Sustainability of Euro-system along with the Stability and Growth Pact

Bernardo Maggi

Department of Statistical Sciences, University of Rome La Sapienza, Faculty of Engineering, Informatics and Statistics

Abstract

We compute the probability of failure to comply with the Stability and Growth Pact (SGP) for the European countries by comparing the tax revenue requested by the SGP with that one which is socially admissible. Both are estimated with a logistic regression where the latter is represented by the corner point and the former is the latent variable satisfying both the efficiency criterion of the SGP, for the control of Public debt with tax revenue, and the rule of the balanced budget. We also take into account the effect of quantitative easing and of an exogenous stable public expenditure prefixed for policy purposes. We apply and adapt the Riccati equation to this stochastic setting in order to understand the stability properties of the benchmark model descending from the SGP.

Keywords: Public Debt and Tax revenue, Stochastic Control, Discrete Riccati Equation, Long-Run Equilibrium, Logistic Regression.

JEL classification: C61, C24, F36, H63.

1. Introduction

In this research we compute the probability of failure to comply with the Stability and Growth Pact (SGP) for the European countries on the base of the comparison between the tax revenue requested by the SGP and the threshold value socially admissible. Both are estimated with a logistic regression. The latter is represented by the corner point - the constant term. The former is the latent variable, obtained as solution of the efficiency criterion that satisfies the recent edition of the SGP for the control of the public debt with tax revenues in order to ensure the compliance with the rule of a balanced budget on average.We also take into account

Email address: `bernardo.maggi@uniroma1.it` (Bernardo Maggi)

the effect of quantitative easing and of a stable public expenditure prefixed at the level desired by policymakers. We formalize the efficiency criterion to which the SGP is inspired with a stochastic optimal control model whose solution represents the prescription of the SGP for tax revenue, public expenditure and public debt, being the first the control variable and the other two state variables.

We show that, if strictly anchored to the SGP and in a long-run phase of low output growth, the Eurosystem may run into troubles once the prescribed tax revenue becomes socially unsustainable. A fact that may be provoked by an unexpected, even though stationary, public expenditure, whose uncertainty is the source of instability of the optimal solution. We also show that the absence of an active monetary policy contributes to an even more critical condition, which justifies the recent European Central Bank (ECB) monetary interventions.[1]

The uncertainty we intend to underline consists in all the economic events not necessarily due to public sector choices but in any case of such an important public utility as to generate public expenses, and therefore to be unexpected for the public sector. In an unpredictable future period, governments may be requested to pay the amount of money necessary for providing the due services deemed important for the collectivity. In this regard, an exemplary case is that one occurred in Italy where the Constitutional court in the recent past judged in favor of refunding the interruption of both the long service bonus and the indexation of pensions for many billions of euros. Still, taking into consideration the large size of the private banks in the Euro zone compared to GDP, Obstfeld (2013) claims that the risk of instability may become serious if governments are forced to provide financial support to private banks in difficulty because of their public utility. As time passes, particularly for countries with low output growth and in absence of money creation, these expenses make the system more and more unstable and vulnerable since the required taxes will tend to become socially unsustainable.

Therefore, the adoption of one-shot policy corrective intervention to reduce public debt (for instance a sale of public goods by means of the financial law) will consist in running a new optimal program with a new initial condition for public debt, which is a palliative treatment

[1]These consist in the so called *quantitative easing* through which governments with budget constraint in financial needs are financed by ECB money allowances (or money creation).

for the achievement of a tax revenue path coherent with the SGP, being just a way to shift forward the problem of excessive tax revenues. Moreover, it may occur an additional risk of lowering the upper limit of the socially sustainable tax rate if the public services provided diminish (following the above example, as a consequence of the sale of public goods). Then, shared structural reforms should be adopted.

Before the European Monetary Union (EMU) foundation occurred with the adoption of the Euro currency, the analogue stability problem occurred under the different context of fixed exchange rate agreements among European countries, when public budgets were in trouble for the excessive taxation and increasing public debt (see among others Alesina et al. (1990), Obstfeld (1986), Obstfeld (1994)). The causal relation consisted in the fact that, if taxation and public debt could not be increased because were too high, additional unforeseen public expenditures should be financed by money creation, which provoked the devaluation of the national currencies and then the crisis of the fixed exchange rate agreement.

Currently, since the prescriptions of SGP should ensure the stability of public debt - as well as that of government deficit - and forbid new money issuance by side of national central banks, the only way to finance unpredictable expenses is to increase further taxation, which, if too high, may reach socially unacceptable levels and therefore engender a crisis of the European countries belonging to EMU, a situation which is referred to as systemic instability of the Euro zone (see Galina and Obstfeld (2014)).

More specifically, this research, for first to our knowledge, addresses the following points: 1) to formalize the SGP control problem accounting for the budget constraint of the European countries and at the same time for the expenditure variables devoted to stable fiscal policies (be they referred to output growth or to social targets); 2) to settle a methodology for assessing the stability, and therefore the dynamics, of the stochastic system composed by the variables of the budget constraint; 3) to settle and estimate a probability model of non complying with the SGP 4) to find a measure to assess and quantify the cost of managing public debt and government expenditure according to the SGP; 5) to study an optimal debt policy for the reimbursement of the debt under the outcomes implied by the SGP.

From the theoretical and methodological point of view, in alternative to the Lagrangian for solving the afore mentioned stochastic control problem, we propose a methodology based on the Riccati equation. A recursive matrix equation that allows to study in a more appropriate way the dynamics of the system. The solution of this equation is that matrix allowing to evaluate the social cost of the SGP implementation - *i.e.*, the value function - and is a useful tool for organizing additional control variables, for financing state variables such as those relative to policy targets. Here, besides the case of an overall spending, we extend this equation to a generic number of different sorts of expenditure and suggest how to introduce additional control variables - *i.e.*, specific taxes and/or categories of debt issuances - for financing them. Still, this solution allows derive the state matrix (filter) to study the dynamics of the state variables, public debt and expenditure, in accordance with the parameters of the model. Then, by comparing the optimal solution for the tax revenue with that one socially admissible, we develop a probability model in order to study the risk of noncompliance with the SGP. Finally, we prove that when an optimal debt-reimbursement policy is followed and the SGP is complied, the short term debt is stationary but not so the long-term one which is proved to be cointegrated with the tax revenue. This is a long-run equilibrium relation which, though optimal, may be dangerous in the absence of output growth for the necessity of collecting increasing tax revenue.

From the empirical point of view, we study the period 1984-2021 for the European countries and estimate the probability model theoretically developed. Still, we adapt the estimated probability to the quality of the public services provided. We find that in periods of crisis, characterized by low output growth, the risk of noncomplying with the SGP rises considerably especially for those countries with less effective public services, a fact that should suggest to pay more attention to the quality (and the relevance) of the public services compared with the costs borne by the community.

The paper is organized as follows. Section 2 formalizes the stochastic optimal control model that reflects the SGP prescriptions in terms of public debt and tax revenue. Section 3 identifies the probability of noncomplying with the SGP on the base of the results in Section 2. Section 4 estimates the probability model developed in the previous section and calculates the probability of noncomplying with the SGP for each European country in the sample. Also, in this section we evaluate the dynamics of the value function which defines the cost for managing the state variables. Section 5 studies a refunding plan for debt according to its maturity and finds the SGP long-run equilibrium condition for the long-term debt. Section 6 comments on the main results of the paper and gives suggestions for future research. Finally, Appendix A-Appendix C report some proofs.

2. Statement of the problem

In this section, we formalize a stochastic optimal control model reflecting the prescription of the SGP to minimize the cost both of collecting tax revenue and of the allocative inefficiency. Hence, without money issuance, the endogenous tax revenue is the control variable and the state variables are the endogenous public debt, the exogenous primary public expenditure and its exogenous constant components. The primary public expenditure is assumed exogenous under the stability requirement prescribed by SGP, which implies the nature of state variable. The public debt is, by its nature of stock, the variable to be controlled and therefore is an endogenous state variable. We move from Maggi (2023) and introduce among the state variables a constant term for the public expenditure, which plays a relevant role in country analysis since may be conceived as a specific exogenous economic policy variable. This apparently innocent additional variable complicates the model in that the order of the matrix equation, involved for representing the system dynamics, increases. We also notice that, though this kind of variable is in its essence experimental, it becomes actual after the implementation of a stable (constant) policy action, and then the model proposed may be useful for studying previously the consequences of a policy with simulations and forecasting of the control and state variables as well as of the probability to comply with the SGP. Furthermore, the methodology presented may be extended to a generic number of economic policy variables.

The public expenditure is supposed to follow a stable stochastic process with additive uncertainty (see Bray (1975), Phelps and Taylor (1977) and Sargent and Wallace (1975))

$$g_t = g_c + \rho g_{t-1} + \varepsilon_t, |\rho| < 1, E_t(\varepsilon_{t+1}) = 0, \tag{1}$$

$$\varepsilon_t \rightsquigarrow W.N.(0, \sigma_\varepsilon^2), \geq 0, g_c \geq 0, g_t \geq 0$$

where g_t is the primary public expenditure, ε_t is an idiosyncratic error, and g_c is the prefixed economic policy variable in terms of public expenditure above mentioned. g_t depends on its past value according to the parameter ρ which represents the effect of the past expenditure on the future periods.

The uncertainty on the public expenditure brings about uncertainty to public debt B_t which obeys to the following budget constraint equation without money issuance for the European countries

$$B_t - B_{t+1} + g_t + B_t r \equiv \tau_t, \tag{2}$$

$$B_t \geq 0,\ B_0 > 0,\ \tau_t \geq 0,\ \tau_0 > 0$$

where r is the average interest rate assumed constant according to the financial stability target of the European Monetary Union, and τ_t is the tax revenue.

Equation (2) shows how the g_t $AR(1)$ process affects the stock of public debt which is kept under control by tax revenue managed efficiently according to the prescriptions contained in the Stability and Growth Pact.

The efficiency criterion is formalized here by the minimization of a quadratic cost function $\Psi(\tau_t)$[2] representing the costs due to dead-weight loss (see Auerbach and Feldstein (1985)) and tax collection (see Barro (1979)).

By imposing the Solvency condition that the discounted expected current value of the public debt conditional on information at the initial time 0, E_0, is null whatever will be its value at infinite time, $B_{+\infty}$ (Hamilton and Flavin (1986))

$$\lim_{t \to +\infty} \delta^t E_0(B_t) = 0, \tag{3}$$

it is possible express the value function with a constraint in static form, *i.e.*, with a unique Lagrange multiplier (though stochastic) rather than with a sequence of costate variables for each

[2]In general, $\Psi(\tau_t)$ is nonnegative and strictly convex function, and may be assumed quadratic (see, among others Tanner and Carey (2005)).

period:

$$V_t(B_t, g_t) = MinE_t \left\{ \sum_{j=0}^{+\infty} \delta^j \Psi(\tau_{t+j}) + \lambda(S)[B_t - \sum_{j=0}^{+\infty} \delta^{j+1}(\tau_{t+j} - g_{t+j})] \right\} \tag{4}$$

given $g_{t+j}\ \forall j,\ B_t$.

In effect, in this case it is possible to prove (see Maggi (2023)) that the Transversality first order condition coincides with the Solvency condition, which ensures that the solution of (4) is optimal.

Since the solution of (4) is a martingala process because the first derivatives are linear, it comes out that

$$\tau_t = r[B_t + \sum_{j=0}^{+\infty} \delta^{j+1} E_t\big(\rho^j g_t + g_c \frac{1-\rho^j}{1-\rho} + \sum_{s=0}^{j-1} \rho^j \varepsilon_{t+j-s}\big)] \tag{5}$$

from which

$$\tau_t^* = rB_t + \frac{rg_t + g_c}{1+r-\rho}. \tag{6}$$

Importantly, being τ_t^* in formula (6) the amount of tax revenue prescribed on the basis of the conditional expectation of the future and current expenses at time t, what matters in this prevision is not only the initial value g_t but also all the other prefixed values from time t on, *i.e.*, g_c, r and ρ.

Also, first the functional form of $V_t(B_t, g_t)$ may be found as indicated in Appendix C and then the cost of managing the state variables may be computed after the estimation of the interest rate and the autoregressive coefficient of the public expenditure.

Now, we are going to prove that the dynamics of public debt turns out to be conditioned by g_c but not so that one of tax revenue which remains affected only for the initial conditions.

Theorem 1. *The dynamics of τ_t, obtained as a solution of problem (4), remains unaffected (apart from the initial condition) by introducing g_c as in (1). Instead, the dynamics of B_t will depend on g_c.*

Proof. As for taxation dynamics, from (6) we get

$$\tau_t^* - \tau_{t-1}^* = r(B_t - B_{t-1}) + \frac{r(g_t - g_{t-1})}{1+r-\rho} \tag{7}$$

and, from (2), after substituting the expression of rB_{t-1} derived from (6), we get

$$B_t = B_{t-1} + \tau_{t-1}^* - \frac{rg_{t-1}}{1+r-\rho} - \frac{g_c}{1+r-\rho} + g_{t-1} - \tau_{t-1}^*, \tag{8}$$

then, plugging it into B_t in (7), we obtain

$$\tau_t^* - \tau_{t-1}^* = \frac{r(g_t - \rho g_{t-1})}{1+r-\rho} - \frac{rg_c}{1+r-\rho} \quad \text{with} \quad g_t - \rho g_{t-1} = \varepsilon_t + g_c \tag{9}$$

from which

$$\tau_t^* = \tau_{t-1}^* + e_t, \; e_t = \frac{r}{1+r-\rho}\varepsilon_t \tag{10}$$

where e_t defines the residual term of the martingala solution. In expression (10), g_c is not present which proves that the dynamics of τ_t^* is unaffected by this term. However, introducing this constant expenditure, and recalculating the initial condition at $t-1$, makes τ_{t-1}^* increase by the term $\frac{g_c}{1+r-\rho}$ according to (6).

As for public debt, from (8) its dynamics is given by

$$B_t = B_{t-1} + \frac{g_{t-1}(1-\rho)}{1+r-\rho} - \frac{g_c}{1+r-\rho} \tag{11}$$

from which, by iterating back the public expenditure to $t = 0$, we obtain

$$B_t = B_{t-1} + g_0\rho^{t-1}\frac{1-\rho}{1+r-\rho} - g_c\rho^{t-1}\frac{1}{1+r-\rho} + \frac{1-\rho}{1+r-\rho}\sum_{j=0}^{t-2}\rho^j\varepsilon_{t-j-1} \tag{12}$$

and, by iterating back the public debt,

$$B_t = B_0 + \sum_{j=0}^{t-1} \rho^j g_0 \frac{1-\rho}{1+r-\rho} - \sum_{j=0}^{t-1} \rho^j \frac{g_c}{1+r-\rho} + \frac{1-\rho}{1+r-\rho} \sum_{s=1}^{t} \sum_{j=0}^{t-(s+1)} \rho^j \varepsilon_{t-s-j}$$

$$B_t = B_0 + \frac{g_0}{1+r-\rho} - \frac{\rho^t g_0}{1+r-\rho} - \frac{g_c}{(1-\rho)(1+r-\rho)} + \frac{\rho^t g_c}{(1-\rho)(1+r-\rho)} + \frac{1-\rho}{1+r-\rho} \sum_{s=1}^{t} \sum_{j=0}^{t-(s+1)} \rho^j \varepsilon_{t-s-j} \tag{13}$$

but, given that $\frac{1-\rho}{1+r-\rho} \sum_{s=1}^{t} \sum_{j=0}^{t-(s+1)} \rho^j \varepsilon_{t-s-j} = \sum_{i=1}^{t-1} \frac{1-\rho^i}{1+r-\rho} \varepsilon_{t-i}$ and that, because of (6), $B_0 = \frac{\tau_0}{r} - \frac{g_0}{1+r-\rho} - \frac{g_c}{r(1+r-\rho)}$, we obtain

$$B_t = \frac{\tau_0}{r} + \rho^t \frac{g_c - g_0(1-\rho)}{(1-\rho)(1+r-\rho)} - \frac{g_c}{r(1-\rho)} + \sum_{i=1}^{t-1} \frac{1-\rho^i}{1+r-\rho} \varepsilon_{t-i}, \tag{14}$$

which proves that the dynamics of public debt is affected by g_c - through ρ^t - and that the initial condition changes for an additional constant term. □

Noticeably, from (12) and (14) we otain that in the presence of g_c (for all values) the public debt implied by the SGP prescriptions is a martingala process analogously with τ_t, (10), and therefore is nonstationary and integrated of order 1, $I(1)$.

However, from (14) it is possible to see that on average public debt has a path that tends to the stability represented by the horizontal asymptote corresponding to $\frac{\tau_0}{r} - \frac{g_c}{r(1-\rho)}$. Also, this horizontal asympote is above or below B_0 when the sign of the expression expression $g_c - g_0(1-\rho)$ is, respectively, negative or positive and accordingly the rate of increase of public debt is positive (and decreasing) or negative (and increasing).

Interestingly, expression (13), shows that g_c has a decreasing effect on B_t for lowering both its slope (the term $\rho^t \frac{g_c}{(1-\rho)(1+r-\rho)}$ has a negative first order derivative) and the horizontal asymptote. The explanation is the following. According to the optimal criterion (4), all current expenses are covered with the constant expected tax revenue at each time t (as indicated in (10)) plus the debt issuance. The former is given by the perpetuity of all the expected discounted expenses as indicated in (6), where g_t is financed by the optimal proportion $\frac{r}{(1+r-\rho)}$ and g_c is financed by $\frac{1}{(1+r-\rho)}$. Then, since the issuance of debt serves to ensure the constancy of taxes, it is equal to the complement to 1 of the above mentioned proportion applied to g_t, *i.e.*, $\frac{1-\rho}{(1+r-\rho)}$, less $\frac{g_c}{(1+r-\rho)}$ (as indicated in (11)) in order to respect the budget constraint (2).

Also, note that interest expenses are entirely financed by taxes in each period, which implies the *statement* that: *no issuance of debt is required for the financial expenses to meet the budget constraint*. The rationale, and the proof, of this result is in that the existence of a minimum and of a unique solution τ_t^* for problem (4) is ensured - respectively by the Weierstrass and the fixed point theorems applied to the quadratic functional form here used - $\forall\ g_c$, ρ and ε_t. Then, for all these terms equal to 0 the only term remaining in (6) is rB_t which becomes constant (as confirmed by (11) where the change in the debt is equal to 0 with these values) and guarantees that tax revenue is also constant as prescribed by the optimum in this case, thus showing that no debt issuance related specifically to the financial expenses is required.

Furthermore, Theorem 1 shows in which way program (4) follows the balanced budgetary rule requested by the SGP. In particular, once approached, the average stationary solution of g_t, $\frac{g_c}{1-\rho}$, brings about an expected zero public debt issuance as shown by (11) in such a case for ΔB_{t+1}. At the same time, expression (6) shows that tax revenues meet fully the budget constraint.[3]

We stress that even though this rule is respected by countries in the best way and the public expenditure is stationary, both tax revenue and public debt are nonstationary, an implication that will be deepened in Section 5. Then, if uncertainty persists, with equation (14) public debt displays always an unstable path also in the long run when public expenditure reaches the stationary solution, and so does tax revenue with equation (6). Explained in another way, the nonstationarity of the budget constraint (2) (due to the accumulation of uncertainty generated by the unitary coefficient of B_{t-1}) induces the nonstationarity of tax revenue though with the minimum cost thanks to program (4).

All these considerations are without taking into account output growth dynamics. Therefore, we can realize that the balanced budgetary rule cannot ensure the respect of the Maastricht parameters - notably, the 3% of the deficit (ΔB_t) out of GDP and the 60% of public debt, B_t, out of GDP - in case of prolonged recession.

We now show that that the same results may be obtained using the Riccati equation applied

[3]In fact, in the limit it should be: $\tau_t^* = rB_t + \frac{g_c}{(1+r-\rho)}$ with $g_t \to \frac{g_c}{(1+r-\rho)}$.

to this stochastic setting. We do this effort is because the proposed approach has the following several advantages: *a*) to frame more appropriately the term g_c for policy purposes by assigning to this term a specific equation, or considering several equations in case of multiple policy targets; *b*) in analogy with *a*), it allows to split g_t in different components for more specific analysis, and the same for B_t in order to distinguish between different types of funding according to maturity; *c*) to address more appropriately the stability and the stochastic properties of the benchmark model descending from the SGP.

In particular, here below we deepen the previous points *a*) and *c*). As for point *b*), this is an extension of point 1) whose specificity is left for future research, given the purposes of the present research.

Theorem 2. *Under condition* (3) *finding a solution to problem* (4) *is equivalent to solving the following LQR time variant stochastic problem (i.e., with matrices depending on time) with three state variables,* B_t, g_t, g_{c_t} *and one control variable* τ_t.

$$V_t(\mathbf{x}_t) = MinE_t\left\{\delta^{t+T}\mathbf{x}'_{t+T}\mathbf{Q}_{t+T}\mathbf{x}_{t+T} + \sum_{j=0}^{T-1}\delta^{t+j}\mathbf{x}'_{t+j}\mathbf{Q}_{t+j}\mathbf{x}_{t+j} + \sum_{j=0}^{T}\mathbf{u}'_{t+j}\mathbf{R}_{t+j}\mathbf{u}_{t+j}\right\}, \; T \to +\infty \tag{15}$$

sub

$$\begin{bmatrix} B_{t+1} \\ g_{t+1} \\ g_{c_{t+1}} \end{bmatrix} = \begin{bmatrix} (1+r) & 1 & 0 \\ 0 & \rho & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_t \\ g_t \\ g_{c_t} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tau_t + \begin{bmatrix} 0 \\ \varepsilon_{t+1} \\ 0 \end{bmatrix}, \tag{16}$$

where, $\mathbf{x}'_{t+j} \equiv [B_{t+j}, g_{t+j}, g_{c_{t+j}}]$, $\mathbf{u}_{t+j} \equiv \tau_{t+j}$, $\mathbf{Q}_{t+j} = \mathbf{0}$, $\mathbf{Q}_{t+T} > \mathbf{0}$, $\mathbf{R}_{t+j} \equiv \delta^j$, *and* $t+j$ *is the period split into the time of evaluation t and that one pertaining to the future periods j.*

Proof. First, realize that problem (15)-(16) corresponds to a time invariant LQR problem with $\mathbf{R} = 1$, $\mathbf{Q} = \mathbf{0}$ and $\mathbf{Q}_{t+T} > \mathbf{0}$ independent of time.

To see this, let us state for problem (15)-(16) the following Bellman equation that defines the Riccati matrix $\mathbf{P}_{t+j}$

$$\delta^{j-1}E_t[V_{t+j-1}(\mathbf{x}_{t+j-1})] = \delta^{j-1}E_t[\mathbf{x}'_{t+j-1}\mathbf{P}_{t+j-1}\mathbf{x}_{t+j-1}]$$
$$= min\{E_t[\delta^j V_{t+j}(\mathbf{x}_{t+j}) + \mathbf{u}'_{t+j-1}\delta^{j-1}\mathbf{u}_{t+j-1}]\}$$
$$= min\{E_t[\delta^j \mathbf{x}'_{t+j}\mathbf{P}_{t+j}\mathbf{x}_{t+j} + \mathbf{u}'_{t+j-1}\delta^{j-1}\mathbf{u}_{t+j-1}]\}. \; \forall\, t \tag{17}$$

By deriving (17) with respect to the control variable at each step $t+j$, taking into account the constraint (16), we get: I) $\mathbf{u}_{t+j-1} = \mathbf{K}_{t+j}\mathbf{x}_{t+j} = -(\mathbf{R}+\delta\mathbf{B}'\mathbf{P}_{t+j}\mathbf{B})^{-1}\delta\mathbf{B}'\mathbf{P}_{t+j}\mathbf{A}\mathbf{x}_{t+j-1}$, being $\mathbf{K}_{t+j}$ the time variant gain matrix; II) the - optimal - value function. Then, according to point II), it is possible to define the following discounted difference time variant Riccati matrix equation

$$\delta^{j-1}\mathbf{P}_{t+j-1} = -\delta^j\mathbf{A}'\mathbf{P}_{t+j}\mathbf{B}(\mathbf{R}_{t+j-1}+\delta^j\mathbf{B}'\mathbf{P}_{t+j}\mathbf{B})^{-1}\delta^j\mathbf{B}'\mathbf{P}_{t+j}\mathbf{A} + \delta^j\mathbf{A}'\mathbf{P}_{t+j}\mathbf{A}, \tag{18}$$

with

$$\mathbf{A} = \begin{bmatrix} (1+r) & 1 & 0 \\ 0 & \rho & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$\mathbf{B} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

where the symmetric matrix $\mathbf{P}_{t+j-1}$ on the l.h.s. is discounted by δ^{j-1} and the r.h.s. contains $\mathbf{R}_{t+j-1} = \delta^{j-1}$, $\forall$ initial time t. Interestingly, the absence of stochastic terms in (18) is due to the *Certainty Equivalence Principle* stated by Simon (1956) and Theil (1957) which consists in formalizing the effect of uncertainty in (17) with a function of a constant second order population moment that disappears after deriving (see Appendix C for the proof). It is also worth noticing that (18) accounts for the Transversality condition since the first derivatives of (17) are calculated also at $t = +\infty$ with infinite horizon (see Kamihigashi (2005)).

Now, after simplifying both sides of (18) by δ^{j-1}, we may get the same Riccati difference equation which is time invariant since it is possible to redefine $\mathbf{R} = 1$

$$\mathbf{P}_{t+j-1} = -\delta\mathbf{A}'\mathbf{P}_{t+j}\mathbf{B}(1+\delta\mathbf{B}'\mathbf{P}_{t+j}\mathbf{B})^{-1}\delta\mathbf{B}'\mathbf{P}_{t+j}\mathbf{A} + \delta\mathbf{A}'\mathbf{P}_{t+j}\mathbf{A} \tag{21}$$

which provides the following Discounted Discrete Algebraic Riccati Equation (DDARE):

$$\mathbf{P} = -\delta\mathbf{A}'\mathbf{P}\mathbf{B}(1+\delta\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\delta\mathbf{B}'\mathbf{P}\mathbf{A} + \delta\mathbf{A}'\mathbf{P}\mathbf{A}. \tag{22}$$

As a second step of the proof, it remains to show that the solution to the time invariant Riccati equation (21) converges to that one of (22), $\mathbf{P}^*$, which enables to find the following stationary gain matrix $\mathbf{K}^*$, and then the optimal control variable as a function of the state variables

$$\mathbf{u}_t^* = \mathbf{K}^*\mathbf{x}_t = -(\mathbf{R}+\delta\mathbf{B}'\mathbf{P}^*\mathbf{B})^{-1}\delta\mathbf{B}'\mathbf{P}^*\mathbf{A}\mathbf{x}_t. \tag{23}$$

We start by observing that equation (23) allows to study the dynamics of the optimal state variable vector given by

$$\mathbf{x}_{t+1} = (\mathbf{A}+\mathbf{B}\mathbf{K}^*)\mathbf{x}_t, \tag{24}$$

which defines the filter $\mathbf{A}+\mathbf{B}\mathbf{K}^*$.

Now, since may be proved (see Appendix A and Appendix B) that there exists a unique *strong* solution to (22), which means that $\mathbf{P}^*$ is *a*) nonnegative and *b*) the filter $\mathbf{A}+\mathbf{B}\mathbf{K}^*$ has at least one root on or inside the unit circle, the difference equation solution, consisting in the l.h.s. of (21), converges to $\mathbf{P}^*$ by virtue of Theorem 4.3 of Wah Chan et al. (1984). It is worth stressing that this result means the convergence of the difference Riccati equation solution to the strong solution of the DDARE and not necessarily the stationarity of the state variables, but in any case it implies that the value function (17) is real. In effect, public debt is in itself a nonstationary variable for the unstable mode $1+r$ of the state matrix $\mathbf{A}$, and in addition in Appendix B we prove that the strong solution is characterized by a filter $\mathbf{A}+\mathbf{B}\mathbf{K}^*$ with two roots on the unit circle of which one is referred to public debt, which therefore continues to be nonstationary - and then uncontrollable - even under the optimal rule of equation (23).[4] In Appendix A, besides confirming the symmetry, we prove that the unique strong solution $\mathbf{P}^*$ is positive in this specific problem. In Appendix C we prove that the value function (4) is real and that the same value is obtainable from (15), that is from $\mathbf{x}_t'\mathbf{P}^*\mathbf{x}_t$ plus a term quantifying uncertainty, which confirms the convergence of $\mathbf{P}_t$ to $\mathbf{P}^*$.

From Appendix A,

$$\mathbf{P}^* = \begin{bmatrix} r(1+r) & \frac{r(1+r)}{1+r-\rho} & \frac{1+r}{1+r-\rho} \\ \frac{r(1+r)}{1+r-\rho} & \frac{r(1+r)}{(1+r-\rho)^2} & \frac{1+r}{(1+r-\rho)^2} \\ \frac{1+r}{1+r-\rho} & \frac{1+r}{(1+r-\rho)^2} & \frac{1+r}{r(1+r-\rho)^2} \end{bmatrix}. \tag{25}$$

Substituting (25) in the following time invariant gain matrix related to equation (22)

$$\mathbf{K}^* = -(\mathbf{R} + \delta\mathbf{B}'\mathbf{P}^*\mathbf{B})^{-1}\delta\mathbf{B}'\mathbf{P}^*\mathbf{A} \tag{26}$$

we get

$$\mathbf{K}^* = \begin{bmatrix} r & \frac{r}{1+r-\rho} & \frac{1}{1+r-\rho} \end{bmatrix} \tag{27}$$

and consequently, being τ_t the control variable, from (27)

$$\tau_t^* = \mathbf{K}^*\mathbf{x}_t = \begin{bmatrix} r & \frac{r}{1+r-\rho} & \frac{1}{1+r-\rho} \end{bmatrix} \begin{bmatrix} B_t \\ g_t \\ g_{c_t} \end{bmatrix} = rB_t + \frac{rg_t}{1+r-\rho} + \frac{g_{c_t}}{1+r-\rho}. \tag{28}$$

which is the same as (6) obtained from (4), once the last constraint for g_{c_t} in (16) is accounted for.

□

Equation (28) shows that this analysis may be extendable to more specific expenditure and public debt state variables, being the former referable to either constant or variable policy ac-

[4] In Appendix B, since the system is unstabilizable, we also prove the existence of another nonnegative - not strong - solution in accordance with Theorem 3.1 of Wah Chan et al. (1984).

tions and the latter to the kind of debt - in terms of maturity and interest rates - that government choose for financing the several categories of expenditures. Accordingly, the number of rows and columns for the matrix **A** and possibly that of matrix **B** would change.[5]

3. Failure probability of complying with the SGP

3.1. Failure probability

In this section we evaluate the failure probability of complying with SGP both with and without the intervention of the ECB. In order to account for the effect of output path, y_t, we may define the average taxation rate as $\upsilon_t = \tau_t / y_t$. In the absence of monetary intervention and in the case of low output growth, the accumulation of the error term on the right hand side of (10) engenders the risk that the average tax rate may reach and overpass the upper limit, $\bar{\upsilon}$, socially tolerable. This means formally that $\upsilon_t \in [0, \bar{\upsilon})$. Working with the optimal solution, for simplicity' sake we omit in this section the star symbol. Hence if, from (6), the average tax rate implicit in the European SGP is defined as

$$\upsilon_t = \frac{rB_t}{y_t} + \frac{rg_t + g_c}{y_t(1 + r - \rho)} \tag{29}$$

the condition to avoid a systemic crisis is

$$\tau_t - \bar{\upsilon} y_t \leq 0 \tag{30}$$

which points out the level of the tax rate over which social conflicts and tensions would be so harsh to make the solution (6), and so the Eurosystem, difficult to be viable. Without a significant output growth in the long-run (*i.e.*, for $t \to +\infty$), this possibility becomes a certainty in our context, given the martingale process (10). In practice, this means that this possibility becomes more and more realistic. It is worth of notice that, in order to reach such a conclusion it just suffices the strict imposition of a stable $AR(1)$ public expenditure and that an even worse result should occur under more permissive (and maybe more plausible) hypotheses on g_t - i.e., υ_t

[5] In this case, the issuances of the new caegories of debt would actually play the role of new control variables of the model.

would surpass $\bar{\upsilon}$ in a shorter period of time. Therefore, it is possible to formulate the following theorem in terms of probability limit ($Plim$).

Theorem 3. *Given program* (4)*, in the absence of growth and even if g_t is a stationary process,* $Plim[\tau_t/y_t - \bar{\upsilon} \geq 0] = 1$.

Proof. This proof is straightforward since in the absence of output growth, $\Delta y_t = 0$, the denominator of υ_t in the $Plim[\tau_t/y_t - \bar{\upsilon} \geq 0]$ is irrelevant. Hence, being $\tau_t \rightsquigarrow I(1)$ with $\sigma_\tau^2 \to +\infty$ because of (10), it follows $0 \leq \tau_t \leq +\infty$, from which $Plim[\tau_t/y_t - \bar{\upsilon} \geq 0] = 1$. □

Theorem 3 formalizes that in an indefinite future period of time the system goes to failure in the absence of growth.

As for the case of the ECB monetary intervention, it consists in the financial support to governments that run into financial troubles - that is, in the present case, when the fiscal pressure necessary to respect the SGP is going to become socially unsustainable. A notable example of a monetary injection is the so called *Quantitative Easing*, through which many billions of euros are allocated in order to buy new government bonds. The following theorem states that even in such a case and in a persisting absence of output growth a systemic crisis is always possible.

Theorem 4. *Even if an attenuation of the Stability and Growth Pact is allowed by means of a monetary intervention, with maximum amount of monetary base out of output given by $\Delta\bar{m}$, the probability limit of* (31) *is* $Plim[(\tau_t/y_t) \geq \bar{\upsilon}]Plim[\frac{rB_t}{y_t} + \frac{rg_t+g_c}{y_t(1+r-\rho)} - \bar{\upsilon} \geq \Delta\bar{m}|(\tau_t/y_t) \geq \bar{\upsilon}] = Plim[\upsilon_t \geq \bar{\upsilon} + \Delta\bar{m}] = 1$.

Proof. If the possibility of a monetary intervention is allowed, the event of a systemic crisis in the long-run is to be evaluated by the limit of the composed probability that condition (30) occurs and that the maximum injection of monetary base provided by the ECB is not enough to cover the positive difference between the upper limit of the fiscal revenue socially sustainable, $\bar{\upsilon}y_t$, and the taxes collected. These two events are dependent and so the failure composed probability is given by

$$P[\tau_t/y_t \geq \bar{\upsilon}]P[\frac{rB_t}{y_t} + \frac{rg_t + g_c}{y_t(1+r-\rho)} - \bar{\upsilon} \geq \Delta\bar{m}|(\tau_t/y_t) \geq \bar{\upsilon}] \tag{31}$$

However, since under our hypotheses the events $\{\tau_t/y_t \geq \bar{\upsilon}\}$ and $\{\bar{\Delta m} = 0\}$ are disjoint it follows that $\{\bar{\Delta m} > 0\}$ implies $\{\tau_t/y_t \geq \bar{\upsilon}\}$, which means that $P[\{(\tau_t/y_t) \geq \bar{\upsilon}\} \bigcap \{\bar{\Delta m} = 0\}] = \emptyset$ and $P[\{(\tau_t/y_t) \geq \bar{\upsilon}\} \bigcup \{\upsilon_t \geq \bar{\upsilon} + \bar{\Delta m}\}] = P[\{(\tau_t/y_t) \geq \bar{\upsilon}\}$, from which equation (31) reduces to

$$P[\upsilon_t \geq \bar{\upsilon} + \bar{\Delta m}]. \tag{32}$$

The probability (32) tends to 1 by virtue of Theorem 3 as time grows indefinitely for the accumulation of unexpected public expenditure pushing tax revenue ratio unavoidably beyond $\bar{\Delta m}$ in the absence of output growth

$$Plim[\upsilon_t \geq \bar{\upsilon} + \bar{\Delta m}] = 1. \tag{33}$$

□

Then, this sort of financing is a temporary remedy even though the probability of a systemic crisis considered in Theorem 3 is undoubtedly reduced by the occurrence of $\{\bar{\Delta m} > 0\}$.

3.2. Discussion

These last results allow drawing some considerations on the rules imposed by the SGP. In particular, the problem of complying with the SGP is not much the attenuation of the strict rules on debt and public deficit but rather the reduction in the ambiguity deriving from laws and policy actions pertaining the public expenditure and so the related uncertainty. In fact, from the previous section we know that the nonstationarity of solution (10) is due to its variance. Therefore, such a solution would be viable if the process, τ_t, would become stable with a new additional hypothesis on the limiting variance of the public expenditure shock, ε_t, consisting in $\sigma^2_{\varepsilon,t} \to 0$. In the absence of this hypothesis and of output growth, the system will tend autonomously to crisis notwithstanding the optimality of the program (4) adopted. Such a result strongly underlines the relevance of reducing the uncertainty of taxation deriving from the public expenditure decisional process and, in practice, calls for clear financial plans both in terms of public investments and services to provide.

Nonetheless, in case of systemic crisis, a rescue way to respect the stability pact, and at the same time to cope with an intolerable level of taxes, is to run a new optimal program (4) with new values for the initial conditions of the state variables, g_t, B_t and, g_c, and for the interest rate r and the parameter ρ. The new initial conditions and parameters may be obtained in several ways such as: reducing exogenously the public expenditure; promoting securizations and privatizations with the aims of reducing the debt initial condition; increasing incentives to boost production in order to reduce the taxation rate and then to calibrate appropriately the initial conditions; changing interest rates and rescheduling the -autoregressive- dynamics of the public expenses.

From a technical point of view the search of the new desired initial conditions may be conducted by inverting the filter of expression (24) in order to obtain the values of tax revenues, debt, deficit and fixed public expenditure that would bring to pursue the economic policy targets.

However, without adventuring in the complexity of the social and economic choices mentioned above, it is clear that these remedies are all provisional devices as theorems 3 and 4 continue to be valid and the time of crisis is just shifted forward. Hence, the economic policy debate should consist in decisions capable to control structurally the fiscal pressure and not in a mere revision of initial conditions and model parameters.

Three issues, which may be important for addressing the reduction of uncertainty in a structural way and, consequently, keeping under control the probability to violate the stability condition (30), may be the followings. 1) A clear and reliable expenditure plan to be carried on in a long-run perspective, whilst, at the present, financial laws typically consider a modifiable expenditure plan with a short-term horizon. The expenditure plan should be based on a broad social consensus in order to be pursued, possibly, independently of the governments in charge. 2) In general, if particularly relevant public services and rights are managed by the private sector, the market risk may potentially be the cause for future and relevant government expenditures. Hence, public rights and services particularly relevant for the collectivity in a country should be under government control. 3) Moreover, high levels of fiscal pressure are necessarily compared with the provided public services, then effectiveness of the public expenditure is a key variable for the determination of the considered failure probability. An improvement of the effectiveness, other than of the quantity, of the public services would raise the upper limit, $\bar{\upsilon}$, and so keeps the condition (30) better under control. Along these lines, Masuch et al. (2017)

provide empirical evidence supporting the positive effect of credibility and soundness of governments institutions on the European economic growth cushioning the negative effect of an high public debt.

4. Empirical analysis

4.1. Estimation of the SGP failure risk probability

The empirical analysis focuses on equation (32) of Theorem 4, which is here estimated and then simulated in order to study the compliance with the SGP. Since the nature of g_c is experimental, in this empirical analysis the real public expenditure will be autoregressive of the first order without constant term, *i.e.*, $g_t = \rho g_{t-1} + \varepsilon_t$, with the same assumptions of equation (1). In effect, according with the empirical literature, it is a proportional amount of public expenditure to GDP which is found to be constant.[6] However, from the theoretical analysis and the empirical results on r and ρ we may calculate the coefficient associated to the activation of g_c necessary for simulations and forecasting, amounting to $\frac{1}{1+r-\rho}$.

In what follows, we want to evaluate the probability that the control variable normalized by output goes beyond the socially tolerable threshold tax revenue rate, $\bar{\upsilon}$, plus the possible financial aid upon output, $\bar{\Delta m}$. Since υ_t is not observable we treat it as a latent variable, of which we also do not its threshold level. Nonetheless, we can reasonably identify the periods in our sample when high rates of υ_t were equal to or greater than the corner point $\bar{\upsilon} + \bar{\Delta m}$. Then, we can build a probability model by assigning value 1 for the occurrence of the failure event and 0 when $\upsilon_t < \bar{\upsilon} + \bar{\Delta m}$. More specifically, by observing that, when $\rho = 1$, υ_t becomes equal to the total government expenditure upon output, $G_t/y_t = rB_t/y_t + g_t/y_t$, which is greater than in the case when $\rho < 1$, we assign value 1 when G_t/y_t is equal to or greater than a reasonably large value $\overline{G/y}$ and 0 otherwise. After trying several values, we settled $\overline{G/y} = 55\%$.[7]

[6] See for all Ginebri et al. (2005) where it is shown that the elasticity to GDP of a relevant part of the public expenditure is about 1. Instead, the case we would consider is a one where the public expenditure is prefixed according to a policy target that, for its importance, would not be affected by the GDP path. Actually, we also tested the presence of a constant public expenditure term in levels with nonsignificant results.

[7] The other, smaller, values we tried provided acceptable results for a sample with a shorter time dimension ending before the quantitative easing periods, when $\bar{\Delta m}$ were negligible. Therefore, we choose to consider only $\overline{G/y} = 55\%$ which revealed consistent with the whole sample period. Larger values are too few in the sample and therefore not admissible.

We consider 12 European countries - Italy, Belgium, Germany, Greece, Spain, France, Ireland, Luxemburg, Netherlands, Austria, Portugal, Finland - and 38 years ranging from 1984 till 2021. We estimate a Logit panel data model with random effects which is validated both by the Hausman test and by $Pseudo-R^2$.[8] We opt to grouping the data in a panel even if the span of time is rather large in order to have a sample with a more pronounced and reliable evidence of the noncompliance occurrences. Nonetheless, we also depart from the classical random effect hypotheses and account for a variance-covariance matrix which allows for residuals autocorrelation besides that one induced by the non-idiosyncratic term.[9]

Hence, our final failure probability model for the i-th country is

$$P[(\zeta_{it}+\upsilon_{it})\geq\bar{\upsilon}+\bar{\Delta m}|\mathbf{x}_{it}]=P[(\underbrace{\zeta_{it}-(\bar{\upsilon}+\bar{\Delta m})+r\frac{B_{it}}{y_{it}}+\frac{r}{1+r-\rho}\frac{g_{it}}{y_{it}}}_{\mathbf{x}'_{it}\beta})\geq 0|\mathbf{x}_{it}], \tag{34}$$
$$\mathbf{x}'_{it}\equiv[1,\ \frac{B_{it}}{y_{it}},\ \frac{g_{it}}{y_{it}}],\ \beta'\equiv[-(\bar{\upsilon},\bar{\Delta m}),\ r,\ r/(1+r-\rho].$$

where ζ_{it} is the random error, and the absolute value of the constant term is the threshold level described in equation (32).[10]

Since g_{it} and B_{it} are real data, they are also exogenous, being the occurrences of the state variables according to which the latent variable, *i.e.*, the tax revenue prescribed by the SGP (the control variable), is determined (in other words a simultaneity problem would have been arisen if data of controlled public debt would have been available). Then, the error term ζ_{it} can be assumed independent of the regressors, which allows to build the following likelihood function whose estimated coefficients have valid diagnostic tests.

When $\zeta_{it}+\mathbf{x}'_{it}\beta\geq 0$ the failure to comply with the SGP occurs and $fail_{it}$ =1, or 0 otherwise in case of compliance. Hence, the corresponding likelihood to be maximized is

[8]This index measures the goodness of fit for nonlinear model and is defined as $0\leq 1-\frac{lnL}{lnL_0}\leq 1$, being lnL_0 the natural logarithm of the likelihood function with only a constant term. As shown in Tunali (1986), its empirical value is typically low in empirical studies with economic data, due to the nonlinearity of the models to which it is applied.

[9]Further, we performed also a robust population averaged estimations obtaining quite similar results.

[10]Due to the meaning of this term, we report the corresponding estimate with positive sign.

$$L_i = P(fail_{i1},...,fail_{iT}|\mathbf{X}) = \int_{inf_{i1}}^{sup_{i1}} ..... \int_{inf_{iT}}^{sup_{iT}} f(\zeta_{i1},...,\zeta_{iT})\, d\zeta_{i1},...,d\zeta_{iT},$$
$$i = 1,...,N; \qquad t = 1,...,T \tag{35}$$

where $\mathbf{X}$ is the dataset of the covariates for every time and country. If $fail_{it} = 1$ we have $(inf_{it}, sup_{it}] = (-\infty, -\mathbf{x}'_{it}\boldsymbol{\beta}]$ while if $fail_{it} = 0$ we have $(inf_{it}, sup_{it}) = (-\mathbf{x}'_{it}\boldsymbol{\beta}, +\infty)$.[11]

The random effect model implies that the residual of the empirical model in (34) is $\zeta_{it} = \eta_{it} + u_i$, where, as usual, the random errors on the right hand side are independent and with the following proprieties

$$E[\eta_{it}|\mathbf{X}] = 0, Cov[\eta_{it}, \eta_{is}|\mathbf{X}] = Var[\eta_{it}|\mathbf{X}] = 1 \quad \text{if} \quad i = j \quad \text{and} \quad t = s, \quad 0 \quad \text{otherwise}$$
$$E[u_i|\mathbf{X}] = 0, Cov[u_i, u_j|\mathbf{X}] = Var[u_i|\mathbf{X}] = \sigma_u^2 \quad \text{if} \quad i = j, \quad 0 \quad \text{otherwise}$$
$$Cov[\eta_{it}, u_j|\mathbf{X}] = 0 \forall i, t, j, \quad E[\zeta_{it}|\mathbf{X}] = 0, Var[\zeta_{it}|\mathbf{X}] = 1 + \sigma_u^2 \forall i, t, \quad Cov[\zeta_{it}, \zeta_{js}|\mathbf{X}] = \sigma_u^2 \; i = j, t \neq s.$$

Expression (35) is first simplified and then maximized by means of the Gauss-Hermite quadrature. The simplification adopted is based on the reformulation of the joint distribution in terms of the integral over u_i of the product of the single independent densities conditioned on u_i -which is the term engendering the temporal dependence for the i-th observation. This, after some manipulations, brings to

$$L_i = P(fail_{i1},...,fail_{iT}|\mathbf{X}) = \int_{-\infty}^{+\infty} \left(\prod_{t=1}^{T} \int_{inf_{iT}}^{sup_{iT}} f(\zeta_{it}|u_i)\, d\zeta_{it} \right) f(u_i)\, du_i. \tag{36}$$

Finally, we use the Butler and Moffitt's method, which consists in assuming a normal distribution for u_i, and obtain, by assigning to $f(.)$ the logistic function,

[11] We tried also a Probit model obtaining very similar results. However, we choose the logistic model for prudential reasons because in this case the probability of complying with the SGP ($fail_{it} = 0$) turns out to be slightly larger when $\mathbf{x}'_{it}\beta$ is very small and vice-versa (see Amemiya (1981)).

$$L_i = P(fail_{i1},...,fail_{iT}|\mathbf{X}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{u_i}{\sigma_u\sqrt{2}}\right)^2} \left[\prod_{t=1}^{T} \frac{e^{(2fail_{it}-1)(\mathbf{x}'_{it}\beta+u_i)}}{1+e^{(2fail_{it}-1)(\mathbf{x}'_{it}\beta+u_i)}} \right] d\left(\frac{u_i}{\sigma_u\sqrt{2}}\right), \tag{37}$$

which is maximized with the above mentioned numerical procedure. We implement this calculus using Stata 12, the remaining part of the statistical analysis has been developed with Matlab R2022B. Data are from the Statistic Bulletin of Bancad'Italia (2022).

In Table 1 we report also the estimation for the case of ρ constrained to 1 since in the unconstrained case the estimation of this parameter is very close to this limiting value.[12] Still, the results for the constrained case are relevant in order to confirm the consistency of the model with the data, being in this case the latent variable υ_t coincident with the total public expenditure normalized to GDP, G_t/y_t. This means that in the constrained case the estimation of $\bar{\upsilon} + \bar{\Delta m}$ should approximate the threshold level $\overline{G/Y} = 55\%$ and that one of r the ratio between the interest payments and the public debt. In fact, when $\rho = 1$ the upper limit of the socially tolerable tax rate - potentially augmented by the monetary financing from the ECB - results to be 54,57%, and the interest rate about 4%, which approximate well the above mentioned values.

However, the unconstrained analysis is of prime interest in that both $r/(1+r-\rho)$ and $\bar{\upsilon} + \bar{\Delta m}$ are highly sensible to small changes in the estimation of ρ.[13]

regressors	*Parameters*	$\overline{G/y} = 55\% : \upsilon_t \geq \bar{\upsilon}$	$\overline{G/y} = 55\% : \upsilon_t \geq \bar{\upsilon}, \rho = 1$
g_{it}/y_{it}	$r/(1+r-\rho)$	0.65***	1
	S.E.	0.09	-
B_{it}/y_{it}	r	0.028 **	0.039***
	S.E.	0.01	0.013
$\bar{\upsilon} + \bar{\Delta m}$	α_0	35.97***	54.57***
	S.E.	5.228	1.323
-	ρ	0.98	1
χ_2 test of $\mu = 0$ (P-value)		0.01***	0.00***
log-likelihood		-75.55	-80.73
$Pseudo-R^2$ (likelihood ratio index)		0.53	0.50

Table 1: Estimation of Logit model. Point estimates, random effects.

Hausman test (unconstrained model) $Prob > \chi_2 = 0.224$. $\mu = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\eta^2}$ is the proportion of the total variance contributed by the random-effect panel-level variance component. ***Significance at 99%, **significance at 95%. S.E. = standard error.

[12] See also Arpaia and Turrini (2008) which confirm for the European countries a value of ρ close to 1.

[13] Of course, the Hausman test reported in Table 1 has been calculated only for the unconstrained case where the coefficients are free to vary allowing the comparison between fixed and random effects.

As for the unconstrained case, the upper limit of the socially tolerable tax rate (included the monetary financing from the ECB) is quite different with an estimated value of 46.43% when also the double of the standard error is accounted for, which means more than 10 percentage points less than the average-tax-rate upper-limit in the constrained case.[14] Looking at the coefficients of g_t/y_t and B_t/y_t, they are equal to 0.65 and 0.028, respectively. As afore mentioned, the reduction in the estimate of ρ associated with these coefficients is very small, 0.02, compared with the constrained case, notwithstanding the significant difference in the other coefficients.

As regards the experimental parameter associated to g_c, $1/(1+r-\rho)$, its magnitude is quite large in force of ρ close to 1. In particular, for the unconstrained case this is 20.83 while for the constrained one is 25.64, confirming that the introduction of a fixed fiscal policy has serious dynamic implications. However, the g_c variable is easily supposed to be very small when related to specific areas of expenditure and even more so when measured as a percentage of GDP as in this probability framework, which justifies the size of these coefficients.

4.2. Qualitative analysis

4.2.1. Path of the failure probability

In this section we analyze the temporal path of the failure probability (32) computed per each European country of our sample. The probability of complying with the SGP is evaluated accounting also for the credibility that governments have to provide effectively public services with a level of quality coherent with the commitments undertaken. We consider such an aspect by comparing the failure probabilities, obtained with the two estimated models in Table 1, with those corrected by the Worldwide Governance Indicator (WGI, WGI (2022)), which is an index reflecting the government effectiveness in terms of policy formulation and implementation - and so it concerns with the perceptions of the quality of public services and government's credibility. We first re-scale the WGI index of each country by that one of the country at the minimum level and calculate the time average, then we normalize the estimated failure probabilities by this term according to the reasoning that the failure probability should decrease in connection with the country's effectiveness degree in providing services. In Figure 1 the blue and the

[14]We tried for this threshold term also country specific effects obtaining that the only country with a significant effect was Greece with about an additional 3%. However, for prudential reasons we consider in the probability simulations of the next subsection the same common threshold also for this country.

red lines represent respectively the unconstrained estimation of the failure probability not normalized and normalized by the WGI, and those yellow and purple the not normalized and normalized cases for the constrained estimation, respectively.

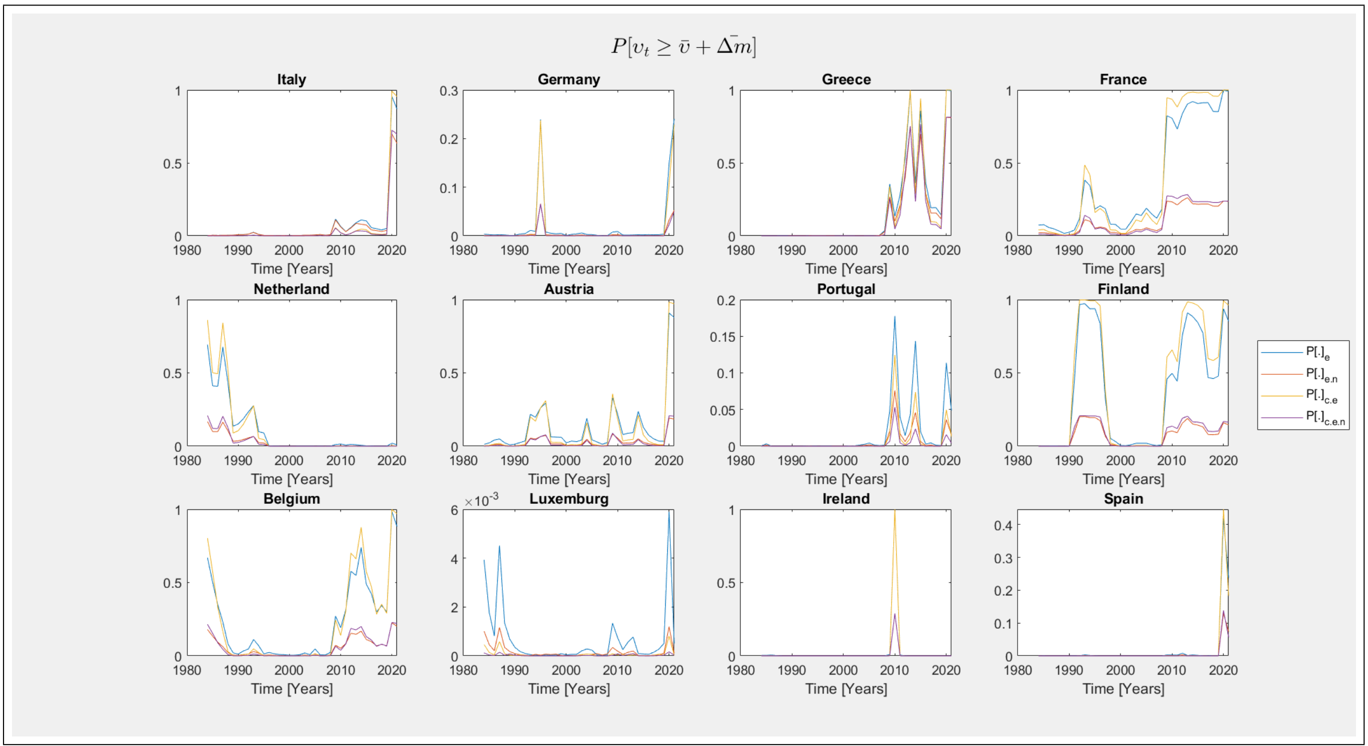

Figure 1: Failure Probability of Complying with the SGP.

$P[.]$ is $P[v_t \geq \bar{v} + \Delta\bar{m}]$. $P[.]_e$= unconstrained estimated $P[.]$, $P[.]_{e.n}$= unconstrained estimated normalized $P[.]$, $P[.]_{c.e}$= constrained estimated $P[.]$, $P[.]_{c.e.n}$= constrained estimated normalized $P[.]$.

It is worthwhile observing that for some countries there is a substantial difference between these two lines, consisting in a much lower normalized failure probability, which means that they may be considered virtuous in terms of qualitative expense capacity. In particular, looking at the nonnormalized probabilities, this fact is more pronounced for France, Finland and Belgium, which are more in critical conditions than Italy notwithstanding it has the worst budget condition after Greece, as reported in the following Table 2.

	rB_t/y_t	g_t/y_t	B_t/y_t	G_t/y_t	τ_t/y_t	$\tau_t/y_t - G_t/y_t$
Austria	3.10	48.72	68.22	51.82	49.33	-2.49
S.E.	0.84	2.15	10.65	2.07	0.73	
Belgium	6.18	47.59	112.13	53.77	49.50	-4.27
S.E.	3.18	3.75	13.56	3.32	1.37	
Finland	1.96	50.35	43.94	52.31	53.61	1.30
S.E.	1.11	5.16	17.32	5.50	1.48	
France	2.64	51.81	66.14	54.45	50.76	-3.69
S.E.	0.60	2.83	25.33	2.46	1.38	
Germany	2.48	44.58	59.49	47.06	45.26	-1.80
S.E.	0.85	2.24	12.83	2.40	0.94	
Greece	6.28	43.22	119.62	49.50	41.19	-8.31
S.E.	2.86	5.66	44.88	4.63	5.20	
Ireland	3.86	34.11	71.44	37.96	34.07	-3.89
S.E.	2.76	7.45	29.98	8.90	4.80	
Italy	6.57	43.04	113.35	49.61	43.94	-5.67
S.E.	2.73	3.15	18.66	2.47	3.45	
Luxemburg	0.46	41.67	12.36	42.13	43.29	1.16
S.E.	0.29	2.45	6.73	2.63	1.09	
The Netherlands	3.33	45.21	62.03	48.54	43.88	-4.66
S.E.	2.01	4.42	10.45	5.97	1.17	
Portugal	4.49	40.69	83.59	45.19	40.15	-5.04
S.E.	1.77	4.41	29.39	3.60	2.50	
Spain	3.03	39.93	64.57	42.96	38.19	-4.76
S.E.	1.09	3.56	25.78	3.68	1.59	

Table 2: Percentage average values over the sample period 1984-2021. S.E. = standard error.

In this table the sample mean and standard deviation of the total deficit and its components upon output are calculated. Finland - together with Luxemburg- is on average the most virtuous country with a positive budget, which underlines that what really matters is the normalized probability. From Figure 1, the country in the most critical condition is Greece since 2009. For such a country, even if the effectiveness of government policies improves from that period on, the failure probability (normalized and not) still keeps on being high. As for Italy, in practice there is no difference between the two lines thus showing that the high taxation, consequent to the high public expenditure, is not perceived as the duly payment for the services provided by government.

As regards the comparison between the cases unconstrained and not, from Figure 1, we may observe that there is a marked positive difference between the values of these probabilities during critical periods such as those relative to the subprime crisis (2007-2009) and the sovereign debt crisis (2010-2012). Hence, particularly in periods of crises and considered that the effect of a small reduction in ρ has a great lowering effect on the parameter of g_t/y_t, a reduction in the effect of the autoregressive components of the public expenditure may provide a quantitatively important contribution for reducing the SGP failure risk probability. However, the exceptional world pandemic crisis (2020-2021) was so painful that this difference becomes minimal in that period.

4.2.2. *Cost of controlling public debt in compliance with the SGP*

In this section, we evaluate the value function, (4), representing the cost of managing the state variables. Since the public expenditure is constrained to be stationary, the management problem is referred in particular to public debt which is, nowadays, the core problem for the European countries. Also, since the form of the value function may be modified in various ways without altering the optimal solution, τ_t^*, it makes more sense to study its dynamics rather than its values in levels. For this reason, in the following Figure 2 we show the rate of growth of $V(\mathbf{x}_t)$, which is computed according both to the functional form found in Appendix C and to the estimation of r and ρ presented Table 1 for the unconstrained case.[15] More in details, σ_ε^2 has been calculated using (10) per each country, and Theorem 2 has been exploited to find the value function per each year.

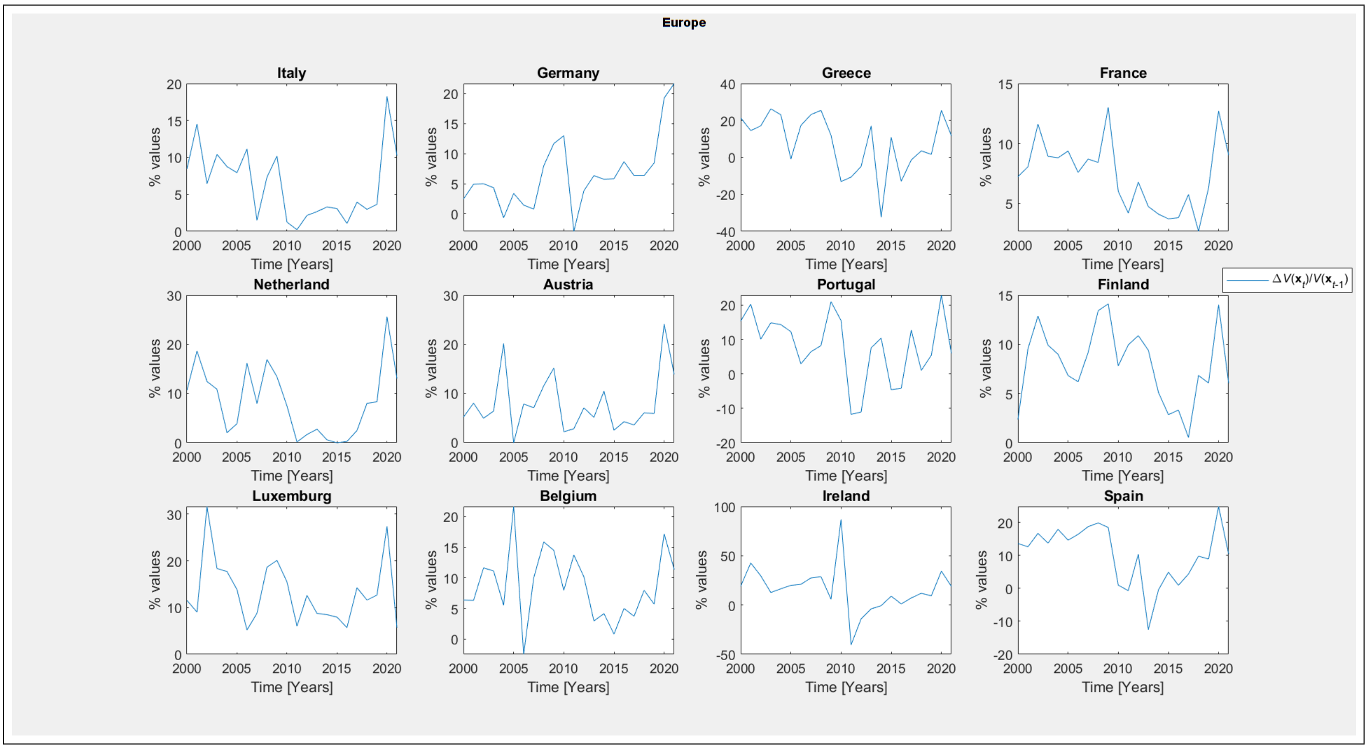

Figure 2: Cost rate of growth of controlling public debt in compliance with the SGP.

$V(\mathbf{x}_t)$ is the value function with initial condition of state variables at each time t.

An empirical evidence for all countries is that, till the European financial crisis occurred during the two years following 2007, there occurred decreasing rates of growth of $V(\mathbf{x}_t)$ due to the initial effort of all countries to reduce the burden of the public sector in terms of public debt and public expenditure, which was one of the crucial target of the SGP. However, once the afore mentioned crisis exacerbated for

[15] To be prudent, we quote the interest rate at its point estimate plus the uncertainty represented by the double of the standard error, which amounts to 4.8% and leaves ρ rounded at 0.98.

the immediately following sovereign debt crisis of some European countries (notably Greece, Ireland, Spain, Portugal, Italy, Cyprus and Slovenia), the cost of managing public debt started an increasing path till 2019, and even more in 2020 when this cost recorded an extremely consistent rate of growth for the pandemic.

Figure 2 shows that reducing the burden of the public sector for the European countries it is hard to do in the presence of periods of great uncertainty on the public expenditure occurring during the above mentioned crises. Then, the structural reforms should first state clearly the services entrusted to the private sector but that deserve public financial aid in case of financial trouble for the relevance that these services have for the community. Second, in order to reduce this source of uncertainty affecting the public expenditure, which is generated outside government budget constraint, the role of governments in the management of these services will have to be decided. In other words, reducing the uncertainty of this kind of unforeseen public expenditures should reduce the initial conditions of public expenditure and debt of the value function and so the burden of the public sector in the economy.

5. Debt dynamics by maturity, optimal refunding plan and long-run equilibrium

In this section we study the optimal dynamics of debt composition by maturity with a two-period model, extendable at any maturity. The criterion here adopted is that of the optimal reimbursement. This technique implies, once the appropriate taxation deriving from program (4) has been found, that government should be in the condition to refund debt or a proportion - here below is 100% for sake of simplicity - in an optimal way if requested, but it does not imply that the reimbursement necessarily should happen.

Since the refunding should occur using the control variable, we first settle the loss (or cost) function of Section 2 to this aim,

$$Min \left\{ l_t(.) = \Psi({}_{t-1}B_t + {}_{t-2}B_t) + \delta\Psi({}_{t-1}B_{t+1}) \right\}, \tag{38}$$

where $l(.)$ is the total loss, ${}_{t-1}B_t$ represents the short term debt ($B_{ST,t}$) issued at the period $t-1$ and expiring at period t, ${}_{t-2}B_t$ is the long-term debt ($B_{LT,t}$) issued at period $t-2$ and expiring at period t and finally ${}_{t-1}B_{t+1}$ is the long-term debt ($B_{LT,t+1}$) of period $t-1$ and expiring at $t+1$. These issuances of debt are additional control variables that should match both the budget constraint (2) and the optimal solution of program (4). The discount factor is applied to the cost of the future loss relative to the taxes collected for the reimbursement at time $t+1$. Then, problem (38) is a control problem over two periods with the following constraint to be respected at any time t

$$B_t =_{t-1} B_t +_{t-2} B_t + \delta \;_{t-1}B_{t+1}. \tag{39}$$

The following theorem shows how to solve this problem, and that, if issuances are regulated in an optimal way for refunding, the short term debt turns out to be stationary and the long-term one nonstationary. Moreover, the long-term debt exhibits a long-run equilibrium with the tax revenue consisting in a cointegration relation, that is the way in which the SGP prescriptions ensure the balance of the budget constraint.

Theorem 5. *If the debt obtained from* (4) *is subdivided into long and short term according to the optimal plan* (38) *sub.* (39)*, then the former will be nonstationary with conditional mean tending to a constant value and the latter will be stationary with conditional mean tending to 0. Moreover, there exists a cointegration relation between long-term debt and tax revenue.*

Proof. The first order conditions are $\forall\, t$

$$\psi'(_{t-1}B_t +_{t-2} B_t) = \psi'(_{t-1}B_{t+1}) \tag{40}$$

which means

$$_{t-1}B_t +_{t-2} B_t =_{t-1} B_{t+1} \tag{41}$$

which, through (39), gives

$$_{t-1}B_{t+1} = (1+\delta)^{-1}B_t$$

that, together with (6), furnishes the long-term debt at time $t+1$, $B_{LT,t+1}$

$$B_{LT,t+1} = [\frac{\tau_t}{r} - \frac{g_t}{1+r-\rho} - \frac{g_c}{r(1+r-\rho)}](1+\delta)^{-1} \tag{42}$$

and, in analogy with (14), it is possible to obtain the following $I(1)$ representation

$$B_{LT,t+1} = B_t(1+\delta)^{-1} = \left[\frac{\tau_0}{r} + \rho^t \frac{g_c - g_0(1-\rho)}{(1-\rho)(1+r-\rho)} - \frac{g_c}{r(1-\rho)} + \sum_{i=1}^{t-1} \frac{1-\rho^i}{1+r-\rho}\varepsilon_{t-i}\right]\frac{1+r}{2+r} \tag{43}$$

and consequently

$$E_0(B_{LT,t+1}) = [\frac{\tau_0}{r} + \rho^t \frac{g_c - g_0(1-\rho)}{(1-\rho)(1+r-\rho)} - \frac{g_c}{r(1-\rho)}]\frac{1+r}{2+r}. \tag{44}$$

$$\lim_{t\to+\infty} E_0(B_{LT,t}) = [\frac{\tau_0}{r} - \frac{g_c}{r(1-\rho)}]\frac{1+r}{2+r} \tag{45}$$

from which it is possible to derive the rule that the long-term debt (B_{LT}), though nonstationary, must reach on average a constant value in the long-run - i.e., in the limit.

Differently, the short term debt ($B_{ST,t}$) is stationary and must tend to 0 on average as time passes. In fact, since (41) says that $B_{ST,t}$ is the first difference of $B_{LT,t}$, it is possible to obtain $B_{ST,t}$ from (42), after differencing and considering that τ_t is a martingala defined by (10),

$$B_{ST,t} = [\frac{e_t}{r} - \frac{g_t - g_{t-1}}{1+r-\rho}]\frac{1+r}{2+r}, \tag{46}$$

which is proved to be $I(0)$.

Still, given that (43) holds at each time t and using again (41), it is possible to obtain

$$\lim_{t\to+\infty} E_0(B_{ST,t}) = 0. \tag{47}$$

Importantly, equation (42) proves the existence of a long-run equilibrium consisting in a cointegration (according to the definition of Engle and Granger (1987)) between long-term public debt and tax revenue, with cointegration coefficients $[(1+\delta)r, -1]'$

$$(1+\delta)rB_{LT,t+1} - \tau_t = -\frac{rg_t}{1+r-\rho} - \frac{g_c}{(1+r-\rho)} \rightsquigarrow I(0) \tag{48}$$

of course, given (6), the same relation is also true for total debt and tax revenue with cointegration coefficients $[r, -1]'$, but by virtue of (48) and (46) it is possible to discern that it depends on the long-term debt in case of an optimal refunding plan. □

In practice, this theorem means that an optimal reimbursement plan, carried out with the taxation implicit in the SGP, implies that the short term debt should be kept on average at a level only temporarily different from 0. This is what it should naturally be according to the original function of the short term debt of financing occasional necessities. The remaining nonstationary part of the debt, consisting in the long-term debt, should be controlled ensuring a cointegration equilibrium with taxation, which is the only control variable envisaged by program (4). This way, European countries face the risk that in the absence of economic growth the tax revenue necessary for such an equilibrium may not be collected.

As regards the viability of the Maastricht parameters, in normal periods the cointegration relation should ensure on average the compliance with the budgetary rule which amounts to meet on average the constraint of the 3% of the deficit over GDP. However, the nonstationarity of public debt still persists thus jeopardizing the respect of the limit of 60% of public debt out of GDP, which ultimately depends on the rate of GDP growth.

In conclusion, the cointegration relation implied by the SGP can ensure (according to phase of the GDP growth) the compliance with the constraints on the flows composing the government deficit but not so for the stock of public debt.

6. Conclusions

In this research a stochastic optimal control model for the European countries has been presented and estimated with a logistic function in order to study the risk of noncompliance with the Stability and Growth Pact. The model formalizes the prescriptions of the SGP allowing to find the benchmarks for public debt and tax revenue. It shows the fragility of those countries with a required heavy level of taxation and low output growth in absence of appropriate services. This makes effective, due to the threat of social tensions, the constraint given by the upper limit of the socially tolerable taxation rate and, as time passes, puts in critical condition the capacity to adhere to the tax revenue path requested by the SGP. The main reason for this risk lies in the countries' political decision-making process on the public expenditure in those sectors of special public interest. In case of unclear or questionable laws, the outcoming uncertainty, amounting to the unforeseen expenditure, will be paid through more and more painful taxes even if optimally smoothed, thus engendering, if outpt growth is not enough to collect the necessary tax revenue, the serious risk of not meeting that budget constraint coherent with the respect of the SGP. Since this mechanism is autonomous, it works even in the privilege of an attenuation of the severity of the SGP consisting in spot monetary interventions or in letting changes in the initial conditions or in the parameters of the optimal program for obtaining the tax revenue and public debt prescribed by the SGP. In particular, the empirical analysis shows that improving the quality of the public expenditure and reducing the effect of the autoregressive components of the public expenditure contribute significantly in reducing the SGP failure risk probability. However, even though these measures are important in that help alleviating the problem, theorems 3 and 4 prove that they are temporary remedies. Therefore, in order to ease the feasibility of the Eurosystem, the structural reforms to adopt are those coping with the sources of public expenditure uncertainty, in general all the expenses generated outside the budget constraint. Whilst in the pre-euro era these expenses could be financed by several control variables such as by changes in debt and money, devaluation and finally taxes supported - hopefully - by output growth, with the European monetary union there remains only the last option linked to output growth. This means that, in case of prolonged recession, the stability of the European countries might face difficulties unless the structural reforms, necessary to reduce the above mentioned uncertainty, are undertaken. Reducing uncertainty should reduce the initial conditions of public expenditure and debt and so the burden of the public sector in the economy represented by the cost of the value function. Formally, we prove with Theorem 5 that the stability underlying the SGP consists in a cointegration relation of long-run equilibrium between tax revenue and public debt, which implies on average the compliance with the budgetary rule and consequently with the constraint of the 3% of the deficit over GDP. However, the nonstationarity of public debt may undermine the respect of the limit of 60% of public debt out of GDP, which ultimately depends on the rate of GDP growth.

As for future research, the model presented may be extended to a more detailed characterization of the state variables for policy purposes. In particular, theorems 1 and 2 prove, respectively, how a constant public expenditure affects the dynamics of both tax revenue and state public debt, and that it may be inserted as an additional state variable and contribute to the Riccati DDARE equation for finding the optimal solution. This fact may be generalized by splitting both the constant, as well as the autoregressive, public expenditure into more state variables representing the several policy actions to be undertaken. Accordingly, the public debt may be subdivided into components of different maturities and interest rates with specific expenditure financing targets, being additional control variables the issuances of this new debt. Finally, a promising area of investigation linked to the stability would be to use the filter of the system to study the initial conditions complying with the SGP and the time required for them to attainable.

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Appendix A.

In this Appendix we prove, for this specific 3×3 case, the uniqueness and the positivity of all terms for the strong solution $\mathbf{P}^*$ of the DDARE equation (22), and verify the symmetry, according to the hypotheses of Theorem 2 on the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, $\mathbf{R}$.

Corollary 1. *Corollary of Theorem 2. Under the hypotheses of Theorem 2, there exists a unique strong solution* $\mathbf{P}^*$ *of equation* (22) *which, other than being symmetric, is also positive for all values of the parameters* r *and* ρ *involved, under the assumptions of equations* (1) *and* (2)

Proof. We first decompose the following matrices in blocks

$$\mathbf{P}^* = \left[\begin{array}{cc|c} P^*_{11} & P^*_{12} & P^*_{13} \\ P^*_{21} & P^*_{22} & P^*_{23} \\ \hline P^*_{31} & P^*_{32} & P^*_{33} \end{array}\right] = \begin{bmatrix} \mathbf{P_{11}}^* & \mathbf{P_{12}}^* \\ \mathbf{P_{21}}^* & \mathbf{P_{22}}^* \end{bmatrix}, \tag{A.1}$$

$$\mathbf{A} = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right] = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix}, \tag{A.2}$$

$$\mathbf{B} = \left[\begin{array}{c} B_{11} \\ B_{21} \\ \hline B_{31} \end{array}\right] = \begin{bmatrix} \mathbf{B_{11}} \\ \mathbf{B_{21}} \end{bmatrix}, \tag{A.3}$$

and then apply the definition of equation (22)

$$\mathbf{P}^* = \begin{bmatrix} \mathbf{P_{11}}^* & \mathbf{P_{12}}^* \\ \mathbf{P_{21}}^* & \mathbf{P_{22}}^* \end{bmatrix} = - \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} \delta \mathbf{P}^* \begin{bmatrix} \mathbf{B_{11}} \\ \mathbf{B_{21}} \end{bmatrix} \left(\begin{bmatrix} \mathbf{B_{11}}' & \mathbf{B_{21}}' \end{bmatrix} \delta \mathbf{P}^* \begin{bmatrix} \mathbf{B_{11}} \\ \mathbf{B_{21}} \end{bmatrix} + \mathbf{R} \right)^{-1} \tag{A.4}$$
$$\times \begin{bmatrix} \mathbf{B_{11}}' & \mathbf{B_{21}}' \end{bmatrix} \delta \mathbf{P}^* \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} + \begin{bmatrix} \mathbf{A_{11}}' & \mathbf{A_{21}}' \\ \mathbf{A_{12}}' & \mathbf{A}'_{22} \end{bmatrix} \delta \mathbf{P}^* \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix}$$

from which, after applying the entries in (19)-(20) for the indexes $ij \neq 11$ and $\mathbf{R} = 1$, we get

$$\mathbf{P_{11}}^* = -\delta \mathbf{A_{11}}'\mathbf{P_{11}}^*\mathbf{B_{11}}(\mathbf{R} + \delta \mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{B_{11}})^{-1}\delta \mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{A_{11}} + \delta \mathbf{A_{11}}'\mathbf{P_{11}}^*\mathbf{A_{11}} \tag{A.5}$$

which is proved to furnish (see Maggi (2023)) after lengthy passages the first 2×2 block

$$\mathbf{P_{11}}^* = \begin{bmatrix} r(1+r) & \frac{r(1+r)}{1+r-\rho} \\ \frac{r(1+r)}{1+r-\rho} & \frac{r(1+r)}{(1+r-\rho)^2} \end{bmatrix} \tag{A.6}$$

which is symmetric and positive.

Now, it remains to show that $\mathbf{P_{12}}^* = \mathbf{P_{21}}^{*'}$, to find $\mathbf{P_{22}}^*$, and prove their positivity.

As for $\mathbf{P_{12}}^*$, from equation (A.4), and for the fact that $\left(\begin{bmatrix} \mathbf{B_{11}}' & \mathbf{B_{21}}' \end{bmatrix} \delta \mathbf{P}^* \begin{bmatrix} \mathbf{B_{11}} \\ \mathbf{B_{21}} \end{bmatrix} + \mathbf{R}\right)^{-1} = (\delta P^*_{11} + 1)^{-1} = \delta$, it follows

$$\mathbf{P_{12}}^* = \begin{bmatrix} P^*_{13} \\ P^*_{23} \end{bmatrix} = \mathbf{\Omega}^{-1}_{pre}\Big\{\underbrace{-\delta^3 \mathbf{A_{11}}\mathbf{P_{11}}^*\mathbf{B_{11}}\mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{A_{12}}}_{\begin{bmatrix} -\frac{r^2}{1+r-\rho} \\ -\frac{r^2}{(1+r-\rho)^2} \end{bmatrix}} + \underbrace{\delta \mathbf{A_{11}}\mathbf{P_{11}}^*\mathbf{A_{12}}}_{\begin{bmatrix} \frac{r(1+r)}{1+r-\rho} \\ \frac{r(1+r)}{(1+r-\rho)^2} \end{bmatrix}}\Big\} \tag{A.7}$$

with $\mathbf{\Omega}_{pre} = \left\{\mathbf{I} - [-\delta^3 \mathbf{A_{11}}\mathbf{P_{11}}^*\mathbf{B_{11}}\mathbf{B_{11}}' + \delta \mathbf{A_{11}}]\right\} = \begin{bmatrix} \frac{r}{1+r} & 0 \\ -\frac{1-\rho}{(1+r)(1+r-\rho} & \frac{1+r-\rho}{1+r} \end{bmatrix}$

and

$$\mathbf{\Omega}^{-1}_{pre} = \begin{bmatrix} \frac{1+r}{r} & 0 \\ \frac{1+r)(1-\rho}{r(1+r-\rho}^2 & \frac{1+r}{1+r-\rho} \end{bmatrix} \tag{A.8}$$

from which

$$\mathbf{P_{12}}^* = \begin{bmatrix} P^*_{13} \\ P^*_{23} \end{bmatrix} = \begin{bmatrix} \frac{1+r}{1+r-\rho} \\ \frac{1+r}{(1+r-\rho)^2} \end{bmatrix}, \tag{A.9}$$

which is positive.

As for $\mathbf{P_{21}}^*$, from equation (A.4)

$$\mathbf{P_{21}}^* = \begin{bmatrix} P_{31}^* & P_{32}^* \end{bmatrix} = \Big\{ \underbrace{-\delta^3 \mathbf{A_{12}}'\mathbf{P_{11}}^*\mathbf{B_{11}}\mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{A_{11}}}_{\begin{bmatrix} -\frac{r^2}{1+r-\rho} & -\frac{r^2}{(1+r-\rho)^2} \end{bmatrix}} + \underbrace{\delta \mathbf{A_{12}}'\mathbf{P_{11}}^*\mathbf{A_{11}}}_{\begin{bmatrix} \frac{r(1+r)}{1+r-\rho} & \frac{r(1+r)}{(1+r-\rho)^2} \end{bmatrix}} \Big\} \boldsymbol{\Omega}_{post}^{-1} \tag{A.10}$$

$$\text{with } \boldsymbol{\Omega}_{post} = \Big\{ \mathbf{I} - [-\delta^3 \mathbf{A}_{22}^* \mathbf{P_{11}}^*\mathbf{B_{11}}\mathbf{B_{11}}'\mathbf{A_{11}} + \delta \mathbf{A_{11}}] \Big\} = \begin{bmatrix} \frac{r}{1+r} & -\frac{1-\rho}{(1+r)(1+r-\rho} \\ 0 & \frac{1+r-\rho}{1+r} \end{bmatrix}$$

and

$$\boldsymbol{\Omega}_{post}^{-1} = \begin{bmatrix} \frac{1+r}{r} & \frac{(1+r)(1-\rho)}{r(1+r-\rho)^2} \\ 0 & \frac{1+r}{1+r-\rho} \end{bmatrix} \tag{A.11}$$

From which

$$\mathbf{P_{21}}^* = \begin{bmatrix} P_{31}^* & P_{32}^* \end{bmatrix} = \begin{bmatrix} \frac{1+r}{1+r-\rho} & \frac{1+r}{(1+r-\rho)^2} \end{bmatrix} = \mathbf{P_{12}}^{*'} \tag{A.12}$$

which confirms the symmetry.

As for $\mathbf{P_{22}}^*$, from equation (A.4)

$$\mathbf{P_{22}}^* = P_{33}^* = \boldsymbol{\Omega}_{22}^{-1} \Big\{ \underbrace{-\delta^3 [\mathbf{A}_{12}'\mathbf{P_{11}}^*\mathbf{B_{11}} + \mathbf{A}_{22}'\mathbf{P_{21}}^*\mathbf{B_{11}}][\mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{A_{22}} + \mathbf{B_{11}}'\mathbf{P_{11}}^*\mathbf{A_{12}}]}_{-\frac{1+r}{(1+r-\rho)^2}} + \tag{A.13}$$

$$+ \underbrace{\delta [\mathbf{A}_{12}'\mathbf{P_{11}}^*\mathbf{A_{12}} + \mathbf{A}_{12}'\mathbf{P_{12}}^*\mathbf{A_{22}} + \mathbf{A}_{22}'\mathbf{P_{21}}^*\mathbf{A_{12}}]}_{\frac{1+r}{(1+r-\rho)^2} + \frac{1}{(1+r-\rho)^2}} \Big\} = \frac{1+r}{r(1+r-\rho)^2}, \text{ with } \boldsymbol{\Omega}_{22} = (1-\delta)$$

which is positive and proves the result (25)

$$\mathbf{P}^* = \begin{bmatrix} r(1+r) & \frac{r(1+r)}{1+r-\rho} & \frac{1+r}{1+r-\rho} \\ \frac{r(1+r)}{1+r-\rho} & \frac{r(1+r)}{(1+r-\rho)^2} & \frac{1+r}{(1+r-\rho)^2} \\ \frac{1+r}{1+r-\rho} & \frac{1+r}{(1+r-\rho)^2} & \frac{1+r}{r(1+r-\rho)^2} \end{bmatrix}. \tag{A.14}$$

.

Furthermore, there exists a second nonnegative solution for (22) given by $\mathbf{P}^* = \mathbf{0}$. This solution is due to the fact that, as proved in Maggi (2023), P_{11}^* admits also - and only - another 0 solution - besides $r(1+r)$ - which implies $\mathbf{P_{11}}^* = \mathbf{0}$ and then, from (A.5), (A.7),(A.10) and (A.13), $\mathbf{P}^* = \mathbf{0}$. Consistent with Theorem 3.1 of Wah Chan et al. (1984), this second solution is due to the presence of the uncontrollable mode, $1+r$, in the state matrix $\mathbf{A}$. However, since, by virtue of (23), the filter $\mathbf{A}+\mathbf{BK}$ in this case would coincide with $\mathbf{A}$, the unstable root would lay outside the unit circle, which means that the solution $\mathbf{P}^*=\mathbf{0}$ is not strong and that the unique strong solution is (25). □

Appendix B.

In this Appendix we prove the result of formula (27) used in the main text.

Proof. Implementing (23) with the entries obtained in Appendix C with $\mathbf{R}\equiv 1$

$$\mathbf{K}=-\left[1+\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\delta\mathbf{P}^*\begin{bmatrix}-1\\0\\0\end{bmatrix}\right]^{-1}\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\delta\mathbf{P}^*\begin{bmatrix}1+r & 1 & 0\\0 & \rho & 1\\0 & 0 & 1\end{bmatrix} \tag{B.1}$$

$$\mathbf{K}=\underbrace{(1+\delta P_{11}^*)^{-1}\delta}_{\delta}\begin{bmatrix}P_{11}^*(1+r) & P_{11}^*+\rho P_{21}^* & P_{12}^*+P_{13}^*\end{bmatrix} \tag{B.2}$$

from which, after substituting the corresponding entries,

$$\mathbf{K}=\begin{bmatrix}r & \frac{r}{1+r-\rho} & \frac{1}{1+r-\rho}\end{bmatrix} \tag{B.3}$$

which is expression (27). Furthermore, it is easy to show that the filter of the autoregressive expression (24) has two roots on the unit circle with an upper triangular representation

$$\mathbf{A}+\mathbf{BK}=\begin{bmatrix}1 & \frac{1-\rho}{1+r-\rho} & \frac{-1}{1+r-\rho}\\0 & \rho & 1\\0 & 0 & 1\end{bmatrix}. \tag{B.4}$$

which shows that the solution $\mathbf{P}^*$ is strong.

□

Appendix C.

In this appendix we show that also in the presence of a constant public expenditure, g_c, the optimal value function (4), or (15), is real and that it is obtainable using the strong solution $\mathbf{P}^*$ of the DDARE (22).

Theorem 6. *The value function (4) or (15), calculated using (28), is real and equal to the value function $V_t(\mathbf{x}_t)$ calculated using the stationary matrix solution to (22), $\mathbf{P}^*$, also when (1) includes the constant term g_c.*

Proof. From (15), (4), (10), (3) and $\mathbf{Q}_{t+j} = \mathbf{0}$, we obtain

$$V_t(\mathbf{x}_t) = E_t\left\{\sum_{j=0}^{T}\delta^j\tau_{t+j}^2\right\} = E_t\left\{\sum_{j=0}^{T}\delta^j[(\tau_t)+(e_{t+1}...e_{t+j-1}+e_{t+j})]^2\right\} \tag{C.1}$$
$$= \underbrace{E_t\left\{2\frac{r}{1+r-\rho}\sum_{j=0}^{T}\delta^j(\tau_t)\times(\varepsilon_{t+1}...\varepsilon_{t+j-1}+\varepsilon_{t+j})\right\}}_{0} +$$
$$+[\frac{r}{1+r-\rho}\sigma_\varepsilon]^2\sum_{j=0}^{T}\delta^j j + \underbrace{\sum_{j=0}^{T}\delta^j\tau_t^2}_{\frac{1+r}{r}[rB_t+\frac{r}{1+r-\rho}g_t+\frac{1}{1+r-\rho}g_c]^2},$$
$$T\to+\infty$$

which is convergent by virtue of the ratio criterion applied to $\sum_{j=0}^{+\infty}\delta^j j$ since

$$\lim_{t\to+\infty}\frac{\frac{T+1}{(1+r)^{T+1}}}{\frac{T}{(1+r)^T}} = \frac{1}{1+r} < 1. \tag{C.2}$$

In particular, being $\delta^{-1}\sum_{j=0}^{+\infty}\delta^j j$ the first derivative, w.r.t. δ, of the series $\sum_{j=0}^{T}\delta^j$, we can get by means of Abel's theorem $\partial\frac{(1-\delta)^{-1}}{\partial\delta} = \delta^{-1}\sum_{j=0}^{+\infty}\delta^j j = \frac{1+r}{r^2}$. So, the minimum of the loss value function, from (C.1), turns out to be

$$V_t(\mathbf{x}_t) = \frac{1+r}{r^2}[\frac{r}{1+r-\rho}\sigma_\varepsilon]^2 + \frac{1+r}{r}[rB_t+\frac{r}{1+r-\rho}g_t+\frac{1}{1+r-\rho}g_c]^2. \tag{C.3}$$

Now, we have to show that the result obtained in (C.3) is the same as that one when the value function $V_t(\mathbf{x}_t)$ is calculated using the stationary matrix solution to (22), $\mathbf{P}^*$ in the presence of the constant term g_c.

We first observe that, consistent with the Bellman (or Riccati in this case) recursion for finding the optimum, the value function may be decomposed in two pieces, one equal to the deterministic case and another one containing the uncertain term. Then, the recursion entailed by (17) may be expressed as

$$V_t(\mathbf{x}_t) = min\{\delta E_t[V_{t+1}(\mathbf{x}_{t+1})] + \mathbf{u}_t'\mathbf{u}_t\} = min\{\delta E_t[\mathbf{x}_{t+1}'\mathbf{P}_{t+1}\mathbf{x}_{t+1}] + \mathbf{u}_t'\mathbf{u}_t + \sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2\}, \; \forall\, t. \quad \text{(C.4)}$$

Deriving (C.4) with respect to $\tau_t \equiv \mathbf{u}_t$ brings about the matrix difference equation (21) which is the same of the deterministic case deterministic because the uncertainty term $\sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2$ disappears after deriving, which is the essence of the Certainty Equivalence principle.

The term $\sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2$ is the result of the backward iteration which adds each time to the value function the - discount - of the term $\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2$ to account for the uncertainty of that future period after the initial time of evaluation t.

Hence, iterating through (C.4) brings to the stationary solution of the deterministic case which verifies the Bellman recursion at each time,

$$V_t(\mathbf{x}_t) = \mathbf{x}_t'\mathbf{P}^*\mathbf{x}_t + \sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2, \; \forall\, t. \quad \text{(C.5)}$$

More explicitly,

$$V_t(\mathbf{x}_t) = \delta E_t[\mathbf{x}_{t+1}^{*'}\mathbf{P}^*\mathbf{x}_{t+1}^*] + \mathbf{u}_t^{*'}\mathbf{u}_t^* = \underbrace{\delta[\mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t^*]'\mathbf{P}^*[\mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t^*] + \mathbf{u}_t^{*'}\mathbf{u}_t^*}_{\mathbf{x}_t'\mathbf{P}^*\mathbf{x}_t} + \sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2. \quad \text{(C.6)}$$

where it is easy (though time consuming) to prove that $\mathbf{x}_t'\mathbf{P}^*\mathbf{x}_t = \frac{1+r}{r}[rB_t + \frac{r}{1+r-\rho}g_t + \frac{1}{1+r-\rho}g_c]^2$, which is the second addend in (C.3) corresponding consistently to the deterministic part of that equation. As for the part containing the uncertainty term it is easy to check that $\sum_{j=1}^{+\infty}\delta^j P_{22_{t+j}} E_t \varepsilon_{t+j}^2 = \frac{1+r}{(1+r-\rho)^2} E_t \varepsilon_{t+j}^2$ since P_{22}^* is an average (constant) solution, which, after substituting, brings to

$$V_t(\mathbf{x}_t) = \underbrace{\mathbf{x}_t'\mathbf{P}^*\mathbf{x}_t}_{\frac{1+r}{r}[rB_t + \frac{r}{1+r-\rho}g_t + \frac{1}{1+r-\rho}g_c]^2} + \frac{1+r}{r^2}[\frac{r}{1+r-\rho}\sigma_\varepsilon]^2 \quad \text{(C.7)}$$

which is equal to (C.3).

□

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