2) = (0.005, 0.005)$ and the second in which $(g_1, g_2) = (0.001, 0.005)$. To specify the model we have still to chose the systemic default intensity λ_{12}; in the following we consider the two extreme cases of independence ($\lambda_{12} = 0$) and maximal correlation ($\lambda_{12} = \min\{g_1, g_2\}$). The corresponding distributions of the number of defaults over a time horizon of 1 year are reported in Table 1.

Table 1: Probability distribution of the number of defaults in the bivariate Marshall-Olkin model for two sets of marginal default intensities (g_1, g_2) in the cases of independence and maximal correlation.

	defaults	$(g_1, g_2) = (0.005, 0.005)$		$(g_1, g_2) = (0.001, 0.005)$	
	defaults	Probab.	Cumulated	Probab.	Cumulated
$\lambda_{12} = 0$	0	0.99005	0.99005	0.99402	0.99402
	1	0.00993	0.99998	0.00598	1.00000
(indepen.)	2	0.00002	1.00000	0.00000	1.00000
$\lambda_{12} = g_1 \wedge g_2$	0	0.99501	0.99501	0.99501	0.99501
	1	0.00000	0.99501	0.00399	0.99900
(max. cor.)	2	0.00499	1.00000	0.00100	1.00000

Notice that:

1. Increasing correlation moves the mass of probability towards the extreme values (no defaults, 2 defaults);

2. in the symmetric case $(g_1 = g_2)$ the distribution is perfectly bimodal with $\mathbf{P}[1 \text{ default}] = 0$ since the maximal attainable correlation is 1, while in the asymmetric case $\mathbf{P}[1 \text{ default}] \neq 0$;

3. in the symmetric case with maximal correlation there is only one Poisson process, with intensity λ_{12} and only 0 or 2 defaults can be observed; in the asymmetric case with maximal correlation there are two Poisson processes, with intensities λ_{12} and $\lambda_2 = g_2 - \lambda_{12}$, thus obligor #1 cannot default *alone* but obligor #2 can;

4. if a confidence level of $\alpha = 0.995$ is used the Value-at-Risk is 0 for the correlated cases, whilst is surely positive for the independent cases, depending on the loss given default of the two obligors; while for a confidence level of $\alpha = 0.9999$ the Value-at-Risk for the independence cases is smaller than that computed in case of dependent risks.

From the above stylized example we infer that:

1. increasing correlation can either lower or increase the Value-at-Risk depending on the confidence level (a well established result in the credit risk literature);

2. when default probabilities are asymmetric, asking maximal systemic risk does not imply that idiosyncratic risk is completely cancelled out.

The above considerations will be useful in explaining the results obtained in the multivariate setting, that otherwise may appear counterintuitive.

2.2.2 The multivariate Marshall-Olkin model

In the multivariate Marshall-Olkin (MO) model there is a set composed of n obligors subject to failure. Let $\Upsilon = \{1, 2, \ldots, n\}$ be the set indexing the n obligors. With n objects it is possible to construct 2^n subsets composed of $0, 1, 2, \ldots, n$ elements each. Excluding the empty set we are left with $2^n - 1$ non-trivial subsets. Let $\Theta \subseteq \Upsilon$ stand for a generic subset of cardinality $|\Theta| = k$, with $k = 1, \ldots, n$. For each of these subsets the model assumes that there is an independent Poisson process N_t^{Θ} with arrival rate λ_{Θ}, such that $N_0^{\Theta} = 0$. A "fatal" shock arrives at the first jump of the process N_t^{Θ}:

$$X_{\Theta} = \inf\{t \geq 0 : N_t^{\Theta} > 0\}, \tag{10}$$

and produces the simultaneous default of all the elements in Θ. Therefore the failure time of the i-th element is:

$$\tau_i = \min\{X_{\Theta} : i \in \Theta\}, \quad i = 1, \ldots, n. \tag{11}$$

The joint distribution of failure times $(\tau_1, \ldots, \tau_n)$ is called the MO multivariate exponential distribution of parameters λ_{Θ}.

To compute the marginal distributions, let first compute the probability that all the components of a given subset Θ of cardinality k survive until time t. Since Poisson processes relative to different subsets are independent this probability is:

$$P(\tau_{\Theta} > t) = e^{-g_{\Theta} t}, \quad \tau_{\Theta} = \tau_1 \wedge \tau_2 \cdots \wedge \tau_k, \tag{12}$$

where the intensity g_{Θ} is the sum of the intensities of all the processes relative to subsets that have at least one obligor in common with those of the subset Θ, *i.e.*:

$$g_{\Theta} = \sum_{\Xi \subseteq \Upsilon : \Xi \cap \Theta \neq \emptyset} \lambda_{\Xi} \tag{13}$$

and $x \wedge y = \min\{x, y\}$.

When Θ has cardinality 1, the above relationship gives the marginal survival rates:

$$g_i = \sum_{\Theta \subseteq \Upsilon : i \in \Theta} \lambda_{\Theta}. \tag{14}$$

For example, when $n = 2$ one has $\Upsilon = \{1, 2\}$ and there are three possible subset, two of cardinality 1, resp. $\{1\}$ and $\{2\}$, and one of cardinality 2, $\{1, 2\}$. Accordingly, there are three Poisson shock processes with arrival rates λ_1, λ_2 and λ_{12}. The marginal rates are:

$$g_1 = \lambda_1 + \lambda_{12}, \quad g_2 = \lambda_2 + \lambda_{12}. \tag{15}$$

Moreover, for the "joint" set $\{1, 2\}$, the rate is $g_{12} = \lambda_1 + \lambda_2 + \lambda_{12}$.

The inverse of eq. (13) allows to compute the arrival rates of the Poisson shocks from the survival rates of the subsets Θ:

$$\lambda_\Theta = \sum_{\Xi \subseteq \Theta} (-1)^{|\Theta| - |\Xi| + 1} g_{\Xi^c}, \tag{16}$$

where Ξ^c is the complementary set of the set Ξ.

For example, when $n = 2$ one has:

$$\lambda_1 = g_{12} - g_1, \lambda_2 = g_{12} - g_2, \lambda_{12} = g_1 + g_2 - g_{12}. \tag{17}$$

The simulation of the survival times in the MO model needs the simulation of up to $2^n - 1$ Poisson processes. For a portfolio with n different obligors and m subsets (that could be *e.g.* economic sectors and geographical areas) $n + m + 1$ Poisson processes are needed, the last one referring to the subset containing all the obligors. A technique to reduce the computational burden of the model when n is large is suggested in [13].

2.2.3 An alternative framework for the Marshall-Olkin model

The alternative framework (for more details and proofs see [8] and [13]) is based on the following proposition.

Proposition 1. Let $\mathcal{S}_t$ be a one-dimensional subordinator, *i.e.* a non-decreasing Lévy process with Laplace exponent $\phi(u; \vec{p})$, and $\mathcal{E}(1)$ an exponential random variable of rate $\lambda = 1$. Define:

$$\tau = \inf\{t \geq 0 : \mathcal{S}_t \geq \mathcal{E}(1)\}. \tag{18}$$

Then the distribution of τ is exponential of parameter $\phi(1, \vec{p})$.

A suitable class of subordinators is provided by the *tempered stable subordinators* with Laplace exponent:

$$\phi(u, \vec{p}) = \begin{cases} -C\,\Gamma(\alpha)\Big[(u+\eta)^\alpha - \eta^\alpha\Big], & \alpha \neq 0 \\ C\,\log(1 + u/\eta), & \alpha = 0 \end{cases} \tag{19}$$

with $\vec{p} = (C, \alpha, \eta)$, where $C > 0$, $\alpha \in (0,1)$ and $\eta > 0$. The special cases of $\alpha = 1/2$ and $\alpha = 0$ correspond respectively to the inverse Gaussian (IG) subordinator and the gamma subordinator. The above proposition suggests that instead of using Poisson processes for the simulation of default times in the MO model, it is possible to use a subordinator processes. The generalization of proposition 1 to dimension n is the following.

Proposition 2. Let $\mathcal{T}$ be a n-dimensional subordinator with Laplace exponent $\phi(\mathbf{u}, \vec{p})$ and $\mathcal{E}_i(1)$ be n independent exponential random variables of parameters $\lambda_i = 1$ $(i = 1, \ldots, n)$. Define:

$$\tau_i = \inf\{t \geq 0 : \mathcal{T}_i \geq \mathcal{E}_i(1)\}, \quad i = 1, \ldots, n. \tag{20}$$

Then the random vector $(\tau_1, \ldots, \tau_n)$ has a joint MO multivariate n-dimensional exponential distribution with parameters:

$$g_\Theta = \phi\Big(\big(\mathbb{1}_\Theta(1), \ldots, \mathbb{1}_\Theta(n)\big), \vec{p}\Big), \tag{21}$$

where $\mathbb{1}_\Theta(k) = 1$ if $k \in \Theta$, 0 otherwise.

To reduce the computational burden it is possible to reduce the effective dimension of the subordinator in a way similar to that factor models are constructed. In practice, one defines only m independent one-dimensional subordinators with Laplace exponent $\phi_j(u; \vec{p}_j)$ $(j = 1, \ldots, m)$ and a mixing matrix A, such that:

$$\mathcal{T}_t^{(i)} = \sum_{j=1}^m A_{ij} \mathcal{S}_t^{(j)}. \tag{22}$$

The Laplace exponent of such subordinator is:

$$\phi(\mathbf{u}, \vec{p}) = \sum_{j=1}^m \phi_j\Big(\sum_{i=1}^n A_{ij} u_i; \vec{p}_j\Big). \tag{23}$$

Thus, we consider a setting where we have as many independent subordinators as obligors, plus one subordinator for each of the n_S risk sectors and a subordinator for the whole set of obligors (the *global* economy sector):

$$\mathcal{T}_t^{(i)} = \mathcal{S}_t^{(id,i)} + \omega_G^{(i)} \mathcal{S}_t^{(G)} + \sum_{k=1}^{n_S} \omega_S^{(i,k)} \mathcal{S}_t^{R,i}, \quad i = 1, \ldots, n. \tag{24}$$

The parameters of the model are:

1. the $n + n_S \times n$ coefficients $\omega_G^{(i)}$ and $\omega_S^{(i,k)}$;

2. the $n + n_S + 1$ (vector) parameters of the subordinators $\vec{p}_i$ $(i = 1, \ldots, n)$, $\vec{p}_{S,k}$ $(k = 1, \ldots, n_S)$, $\vec{p}_G$.

Constraints are imposed by the knowledge of one-year default probabilities $p_d^{(i)}$ of the obligors:

$$
\begin{aligned}
g_i &= \phi(1; \vec{p}_i)^{(i)} + \phi^{(G)}(\omega_G^{(i)}; \vec{p}_G) + \sum_{k=1}^{n_S} \phi^{(k)}(\omega_S^{(i,k)}; \vec{p}_{S,k}) \\
&= -\log(\mathbf{P}[\tau_i > 1]) = -\log(1 - p_d^{(i)}).
\end{aligned} \tag{25}
$$

As a further simplification we define $\alpha_G \in (0, 1)$ and $\alpha_S^{(k)} \in (0, 1)$ $(k = 1, \ldots, n_S)$ such that:

$$\phi^{(G)}(\omega_G^{(i)}; \vec{p}_G) = \alpha_G\, g_i, \quad \phi^{(k)}(\omega_S^{(i,k)}; \vec{p}_{S,k}) = \alpha_S^{(k)}\, g_i. \tag{26}$$

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with :

$$\alpha_G + \sum_{k=1}^{n_S} \alpha_S^{(k)} \leq 1. \tag{27}$$

Notice that the α coefficients do not depend on the obligor index i; in practice we are assuming that the default rate of each obligor is receiving a contribution from non-idiosyncratic components that is the same, in relative size, for all obligors. The idiosyncratic component is:

$$\alpha_{id} = 1 - \alpha_G - \sum_{k=1}^{n_S} \alpha_S^{(k)}, \tag{28}$$

thus, when $\alpha_{id} = 1$ the default times are all independent.

For a given set of $p_d^{(i)}$ and for given parameters of the Laplace exponents of the subordinators, the choice of α_G and $\alpha_S^{(k)}$ univocally determines the coefficients $\omega_G^{(i)}$ and $\omega_S^{(i,k)}$ by inverting eq. (26), from which, using (21), all the rates g_Θ are determined.

The simulation of default times can be performed using an algorithm defined in [13] composed by the following steps:

1) define a partition $\{t_0, t_1, \ldots, t_{nT} = T\}$ of the time horizon $[t_0, T]$;

2) simulate the increments of the $n_S + 1$ common subordinators over each time interval (t_{k-1}, t_k): $\left(\mathcal{S}_{t_k}^{(G)} - \mathcal{S}_{t_{k-1}}^{(G)}\right)$ and $\left(\mathcal{S}_{t_k}^{(S,j)} - \mathcal{S}_{t_{k-1}}^{(S,j)}\right)$ $(j = 1, \ldots, n_S)$;

3) compute the probability $q_i^c(t_{k-1}, t_k)$ of each obligor $(i = 1, \ldots, n)$ to survive from time t_{k-1} to time t_k, conditionally of having survived up to time t_{k-1}:

$$q_i^c(t_{k-1}, t_k) = e^{-g_i \left(t_{k-1} - t_k\right)} \times$$

$$\times \exp\left\{ -\omega_G^{(i)} \left(\mathcal{S}_{t_k}^{(G)} - \mathcal{S}_{t_{k-1}}^{(G)}\right) - \sum_{j=1}^{n_S} \omega_S^{(i,j)} \left(\mathcal{S}_{t_k}^{(S,j)} - \mathcal{S}_{t_{k-1}}^{(S,j)}\right) \right\} \times$$

$$\times \exp\left\{ \phi^{(G)}(\omega_G^{(i)}; \vec{p}_G)\left(t_k - t_{k-1}\right) + \sum_{j=1}^{n_S} \phi^{(j)}(\omega_S^{(i,j)}; \vec{p}_{S,j})\left(t_k - t_{k-1}\right) \right\}. \tag{29}$$

4) generate a uniform random variable U_i for each obligor and declare the default if $q_i^c(t_{k-1}, t_k) < U_i$.

Notice that the above algorithm needs to simulate only the $n_S + 1$ subordinators and does not need to simulate the n idiosyncratic subordinators, since their contribution is already included in eq. (29).

Similarly to CreditRisk$^+$ moments of the distribution of the number of defaults over the time horizon $[0, t]$:

$$N_t = \sum_{i=1}^{n} \mathbb{1}_i(t),$$ (30)

can be computed analytically and used to control the precision of the Monte Carlo procedure. The expected number of defaults is:

$$\mathbf{E}[N_t] = \sum_{i=1}^{n} \mathbf{E}[\mathbb{1}_i(t)] = \sum_{i=1}^{n} \left(1 - e^{-g_i t}\right) = n - \alpha_1(t),$$ (31)

where $\alpha_1(t) = \sum_{i=1}^{n} e^{-g_i t}$. Notice that $\mathbf{E}[N_t]$ depends only on the default rates g_i. The variance of N_t is:

$$\mathbf{Var}[N_t] = \alpha_1(t) + 2\alpha_2(t) - \alpha_1(t)^2,$$ (32)

where:

$$\alpha_2(t) = \mathbf{E}\left[\sum_{i=1}^{n-1} \sum_{j=i}^{n} \bar{\mathbb{1}}_i(t)\bar{\mathbb{1}}_j(t)\right], \quad \text{with } \bar{\mathbb{1}}_i(t) = (\mathbb{1}_i(t) - 1).$$ (33)

In case of independence between defaults:

$$2\alpha_2(t) = 2\sum_{i=1}^{n} \sum_{j>i}^{n} e^{-(g_i + g_j)t} = \alpha_1(t)^2 - \sum_{i=1}^{n} e^{-2g_i t},$$ (34)

so that:

$$\mathbf{Var}[N_t] = \sum_{i=1}^{n} \left[e^{-g_i t}\left(1 - e^{-g_i t}\right)\right],$$ (35)

that could have been easily anticipated since $\mathbf{Var}[\mathbb{1}_i(t)] = e^{-g_i t}(1 - e^{-g_i t})$. Differently, as shown in [13], $\alpha_2(t)$ can be computed using the Laplace exponent of the subordinator $\mathcal{T}_t$:

$$\alpha_2(t) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \exp\left\{\alpha_{id}(g_i + g_j) + \phi^{(G)}(\omega_G^{(i)} + \omega_G^{(j)}; \vec{p}_G) + \sum_{k=1}^{n_S} \phi^{(k)}\right\}t.$$ (36)

Equation (32) can be used both to control the numerical accuracy of the Monte Carlo determination of the loss distribution, and as a constraint in the calibration of the parameters of the Laplace exponents of the elementary subordinators $\mathcal{S}_t^{(G)}$ and $\mathcal{S}_t^{(S,j)}$, resp. $\vec{p}_G$ and $\vec{p}_{S,k}$ ($j = 1, \ldots, n_S$), when the standard deviation of default frequencies are known.

3 Application to Italian life insurance portfolios

3.1 Reference portfolios

We considered a reference portfolio composed of 199 fixed rate bonds all with maturity larger than one year. The bonds are issued by 43 different obligors, both sovereign and corporate, and are actively traded on the market. All obligors belong to the euro area with corporate obligors being mainly Italian.

We define 5 different asset allocation schemes, ranging from a simple equally weighted portfolio to a very high concentrated portfolio almost exclusively composed of Italian government bonds (BTP's). With the exclusion of the equally weighted portfolio, the other are mimicking Italian life insurance segregated fund portfolios. The Macaulay durations of the five portfolios range between 2.5 and 6 years. The asset allocation is resumed by giving the fraction of value invested in one of the three economic sectors we considered, namely "financial", "government" and "other", as reported in Table 2. The normalised Herfindahl index (H-index) is used to identify the five portfolios.

Table 2: Composition of the five reference portfolios in terms of allocation on the three economic sectors. For more details see Table 5.

Asset allocation (% of portfolio market value)					
Sector		H-index (%)			
	0.0	7.7	17.6	29.9	47.8
GOVT	16,28	66,83	46,20	65,58	83,38
FIN	65,12	27,14	47,39	31,39	12,67
OTHER	18,60	6,03	6,41	3,03	3,94

3.2 Results

We have computed the loss distribution for each portfolio using a $T = 1$ year time horizon and different configurations of CreditRisk$^+$ and the Marshall-Olkin model. Table 4 reports the expected loss and the $\alpha = 0.995$ confidence level quantile of the distributions from which the Solvency Capital Requirement (SCR) is easily derived.

Regarding the effects of concentration in the determination of the SCR, we recall that – in the case of default risk – highly concentrated portfolios can have lower risk capital than diversified portfolios (see, *e.g.*, the stylized example in [6, §2.3]).

For CreditRisk$^+$ we have used default frequencies and their standard deviations from Moody's reports based on the ratings of the obligor. The standard deviations are used to infer the parameters of the gamma-distributed risk factors. The values for each obligor are reported in Table 5. Similarly, exposures

have been computed with the hypothesis of deterministic recovery rates fixed to average Moody's values. For what concerns risk factors, we have considered the two extreme cases of independence and maximal dependence, thus in eq. (1) either $\omega_{i0} = 1$ or $\omega_{i1} = 1$ for all obligors ($i = 1, \ldots, 43$).

For the Marshall-Olkin model we have used the same set of default probabilities and recovery rates. For what concern risk factors we have considered four settings: the independence case ($\alpha_{id} = 1$), and three different cases with maximal systemic risks, namely the "all sector" ($\alpha_S = 1$), the "all global" ($\alpha_G = 1$) and the "equally mixed" ($\alpha_S = \alpha_G = 0.5$) cases. The parameters of the subordinators are reported in Table 3.

The loss distributions have been determined by Monte Carlo 100,000 simulations with $\Delta t = 1$ day. We have compared the values of the expected number of defaults and its variance obtained by simulation and by eqs. (31),(32) finding good agreement.

Analysing the results of Table 4 we observe that:

1. in case of independence the CreditRisk$^+$ and the Marshall-Olkin model give very similar results;

2. the effects of correlations in CreditRisk$^+$ are small, and, in particular, become negligible for portfolios with higher concentrations; this is explained on the basis of comments to eq. (2);

3. the effects of correlations in the Marshall-Olkin model depends on the concentration of the portfolios and become more important when the concentration is lower; this is explained by the structure of the model: systemic effects are more important when the diversification is larger;

4. in the Marshall-Olkin model increasing the effects of the systemic component of the sectors is more important than increasing the effects of the "global component"; this is explained by the fact that the global component is limited by the smallest default probabilities as discussed in the example of the bivariate case;

5. the risk capital obtained with the Marshall-Olkin model with correlations is found to be both larger and smaller than the value obtained in case of independence; this is explained by the fact that with low concentration and correlations the loss distributions is likely to become bimodal; again, as discussed for the bivariate case, increasing correlations can lower the risk capital;

6. concerning absolute values of the risk capital, the Marshal-Olkin model gives values as large as four times that of CreditRisk$^+$ when the portfolio is highly diversified, however gives lower values when the portfolio is concentrated; this is explained, as for the previous comment, as due to the effects of correlations.

3.2.1 Subordinators' parameters calibration issues

The results reported in Table 4 have been obtained with the subordinators' parameters reported in Table 3.

Changes in parameters' values have two effects:

1. the distribution of subordinators increments $(\mathcal{S}_{t+\Delta t} - \mathcal{S}_t)$ over the time interval $(0, t)$ changes; recalling that increments are gamma-distributed with shape parameter $\alpha_\gamma = C\Delta t$ and scale parameter $\beta_\gamma = 1/\eta$ and that the expected value (resp. variance) of a gamma distribution is $\alpha_\gamma \beta_\gamma$ (resp. $\alpha_\gamma \beta_\gamma^2$) increasing (resp. decreasing) the C parameters results in increments with larger mean and larger variance while the effects for η are opposite;

2. the value of the loading factors $\omega_G^{(i)}$ and $\omega_S^{(i,k)}$ $(k = 1, \ldots, n_S)$ changes since they are constrained by eq. (26); for example, in the case of $\omega_G^{(i)}$, inverting eq. (26) gives:

$$\omega_G^{(i)} = \eta\Big(e^{\alpha_G g_i/C} - 1\Big), \tag{37}$$

that is monotonically decreasing in C and monotonically increasing in η.

Therefore, the net effects on the increment of the multivariate $\mathcal{T}$ subordinator in eq. (24) are partially compensating, variations of C being more important due to the exponential dependence in eq. (37) and variations of η cancelling out in the mean value of $\mathcal{T}$.

Variations of the distribution of increments of the multivariate $\mathcal{T}$ subordinator modify the probability that its marginal component $\mathcal{T}_i$ simultaneously cross the trigger levels $\mathcal{E}_i(1)$: loosely speaking the smaller the increments the more the Marshall-Olkin distribution tends towards the case of independence. Thus, $C \to \infty$, results in transforming the distribution in one similar to that obtained in case of independence, while variations of η have only second order effects.

We have investigated numerically the sensitivity to the values of the parameters by varying the sector coefficients C and η. For example, considering the portfolio with a H-index value of $H = 29.9\%$, with a value of C_R 10 times larger than that of Table 3 we obtain a quantile of 1.7836% (that is exactly what obtained in case of independence), while with a reduction of C_R by a factor of 10 the quantile is reduced to 0.5416%. Differently, with a value of η_R 10 times larger or smaller than that of Table 3 the quantile remains unaffected at the fourth digit.

We conclude that the qualitative picture emerging from the results obtained in Table 4 remains unchanged with respect to changes in subordinators' parameters.

Table 3: Common subordinators' parameters values.

| parameter | Global | Sectors | | |
		Financial	Government	Other
C	0.0012	0.001	0.001	0.001
η	0.005	0.01	0.01	0.01

Table 4: Statistics (in % of portfolio market value) of the loss distribution. CreditRisk$^+$ results are obtained in a semi-analytic way using the Panjer algorithm. Marshall-Olkin results are obtained with 100,000 Monte Carlo simulations with $\Delta t = 1$ day. For CreditRisk$^+$ the "with correlation" case corresponds to a single economic sector without idiosyncratic risk (maximal correlation).

| Statistics | without correlations | | with correlations | | | |
| | CR$^+$ | MO | CR$^+$ | MO | MO | MO |
		$\alpha_{id}=1$		$\alpha_S=1$	$\alpha_G=0.5$ $\alpha_S=0.5$	$\alpha_G=1$
H = 0 (minimum concentration)						
$\mathbf{E}[L]$	0.2155	0.2168	0.2155	0.2154	0.2154	0.2122
99.5% quantile	2.2900	2.1218	2.9717	8.5242	8.5550	5.9881
H = 7.7% (low concentration)						
$\mathbf{E}[L]$	0.1263	0.1275	0.1263	0.1240	0.1251	0.1246
99.5% quantile	2.2436	2.2733	2.3008	4.1144	4.1144	2.8018
H = 17.6% (moderate concentration)						
$\mathbf{E}[L]$	0.0575	0.0569	0.0575	0.0571	0.0582	0.0565
99.5% quantile	2.7134	2.6803	2.7134	1.5409	1.5367	0.3279
H = 29.9% (high concentration)						
$\mathbf{E}[L]$	0.0507	0.0492	0.0507	0.0488	0.0504	0.0491
99.5% quantile	1.8141	1.7836	1.8141	0.9301	0.9399	0.2208
H = 47.8% (very high concentration)						
$\mathbf{E}[L]$	0.0466	0.0442	0.0466	0.0431	0.0454	0.0444
99.5% quantile	0.7081	0.6470	0.7081	0.7454	0.7307	0.1312

Table 5: Portfolio composition. For each of the 43 obligors the table shows the economic sector, the default frequency and its standard deviation, and the asset allocation for each of the five portfolios identified by the H-index.

	Obl.	Sector	$p_d(\%)$	$\sigma_p(\%)$	fraction of port. market value (%)				
					H-index (%)				
					0.0	7.7	17.6	29.9	47.8
1	GR	GOVT	16,0633	23,4753	2,33	1,51	0,03	0,04	0,08
2	ES	GOVT	0,2480	0,0431	2,33	8,04	0,42	0,60	0,79
3	NL	GOVT	0,0001	0,0007	2,33	5,03	0,49	0,69	0,74
4	BE	GOVT	0,0019	0,0027	2,33	5,53	0,34	0,48	0,62
5	DE	GOVT	0,0001	0,0007	2,33	12,56	4,00	5,68	6,83
6	IT	GOVT	0,0950	0,0773	2,33	23,12	37,91	53,81	69,16
7	FR	GOVT	0,0001	0,0007	2,33	11,06	3,01	4,27	5,17
8	N1	FIN	0,0950	0,0773	2,33	1,01	0,04	0,03	0,01
9	N2	FIN	0,0001	0,0007	2,33	0,50	0,07	0,04	0,03
10	N3	FIN	0,4160	0,0208	2,33	0,50	0,14	0,10	0,04
11	N4	FIN	0,1370	0,1234	2,33	1,51	4,27	2,83	0,97
12	N5	FIN	0,0061	0,0084	2,33	0,50	0,04	0,03	0,03
13	N6	FIN	0,4160	0,0208	2,33	0,50	0,36	0,24	0,08
14	N7	FIN	0,4160	0,0208	2,33	0,50	0,11	0,08	0,03
15	N8	FIN	0,2940	0,1812	2,33	0,50	0,02	0,01	0,00
16	N9	FIN	0,0950	0,0773	2,33	0,50	0,02	0,01	0,01
17	N10	FIN	1,9780	1,0555	2,33	2,01	0,00	0,00	0,00
18	N11	FIN	0,0001	0,0007	2,33	1,01	0,94	0,62	0,20
19	N12	FIN	0,1370	0,1234	2,33	0,50	1,77	1,17	0,43
20	N13	FIN	0,4160	0,0208	2,33	0,50	0,04	0,02	0,04
21	N14	FIN	0,0061	0,0084	2,33	0,50	0,78	0,52	0,15
22	N15	FIN	0,4160	0,0208	2,33	1,51	0,25	0,17	0,06
23	N16	FIN	0,4160	0,0208	2,33	0,50	0,16	0,10	0,03
24	N17	FIN	0,4160	0,0208	2,33	0,50	0,15	0,10	0,03
25	N18	FIN	0,0950	0,0773	2,33	3,52	12,97	8,59	3,87
26	N19	FIN	0,0001	0,0007	2,33	1,01	1,44	0,95	0,37
27	N20	FIN	0,0950	0,0773	2,33	0,50	1,69	1,12	0,68
28	N21	FIN	0,2480	0,0431	2,33	0,50	0,28	0,18	0,06
29	N22	FIN	0,0001	0,0007	2,33	1,01	0,39	0,26	0,11
30	N23	FIN	0,2780	0,2524	2,33	0,50	0,00	0,00	0,00
31	N24	FIN	0,4160	0,0208	2,33	0,50	0,04	0,02	0,03
32	N25	FIN	0,1370	0,1234	2,33	1,51	5,63	3,73	1,29
33	N26	FIN	0,0950	0,0773	2,33	3,52	15,73	10,42	4,13
34	N27	FIN	0,1370	0,1234	2,33	0,50	0,03	0,02	0,01
35	N28	FIN	0,1370	0,1234	2,33	1,01	0,03	0,02	0,01
36	N29	OTHER	0,2940	0,1812	2,33	0,50	0,91	0,43	0,32
37	N30	OTHER	0,2940	0,1812	2,33	0,50	0,05	0,02	0,08
38	N31	OTHER	0,7850	0,3444	2,33	1,01	0,15	0,07	0,08
39	N32	OTHER	0,1370	0,1234	2,33	1,01	0,99	0,47	0,95
40	N33	OTHER	0,0950	0,0773	2,33	0,50	1,08	0,51	0,98
41	N34	OTHER	0,1370	0,1234	2,33	0,50	0,01	0,00	0,02
42	N35	OTHER	0,1370	0,1234	2,33	0,50	1,28	0,61	0,67
43	N36	OTHER	0,1370	0,1234	2,33	1,51	1,95	0,92	0,84

4 Conclusions

Motivated by the possibility of a systemic crisis in the quality of sovereign credit standings, we have investigated the effects of systemic components in default correlations for the valuation of the internal model counterparty default risk Solvency Capital Requirement (SCR) of typical Italian life insurance segregated fund portfolios.

In our investigation, we have modelled the systemic sovereign risk with a multidimensional Marshall-Olkin model, that is particularly suited to describe systemic market shocks, comparing with the classical CreditRisk$^+$ model as a benchmark.

We have applied both models to five different portfolios obtained from a basket of 199 actively traded bonds of the euro area.

As might have been expected, we have shown that for highly concentrated portfolios of high credit standing, *e.g.* composed of government bonds, the CreditRisk$^+$ model is unable to account for high default correlations.

Differently, we found that the range of SCR obtainable with the Marshall-Olkin model, depending on the market segmentation specification (sector composition) is considerably wide. In absolute terms the SCR obtained with the Marshall-Olkin model does not dominate the values obtained with the CreditRisk$^+$ model, but rather the comparison depends on the concentration of the portfolios, systemic effects being more important for more diversified portfolios.

Concerning Italian life insurance segregated funds, at least those similar in composition to those analysed in this work, for the present level of default probabilities and the actual high concentration of these portfolios, the analysis suggests that the use of CreditRisk$^+$ model appears sufficiently prudential, even in presence of systemic effects, for the determination of counterparty default risk when measured with the internal modelling approach.

References

[1] *CreditRisk$^+$ - A risk management framework*, CSFB, 1997,
Available at http://www.csfb.com/institutional/research/assets/creditrisk.pdf.

[2] D'Agostino A., Ehrmann M., *The Pricing of G7 Sovereign bond Spreads. The times, they are a-changin′*, European Central Bank, Working Paper Series, no 1520, March 2013.

[3] Di Cesare A., Grande G., Manna M., Taboga M., *Recent estimates of sovereign risk premia for euro-area countries*, Banca d'Italia, Questioni di Economia e Finanza, 128, 2012.

[4] EIOPA, *Technical Specification on the Long Term Guarantee Assessment (Part I)*, 28 January 2013.

[5] Elouerkhaoui Y.,*Marshall-Olkin Copula-Based Models*, in *The Oxford Handbook of Credit Derivatives*, Oxford University Press, Oxford, 2013.

[6] Frey R., McNeil A.J. *VaR and expected shortfall in portfolios of dependent credit risks: Conceptual and practical insights*, Journal of Banking & Finance, 26 (7) (2002), 1317-1334.

[7] Gundlach M., Lehrbass F., *CreditRisk$^+$ in the banking industry*, Springer, Berlin, 2004.

[8] Mai, J. and Scherer, M., *Reparameterizing Marshall-Olkin copulas with applications to high-dimensional sampling*, Journal of Statistical Computation and Simulation, 81(1) (2011), 59-78.

[9] Marshall A.W., Olkin I., *A generalized bivariate exponential distribution*, Journal of Applied Probability 4 (1967), 291-302.

[10] Marshall A.W., Olkin I., *A multivariate exponential distribution*, Journal of the American Statistical Association, 62 (1967), 30-44.

[11] McNeil A.J., Frey R., Embrechts P., *Quantitative Risk Management*, Princeton University Press, Princeton, 2005.

[12] Mottura C.D., Passalacqua L., *Default dependence structure effects on the valuation of government guarantees*, Roma Tre University, Collana del Dipartimento di Economia, working paper (2013), in print.

[13] Sun Y., Mendoza-Arriaga R., Linetsky V., *Marshall-Olkin multivariate exponential distributions. Multidimensional Lévy subordinators, efficient simulation and applications to credit risk*, working paper (2011).
Available at http://ssrn.com/abstract=1702087