A market consistent calibration of the Jarrow-Yildirim model

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Abstract

For some years now, financial institutions have been involved in several pricing and market consistent valuations for their assets and liabilities. In this regard, risk-neutral models have become more and more popular both in the banking and insurance business. The Jarrow-Yildirim model is the most famous risk-neutral model for inflation and it is the main reference technique adopted in the inflation market. At the same time, this model considers a one-factor process for the nominal short rate, real short rate and consumer price index. In this paper, we present a market consistent calibration of the Jarrow-Yildirim model on Euro market data, such as year-on-year inflation-indexed swaps and inflation-indexed caps.

Keywords: Jarrow-Yildirim model; Risk-neutral probabilities; Inflation-indexed derivatives; Market consistent calibration

1 Introduction

During the last decades, risk-neutral scenarios have spread in all the financial business. Banking and insurance companies deal very often with risk-neutral probabilities and sometimes they are requested to use them by the law. For instance, according to Solvency II (European Parliament and Council of the European Union, 2009) insurance companies are required to build a market consistent balance sheet using risk-neutral probabilities. On the other hand, according to IFRS 17 (International Accounting Standards Board, 2017) they are required to calculate market consistent technical provisions, which will be likely carried out using risk-neutral probabilities as well.

In this regard, the financial literature gave a lot of attention to the pricing of interest-rate derivatives through interest-rate models (Andersen and Piterbarg, 2010; Brigo and Mercurio, 2006). These models are distinguished in two main categories, i.e. the equilibrium models and the arbitrage-free models. Equilibrium models produce a term structure as output and hence they do not match the current term structure, observed in the market. Arbitrage-free models take the observed term structure as an input and hence they match the current term structure, observed in the market. Some well-known equilibrium models have been introduced by Vasicek (1977), Cox et al (1985) and Duffie and Kan (1996), and some well-known arbitrage-free models have been introduced by Hull and White (1990) and Heath et al (1992).

During the last months, inflation has reached a high level and volatility. For this reason, we can not neglect it anymore, if we want to make reasonable market consistent valuations. This is especially true for those institutions with a large exposure to inflation-indexed derivatives, e.g. life insurance companies selling guaranteed contracts covered by inflation-indexed financial instruments. Consequently, considering inflation and an inflation model, the income distribution and the time value of options and guarantees of the contracts could drastically change.

In this paper, we give our contribution to the literature about risk-neutral models for inflation, because relative little attention has been given to them over the past years. Dealing with risk-neutral probabilities for inflation is a very challenging task, because the risk-neutrality concept is related to the time value of money and inflation is just the conversion instrument to pass from the nominal value to the real one or vice versa. The most famous inflation model has been introduced by Jarrow and Yildirim (2003) and it is still the main reference technique adopted in the inflation market (Cipollini and Canty, 2013). Other popular models have been introduced by Mercurio (2005). The pricing formulas for inflation-indexed derivatives are here available and they are typically used for calibration purposes. The Jarrow-Yildirim model is a nominal risk-neutral arbitrage-free model that, at the same time, describes the nominal short rate, real short rate and consumer price index (CPI), using a one-factor process for each of them, so it is possible to derive the entire nominal, real and inflation term structures. The aim of this paper is to propose a market consistent calibration of the Jarrow-Yildirim model on Euro market data on December 31, 2021.

The paper is organized in the following way. In Section 2 we describe the Jarrow-Yildirim model and in Section 3 we present the main inflation-indexed derivatives and their pricing formulas. In Section 4, we propose a numerical example in which we calibrate the model. Finally, in Section 5 we conclude the research.

2 Jarrow-Yildirim model

We assume that the market is frictionless, meaning that all securities are perfectly divisible and that there are no short-sale restrictions, transaction costs, or taxes. The security trading is continuous and no riskless arbitrage opportunities are present.

We assume that the nominal short rate follows a one-factor Gaussian model (i.e. G1++ model) that is given by:

$$n(t) = x_n(t) + \varphi_n(t)$$

where $x_n(t)$ is the nominal state variable and $\varphi_n(t)$ is a deterministic function of time that allows the model to fit perfectly the nominal term structure observed in the market.

The nominal state variable under the nominal risk-neutral measure Q_n satisfies the following stochastic dynamic:

$$dx_n(t) = -a_n\, x_n(t)\, dt + \sigma_n\, dW_n^{x_n}(t) \quad \text{with} \quad x_n(0) = 0$$

where a_n and σ_n are positive constants and $W_n^{x_n}(t)$ is a standard Brownian motion.

The deterministic function of time that allows the model to fit perfectly the nominal term structure observed in the market is given by:

$$\varphi_n(t) = f_n^M(0,t) + \frac{\sigma_n^2}{2a_n^2}\left(1 - e^{-a_n t}\right)^2$$

where $f_n^M(0,t)$ is the instantaneous forward rate at initial time for the maturity t implied by the nominal term structure observed in the market.

The stochastic dynamic above admits an explicit solution:

$$x_n(t) = x_n(s)\, e^{-a_n(t-s)} + \sigma_n \int_s^t e^{-a_n(t-u)}\, dW_n^{x_n}(u)$$

Hence, the nominal state variable under the nominal risk-neutral measure Q_n and conditional on the sigma-field $\mathcal{F}_s$ is normally distributed, with mean:

$$x_n(s)\, e^{-a_n(t-s)}$$

and variance:

$$\frac{\sigma_n^2}{2a_n}\left(1 - e^{-2a_n(t-s)}\right)$$

The price at time t (conditional on the sigma-field $\mathcal{F}_t$) of a nominal zero-coupon bond with maturity in $T > t$ is thus found to be:

$$P_n(t,T) = \exp\left\{ -\int_t^T \varphi_n(u)\, du - \frac{1 - e^{-a_n(T-t)}}{a_n}\, x_n(t) + \frac{1}{2}\, V_n(t,T) \right\}$$

The integral admits an explicit solution:

$$\exp\left\{-\int_t^T \varphi_n(u)\,du\right\} = \frac{P_n^M(0,T)}{P_n^M(0,t)}\,\exp\left\{-\frac{1}{2}\,V_n(0,T) + \frac{1}{2}\,V_n(0,t)\right\}$$

where $P_n^M(0,t)$ is the price at initial time of a zero-coupon bond with maturity in t implied by the nominal term structure observed in the market. Moreover, we have:

$$V_n(t,T) = \frac{\sigma_n^2}{a_n^2}\left[T - t + \frac{2}{a_n}\,e^{-a_n(T-t)} - \frac{1}{2a_n}\,e^{-2a_n(T-t)} - \frac{3}{2a_n}\right]$$

We assume that also the real short rate follows a G1++ model that is given by:

$$r(t) = x_r(t) + \varphi_r(t)$$

where $x_r(t)$ is the real state variable and $\varphi_r(t)$ is a deterministic function of time that allows the model to fit perfectly the real term structure observed in the market.

The real state variable under the real risk-neutral measure Q_r satisfies the following stochastic dynamic:

$$dx_r(t) = -a_r\,x_r(t)\,dt + \sigma_r\,dW_r^{x_r}(t) \qquad \text{with} \quad x_r(0) = 0$$

where a_r and σ_r are positive constants and $W_r^{x_r}(t)$ is a standard Brownian motion. We assume that the instantaneous correlation between the nominal and real state variables is given by $-1 \leq \rho_{x_n,x_r} \leq 1$.

The deterministic function of time that allows the model to fit perfectly the real term structure observed in the market is given by:

$$\varphi_r(t) = f_r^M(0,t) + \frac{\sigma_r^2}{2a_r^2}\left(1 - e^{-a_r t}\right)^2$$

where $f_r^M(0,t)$ is the instantaneous forward rate at initial time for the maturity t implied by the real term structure observed in the market.

The explicit solution, mean and variance of the real state variable and the price of a real zero-coupon bond are analogous to the nominal case and they can be found by replacing the sub and superscripts n with r.

The CPI under the nominal risk-neutral measure Q_n satisfies the following stochastic dynamic:

$$dI(t) = (n(t) - r(t))\,I(t)\,dt + \sigma_I\,I(t)\,dW_n^I(t) \qquad \text{with} \quad I(0) = I_0$$

where σ_I and I_0 are positive constants and $W_n^I(t)$ is a standard Brownian motion. We assume that the instantaneous correlation between the CPI and real state variable is given by $-1 \leq \rho_{x_r,I} \leq 1$.

The stochastic dynamic above admits an explicit solution:

$$I(t) = I(s)\,\exp\left\{\int_s^t (n(u) - r(u))\,du - \frac{\sigma_I^2}{2}\,(t-s) + \sigma_I\left(W_n^I(t) - W_n^I(s)\right)\right\}$$

We can apply the change-of-numeraire technique and we obtain that the real state variable under the nominal risk-neutral measure Q_n satisfies the following stochastic dynamic:

$$dx_r(t) = (-a_r\,x_r(t) - \rho_{x_r,I}\,\sigma_r\,\sigma_I)\,dt + \sigma_r\,dW_n^{x_r}(t)$$

where the standard Brownian motion keeps the same correlation structure described above.

3 Inflation-indexed derivatives

3.1 Zero-coupon inflation-indexed swap

In a zero-coupon inflation-indexed swap (ZCIIS), at the final time T_M (assuming $T_M = M$ years), one party pays a fixed amount:

$$N\left[(1+K)^M - 1\right] \tag{1}$$

where N is the nominal value and K is the fixed rate of the contract.

In exchange for the fixed payment, at the final time T_M, another party pays a floating amount:

$$N\left[\frac{I(T_M)}{I_0} - 1\right]$$

The no-arbitrage price at time t, $0 \le t < T_M$, of the ZCIIS floating leg under the nominal risk-neutral measure Q_n is given by:

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N\,E_n\left\{e^{-\int_t^{T_M} n(u)du}\left[\frac{I(T_M)}{I_0} - 1\right] \mid \mathcal{F}_t\right\}$$

By the foreign-currency analogy, for each $t < T$, we have the following relation:

$$I(t)\,P_r(t,T) = I(t)\,E_r\left\{e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right\} = E_n\left\{e^{-\int_t^T n(u)du}\,I(T) \mid \mathcal{F}_t\right\}$$

Therefore, we have:

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N\left[\frac{I(t)}{I_0}\,P_r(t, T_M) - P_n(t, T_M)\right] \tag{2}$$

which at time $t = 0$ simplifies to:

$$\mathbf{ZCIIS}(0, T_M, N) = N\left[P_r(0, T_M) - P_n(0, T_M)\right]$$

We can use ZCIISs to easily derive the real term structure. Let $K = K(T_M)$ be the fixed rate of the contract for a given maturity T_M. The nominal discounted value of equation 1 shall be equal to equation 2, so that the price at time $t = 0$ of a real zero-coupon bond with maturity in T_M is found to be:

$$P_r(0, T_M) = P_n(0, T_M)\,(1 + K(T_M))^M \tag{3}$$

We have shown that the price of a ZCIIS does not depend on the assumptions on the evolution of the interest-rate market.

3.2 Year-on-year inflation-indexed swap

Given a set of payment dates $T_1, ..., T_M$, in a year-on-year inflation-indexed swap (YYIIS), at each time T_i, one party pays a fixed amount:

$$N\,\varphi_i\,K$$

where N is the nominal value, K is the fixed rate of the contract and φ_i is the contract fixed-leg year fraction between T_i and T_{i-1} (hence we have that $\varphi_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

In exchange for the fixed payment, at each time T_i, another party pays a floating amount:

$$N\,\psi_i\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]$$

where ψ_i is the contract floating-leg year fraction between T_i and T_{i-1} (hence we have that $\psi_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

The no-arbitrage price at time $t < T_{i-1}$ of the payoff at time T_i of the YYIIS floating leg under the nominal risk-neutral measure Q_n is found to be:

$$\begin{aligned}\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\,\psi_i\,E_n\left\{e^{-\int_t^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] \mid \mathcal{F}_t\right\} \\ &= N\,\psi_i\,E_n\left\{e^{-\int_t^{T_{i-1}} n(u)du}\,P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\right\} - N\,\psi_i\,P_n(t, T_i)\end{aligned}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_{i-1}}$ we have:

$$\begin{aligned}\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = {} & N\,\psi_i\,P_n(t, T_{i-1})\,E_n^{T_{i-1}}\{P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\} \\ & - N\,\psi_i\,P_n(t, T_i)\end{aligned}$$

The expected value above depends on the assumptions on the evolution of the interest-rate market, because real rates are stochastic. According to the Jarrow-Yildirim model, we have:

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\,\psi_i\,P_n(t, T_{i-1})\,\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}\,e^{C(t, T_{i-1}, T_i)}$$

$$- N \psi_i P_n(t, T_i)$$

where:

$$C(t, T_{i-1}, T_i) = \sigma_r B(a_r, T_{i-1}, T_i) \Bigg[B(a_r, t, T_{i-1}) \left(\rho_{x_r, I} \sigma_I - \frac{\sigma_r}{2} B(a_r, t, T_{i-1}) \right.$$
$$\left. + \frac{\rho_{x_n, x_r} \sigma_n}{a_n + a_r} \left(1 + a_r B(a_n, t, T_{i-1}) \right) - \frac{\rho_{x_n, x_r} \sigma_n}{a_n + a_r} B(a_n, t, T_{i-1}) \right) \Bigg]$$

If real rates were not stochastic, the parameter σ_r would be equal to zero and the correction term C would be null.

3.3 Inflation-indexed cap and floor

An inflation-indexed cap (IIC) is a call option, which depends on the CPI. The corresponding put option is an inflation-indexed floor (IIF).

A year-on-year inflation-indexed cap or floor (YYIICF) can be decomposed in a stream of year-on-year inflation-indexed caplets or floorlets (YYIICFlts) with the set of payment dates $T_1, ..., T_M$. Their payoff at time T_i is given by:

$$N \zeta_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \quad \text{with} \quad K = 1 + \kappa$$

where $\omega = 1$ ($\omega = -1$) for a cap (floor), N is the nominal value, κ is the strike rate of the contract and ζ_i is the year fraction between T_i and T_{i-1} (hence we have that $\zeta_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers).

The no-arbitrage price at time $t \leq T_{i-1}$ of the YYIICFlt payoff at time T_i under the nominal risk-neutral measure Q_n is given by:

$$\mathbf{YYIICFlt}(t, T_{i-1}, T_i, \zeta_i, K, N, \omega)$$
$$= N \zeta_i E_n \left\{ e^{-\int_t^{T_i} n(u) du} \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \mid \mathcal{F}_t \right\}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_i}$ we have:

$$\mathbf{YYIICFlt}(t, T_{i-1}, T_i, \zeta_i, K, N, \omega)$$
$$= N \zeta_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \mid \mathcal{F}_t \right\} \qquad (4)$$

We can observe that the expected value above depends on the assumptions on the evolution of the interest-rate market, because nominal and real rates are stochastic.

The ratio $I(T_i)/I(T_{i-1})$ (i.e. CPI ratio) under the nominal forward measure $Q_n^{T_i}$ and conditional on the sigma-field $\mathcal{F}_t$ is lognormally distributed. For this reason, equation 4 can be solved using the expected value of the CPI ratio and the variance of its logarithm. Let X be a lognormal random variable with $E(X) = m$ and $\mathrm{Std}(\ln X) = v$, we thus have:

$$E\big[(\omega\,(X-K))^+\big] = \omega\, m\, \Phi\left(\omega\,\frac{\ln\frac{m}{K}+\frac{v^2}{2}}{v}\right) - \omega\, K\, \Phi\left(\omega\,\frac{\ln\frac{m}{K}-\frac{v^2}{2}}{v}\right) \tag{5}$$

where Φ is the standard normal cumulative distribution function.

According to the Jarrow-Yildirim model, the expected value of the CPI ratio is given by:

$$E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\right\} = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\,\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\,e^{C(t,T_{i-1},T_i)}$$

Furthermore, the variance of the logarithm of the CPI ratio is given by:

$$\begin{aligned}
\mathrm{Var}_n^{T_i}\left\{\ln\frac{I(T_i)}{I(T_{i-1})} \mid \mathcal{F}_t\right\} &= \frac{\sigma_n^2}{2a_n^3}\left(1-e^{-a_n\zeta_i}\right)^2\left(1-e^{-2a_n(T_{i-1}-t)}\right) \\
&+ \frac{\sigma_n^2}{a_n^2}\left[\zeta_i + \frac{2}{a_n}\,e^{-a_n\zeta_i} - \frac{1}{2a_n}\,e^{-2a_n\zeta_i} - \frac{3}{2a_n}\right] \\
&+ \frac{\sigma_r^2}{2a_r^3}\left(1-e^{-a_r\zeta_i}\right)^2\left(1-e^{-2a_r(T_{i-1}-t)}\right) \\
&+ \frac{\sigma_r^2}{a_r^2}\left[\zeta_i + \frac{2}{a_r}\,e^{-a_r\zeta_i} - \frac{1}{2a_r}\,e^{-2a_r\zeta_i} - \frac{3}{2a_r}\right] + \sigma_I^2\,\zeta_i \\
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r\,(a_n+a_r)}\left(1-e^{-a_n\zeta_i}\right)\left(1-e^{-a_r\zeta_i}\right)\left(1-e^{-(a_n+a_r)(T_{i-1}-t)}\right) \\
&- \frac{2\rho_{x_n,x_r}\,\sigma_n\,\sigma_r}{a_n\,a_r}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n} - \frac{1-e^{-a_r\zeta_i}}{a_r} + \frac{1-e^{-(a_n+a_r)\zeta_i}}{a_n+a_r}\right] \\
&+ \frac{2\rho_{x_n,I}\,\sigma_n\,\sigma_I}{a_n}\left[\zeta_i - \frac{1-e^{-a_n\zeta_i}}{a_n}\right] - \frac{2\rho_{x_r,I}\,\sigma_r\,\sigma_I}{a_r}\left[\zeta_i - \frac{1-e^{-a_r\zeta_i}}{a_r}\right]
\end{aligned}$$

In a zero-coupon inflation-indexed cap or floor (ZCIICF), at the final time T_M (assuming $T_M = M$ years), the payoff is given by:

$$N\left[\omega\left(\frac{I(T_M)}{I_0} - K\right)\right]^+ \quad \text{with} \quad K = (1+\kappa)^M$$

The no-arbitrage price at time t, $0 \leq t < T_M$, of the ZCIICF payoff under the nominal risk-neutral measure Q_n is given by:

$$\begin{aligned}&\mathbf{ZCIICF}(t, T_M, I_0, K, N, \omega)\\&\quad = N\, E_n\left\{ e^{-\int_t^{T_M} n(u)du} \left[\omega \left(\frac{I(T_M)}{I_0} - K \right) \right]^+ \mid \mathcal{F}_t \right\}\end{aligned}$$

We apply the change-of-numeraire technique, so that under the nominal forward measure $Q_n^{T_M}$ we have:

$$\begin{aligned}&\mathbf{ZCIICF}(t, T_M, I_0, K, N, \omega)\\&\quad = N\, P_n(t, T_M)\, E_n^{T_M}\left\{ \left[\omega \left(\frac{I(T_M)}{I_0} - K \right) \right]^+ \mid \mathcal{F}_t \right\}\end{aligned}$$

The CPI ratio related to the ZCIICF under the nominal forward measure $Q_n^{T_M}$ and conditional on the sigma-field $\mathcal{F}_t$ is again lognormally distributed and the solution form is the same as in equation 5. The expected value of the CPI ratio is now given by:

$$E_n^{T_M}\left\{ \frac{I(T_M)}{I_0} \mid \mathcal{F}_t \right\} = \frac{I(t)}{I_0}\, E_n^{T_M}\left\{ \frac{I(T_M)}{I(t)} \mid \mathcal{F}_t \right\}$$

Moreover, the variance of the logarithm of the CPI ratio is now given by:

$$\mathrm{Var}_n^{T_M}\left\{ \ln \frac{I(T_M)}{I_0} \mid \mathcal{F}_t \right\} = \mathrm{Var}_n^{T_M}\left\{ \ln \frac{I(T_M)}{I(t)} \mid \mathcal{F}_t \right\}$$

We can easily obtain analytical solutions for the expected value and variance above, by replacing T_i with T_M and T_{i-1} with t (consequently ζ_i becomes the year fraction between T_M and t) in the formulas about the YYIICFlt.

4 Calibration

In this section, we calibrate the Jarrow-Yildirim model on Euro market data on December 31, 2021, in order to obtain a market consistent result. We assume that the interest-rate swap term structure is our reference risk-free nominal interest-rate curve (see Table 1). We instead derive our reference risk-free real interest-rate curve using ZCIIS fixed rates and equation 3.

We firstly calibrate the nominal parameters that we will use as an input to calibrate the remaining ones. In this regard, we now look for the set of parameters that minimizes the sum of squared differences between market and model nominal interest-rate derivative prices. The optimization problem can

Table 1 Risk-free nominal and real term structures (expressed in %) at December 31, 2021

Maturity	Nominal rate	Real rate
1y	−0.488	−3.826
2y	−0.299	−2.859
3y	−0.150	−2.452
5y	0.015	−2.107
7y	0.128	−1.925
10y	0.302	−1.727
15y	0.496	−1.598
20y	0.552	−1.589

be formalized as follows:

$$\text{argmin}_{a_n,\sigma_n} \sum_i \left(\text{price}_i^{\text{market}} - \text{price}_i^{\text{model}}\right)^2$$

The main nominal interest-rate derivatives are the interest-rate cap or floor (CF) and the European payer or receiver swaption (ES). These instruments are typically quoted in terms of their volatility.

According to the Bachelier's model (Bachelier, 1900), the price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a CF is given by:

$$\begin{aligned}
&\mathbf{CF}^{\text{Bachelier}}(t, \mathcal{T}, \tau, N, X, \sigma_{\alpha,\beta}) \\
&= \sum_{i=\alpha+1}^{\beta} \left[\omega \left(\frac{P_n(t,T_{i-1}) - P_n(t,T_i)}{\tau_i \, P_n(t,T_i)} - X \right) \Phi\left(\omega \, \frac{\frac{P_n(t,T_{i-1}) - P_n(t,T_i)}{\tau_i \, P_n(t,T_i)} - X}{\sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}} \right) \right. \\
&\quad \left. + \sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}\,\phi\left(\frac{\frac{P_n(t,T_{i-1}) - P_n(t,T_i)}{\tau_i \, P_n(t,T_i)} - X}{\sigma_{\alpha,\beta}\sqrt{T_{i-1}-t}} \right) \right] N \, \tau_i \, P_n(t,T_i)
\end{aligned}$$

where $\omega = 1$ ($\omega = -1$) for a cap (floor), $\mathcal{T} = \{T_\alpha, ..., T_\beta\}$ is the set of payment and/or reset dates, $\tau = \{\tau_{\alpha+1}, ..., \tau_\beta\}$ is the set of corresponding year fractions, meaning that τ_i is the year fraction between T_i and T_{i-1} (hence we have that $\tau_i = T_i - T_{i-1}$ when T_i and T_{i-1} are real numbers), N is the nominal value, X is the strike rate of the contract and $\sigma_{\alpha,\beta}$ is the volatility parameter for the CF. Moreover, Φ and ϕ are respectively the standard normal cumulative and probability distribution functions.

The price at time $t \leq T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a CF under the G1++ model is given by (Brigo and Mercurio, 2006):

$$\begin{aligned}
&\mathbf{CF}^{\text{G1++}}(t, \mathcal{T}, \tau, N, X) \\
&= \sum_{i=\alpha+1}^{\beta} \omega \left[P_n(t,T_{i-1}) \, N \, \Phi\left(\omega \, \frac{\ln \frac{P_n(t,T_{i-1})}{(1+X\,\tau_i)\,P_n(t,T_i)} + \frac{1}{2}\,\Sigma(t,T_{i-1},T_i)^2}{\Sigma(t,T_{i-1},T_i)} \right) \right.
\end{aligned}$$

$$- N\,(1 + X\,\tau_i)\,P_n(t,T_i)\,\Phi\left(\omega\,\frac{\ln\frac{P_n(t,T_{i-1})}{(1+X\,\tau_i)\,P_n(t,T_i)} - \frac{1}{2}\,\Sigma(t,T_{i-1},T_i)^2}{\Sigma(t,T_{i-1},T_i)}\right)\Bigg]$$

where:

$$\Sigma(t,T_{i-1},T_i)^2 = \frac{\sigma_n^2}{2a_n^3}\left(1 - e^{-a_n\tau_i}\right)^2\left(1 - e^{-2a_n(T_{i-1}-t)}\right)$$

According to the Bachelier's model (Bachelier, 1900), the price at time $t \le T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a ES is given by:

$$\begin{aligned}
&\mathbf{ES}^{\text{Bachelier}}(t,\mathcal{T},\tau,N,X,\sigma_{\alpha,\beta}) \\
&= \Bigg[\omega\left(\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X\right)\Phi\left(\omega\,\frac{\frac{P_n(t,T_\alpha)-P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X}{\sigma_{\alpha,\beta}\sqrt{T_\alpha - t}}\right) \\
&\quad + \sigma_{\alpha,\beta}\sqrt{T_\alpha - t}\,\phi\left(\frac{\frac{P_n(t,T_\alpha)-P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X}{\sigma_{\alpha,\beta}\sqrt{T_\alpha - t}}\right)\Bigg]\sum_{i=\alpha+1}^{\beta} N\,\tau_i\,P_n(t,T_i)
\end{aligned}$$

where $\omega = 1$ ($\omega = -1$) for a payer (receiver) swaption and $\sigma_{\alpha,\beta}$ is the volatility parameter for the ES.

According to Schrager and Pelsser (2006), an approximation of the price at time $t \le T_\alpha$ (conditional on the sigma-field $\mathcal{F}_t$) of a ES under the G1++ model is given by:

$$\begin{aligned}
&\mathbf{ES}^{\text{G1++}}(t,\mathcal{T},\tau,N,X) \\
&= \Bigg[\omega\left(\frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X\right)\Phi\left(\omega\,\frac{\frac{P_n(t,T_\alpha)-P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X}{S(t,T_\alpha,T_\beta)}\right) \\
&\quad + S(t,T_\alpha,T_\beta)\,\phi\left(\frac{\frac{P_n(t,T_\alpha)-P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - X}{S(t,T_\alpha,T_\beta)}\right)\Bigg]\sum_{i=\alpha+1}^{\beta} N\,\tau_i\,P_n(t,T_i)
\end{aligned}$$

where:

$$\begin{aligned}
&S(t,T_\alpha,T_\beta)^2 \\
&= \frac{\sigma_n^2}{2a_n^3}\left(e^{2a_n(T_\alpha-t)} - 1\right)\Bigg[\frac{e^{-a_n(T_\alpha-t)}\,P_n(t,T_\alpha)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} - \frac{e^{-a_n(T_\beta-t)}\,P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)} \\
&\quad - \frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)}\sum_{i=\alpha+1}^{\beta}\frac{\tau_i\,e^{-a_n(T_i-t)}\,P_n(t,T_i)}{\sum_{i=\alpha+1}^{\beta}\tau_i\,P_n(t,T_i)}\Bigg]^2
\end{aligned}$$

In this numerical analysis, we use derived at-the-money (ATM) interest-rate cap prices with maturity from one to twenty years (see Table 2) and derived

Table 2 ATM interest-rate cap prices (expressed in %) at December 31, 2021

Maturity	Market price
1y	0.05
2y	0.32
3y	0.70
5y	1.89
7y	3.28
10y	5.74
15y	10.09
20y	14.55

Table 3 ATM European payer swaption prices (expressed in %) at December 31, 2021

Maturity / Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	0.19	0.44	0.70	0.96	1.22	1.48	1.74	2.00	2.25	2.51
2y	0.35	0.72	1.08	1.43	1.76	2.12	2.48	2.82	3.16	3.49
3y	0.47	0.94	1.38	1.79	2.18	2.60	3.02	3.43	3.82	4.21
5y	0.60	1.20	1.76	2.29	2.80	3.32	3.84	4.33	4.81	5.27
7y	0.69	1.38	2.03	2.66	3.26	3.85	4.43	4.98	5.53	6.06
10y	0.77	1.54	2.27	2.98	3.66	4.33	5.00	5.65	6.28	6.91

ATM European payer swaption prices with maturity and tenor combination from one to ten years (see Table 3).

Given the nominal parameters, the remaining ones are calibrated looking for the set of parameters that minimizes the squared differences between market and model inflation-indexed derivative quotes. The optimization problem can be formalized as follows:

$$\text{argmin}_{a_r,\sigma_r,\rho_{x_n,x_r},\sigma_I,\rho_{x_n,I},\rho_{x_r,I}} \sum_i \left(\text{quote}_i^{\text{market}} - \text{quote}_i^{\text{model}}\right)^2$$

In this numerical analysis, we use YYIIS fixed rates with maturity from one to twenty years (see Table 4) and IIC prices with different strike rates and with maturity from one to twenty years (see Table 5).

The calibrated parameters are shown in Table 6 and the differences between resulting model quotes and market quotes are shown in Figure 1, 2, 3 and 4.

We can observe that the calibration is overall well performed, because the errors are acceptable. In the cases of interest-rate caps and European payer swaptions, the absolute value of the differences between market and model prices is always lower than 0.25% and 0.15% respectively and its maximums are found near to the shortest or longest maturities. In the cases of YYIISs and IICs, the absolute value of the differences between market and model quotes is always lower than 0.10% and 1.50% respectively and its maximums are found near to the longest maturities.

Table 4 YYIIS fixed rates (expressed in %) at December 31, 2021

Maturity	Market rate
1y	3.470
2y	2.637
3y	2.360
5y	2.168
7y	2.094
10y	2.065
15y	2.126
20y	2.172

Table 5 IIC prices (expressed in %) at December 31, 2021

	Zero-coupon				Year-on-year			
Maturity / Strike	1.00%	2.00%	3.00%	4.00%	1.00%	2.00%	3.00%	4.00%
1y	2.49	1.49	0.59	0.11	2.49	1.49	0.59	0.11
2y	3.38	1.47	0.31	0.07	3.42	1.81	0.69	0.14
3y	4.32	1.59	0.32	0.08	4.36	2.13	0.80	0.19
5y	6.38	1.89	0.33	0.09	6.50	3.01	1.21	0.44
7y	8.58	2.39	0.45	0.13	8.84	4.13	1.81	0.82
10y	12.45	4.15	0.88	0.18	12.74	6.16	2.97	1.60
15y	20.44	7.16	1.58	0.33	19.99	9.95	4.98	2.89
20y	29.72	10.85	2.52	0.59	27.27	13.74	7.00	4.23

Table 6 Calibrated parameters at December 31, 2021

Parameter	Calibrated value
α_n	0.02007
σ_n	0.00711
α_r	0.15626
σ_r	0.01348
ρ_{x_n,x_r}	0.79816
σ_I	0.00989
$\rho_{x_n,I}$	−0.76074
$\rho_{x_r,I}$	−0.21617

The magnitude order of the errors in IICs is higher if compared to the other derivatives. This is because the IICs are the most parametrized instruments we have and because errors in absolute terms are affected by the magnitude order of the quotes.

In conclusion, the value of the errors in ZCIICs is smaller than in the corresponding YYIICs, because less elements are considered in their pricing formula.

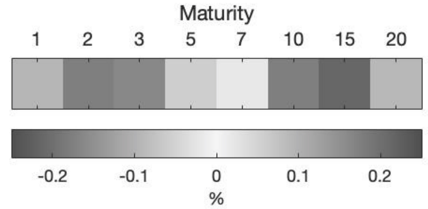

Fig. 1 ATM interest-rate cap model errors (expressed in %) at December 31, 2021

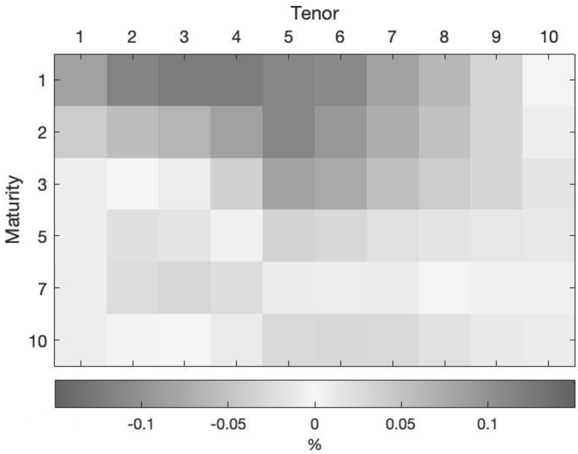

Fig. 2 ATM European payer swaption model errors (expressed in %) at December 31, 2021

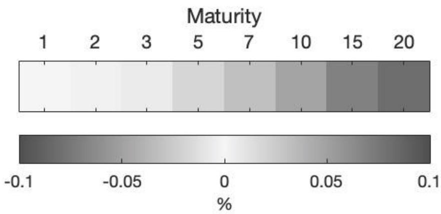

Fig. 3 YYIIS model errors (expressed in %) at December 31, 2021

5 Conclusion

In this paper, we described the Jarrow-Yildirim model, that is the most famous risk-neutral model for inflation. We then presented the main inflation-indexed derivatives, i.e. inflation-indexed swaps and inflation-indexed caps.

We finally proposed a numerical example in which we calibrated the model on Euro market data on December 31, 2021. As a consequence, our calibration procedure is market consistent and it could be used for valuation purposes. We determined the parameters using a two-step process. We firstly calibrated the nominal parameters on ATM interest-rate cap prices and ATM European

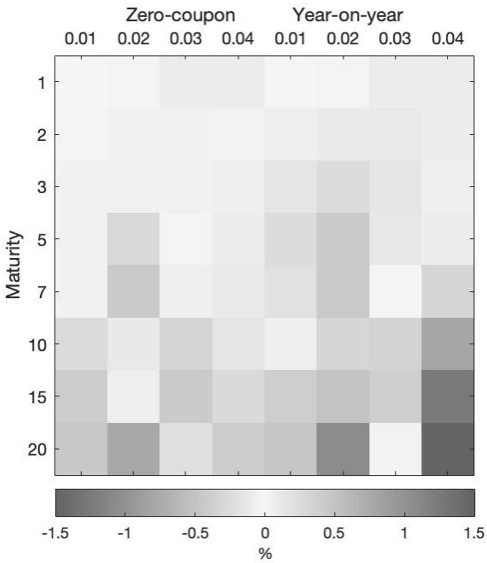

Fig. 4 IIC model errors (expressed in %) at December 31, 2021

payer swaption prices (both derived from their market volatilities). We then calibrated the remaining parameters on YYIIS market fixed rates and IIC market prices with different strike rates.

We observed that the differences between market and model quotes are quite small and the highest peaks are found in proximity to the shortest or longest maturities.

References

Andersen LBG, Piterbarg VV (2010) Interest rate modeling. Atlantic Financial Press

Bachelier L (1900) Théorie de la spéculation. Annales Scientifiques de l'École Normale Supérieure 17:21–86

Brigo D, Mercurio F (2006) Interest rate models: theory and practice - with smile, inflation and credit. Springer

Cipollini A, Canty P (2013) Inflation breakeven in the Jarrow and Yildirim model and resulting pricing formulas. Quantitative Finance 13(2):205–226

Cox JC, Ingersoll JE, Ross SA (1985) A theory of the term structure of interest rates. Econometrica 53(2):385–407

Duffie D, Kan R (1996) A yield-factor model of interest rates. Mathematical Finance 6(4):379–406

European Parliament, Council of the European Union (2009) Directive 2009/138/EC - on the taking-up and pursuit of the business of insurance and reinsurance (Solvency II)

Heath D, Jarrow R, Morton A (1992) Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. Econometrica 60(1):77–105

Hull JC, White AD (1990) Pricing interest rate derivative securities. The Review of Financial Studies 3(4):573–592

International Accounting Standards Board (2017) IFRS 17 insurance contracts

Jarrow R, Yildirim Y (2003) Pricing treasury inflation protected securities and related derivatives using an HJM model. Journal of Financial and Quantitative Analysis 38(2):337–359

Mercurio F (2005) Pricing inflation-indexed derivatives. Quantitative Finance 5(3):289–302

Schrager DF, Pelsser A (2006) Pricing swaptions and coupon bond options in affine term structure models. Mathematical Finance 16(4):673–694

Vasicek O (1977) An equilibrium characterization of the term structure. Journal of Financial Economics 5(2):177–188