ASSET ALLOCATION TO PREVENT UNEXPECTED LARGE LOSSES IN AN EXTREME VALUE THEORY FRAMEWORK

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ASSET ALLOCATION PROBLEM:
The aim of this work is to examine the optimal portfolio selection problem for a risk-adverse investor who wants to prevent large losses:

• minimizing quantile-based risk measure (for large quantile)
• under the Extreme Value Theory (EVT) framework.

EXTREMAL DEPENDENCE ANALYSIS
We also analyse the role of extremal dependence in this problem.
Several crises have overtaken the financial markets: forecasting risk and risk management, has thus become a major concern for both financial institutions and market regulators.

**Risk models**

- Combination of:
  - probability distribution model
  - risk measure

  It provides a measure of risk that could be employed in portfolio selection, risk management, derivatives pricing and so forth.

**Risk measure**

- Any functional (mapping) $\rho(X)$ that sets a real number to any random variable.
Choosing the proportions of various assets to minimize risk for a given level of expected return, or equivalently maximize portfolio expected return for a given amount of portfolio risk.

Firstly addressed by Markowitz (1952):
- mean-variance model
- normally distributed returns
- variance as a risk measure.

\[
\min_{\mathbf{w} \in \mathbb{R}^n} \sigma_P^2(\mathbf{w})
\]

\[\text{constraints:}\]
\[\mu_P(\mathbf{w}) = \mu_0\]
\[\sum_{k=1}^{n} w_k = 1\]
Asymmetry and heavy tails

Empirical evidence:
- The distribution of financial assets return is actually skewed and fat-tailed
- Normal distributed return assumption not reliable

Extreme value Theory
- Concerned with the asymptotic distribution of extreme events
- Models the tails, without making assumptions on the underlying data distribution

Empirical histogram and normal density

Focus on the right tail

Data source: Data stream
Nobs: 5834
Min: -20.5%; Max: 9.1%
Mean: 0.05%; sd: 1.02%
Skew: -1.51; Curtosi: 33.2
Drawbacks of the mean-variance and our approach

Variance as a risk measure

- It measures the spread of the distribution around the mean
- It assigns the same weight to gains as well as losses

Underweights extreme events and might lead to an optimistic asset allocation

Quantile-based risk measures

\[
VaR_p(X) = F_X^{-1}(p) = \inf \{ x \in \mathbb{R} \mid F_X(x) \geq p \}
\]

\[
CTE_p(X) = \mathbb{E}[X \mid X \geq VaR_p(X)]
\]

\[
ES_p(X) = \frac{1}{1 - p} \int_p^1 VaR_u(X) \, du.
\]
Threshold excess method
F. Balkema and de Haan (1974) and Pickands (1975)

Let $Z_1, Z_2, ...$ i.i.d. r.v with F distribution function, consider the distribution of all the amounts exceeding some large threshold $u$

$$F_u(z) = P\{Z - u \leq z | Z > u\}, \ 0 < z < z_F - u$$

where $z_F$ right end point of the support of the distribution.

It can be shown that is approximately a generalized Pareto distribution (GPD):

$$H_{\xi, \sigma}(z) = 1 - \left[1 + \xi \left(\frac{z}{\sigma}\right)\right]^{-\frac{1}{\xi}}, \begin{cases} z \geq 0 \text{ for } \xi \geq 0 \\ 0 \leq z \leq \sigma / \xi \text{ otherwise} \end{cases}$$
Model Approach
Asset allocation problem definition

\[ X = (X_1, X_2, \ldots, X_d) \text{ negative returns of } d \text{ assets in our universe} \]

**Loss** of a linear portfolio \( Z(w) \) with allocation \( w = (w_1, w_2, \ldots, w_d) \) is:

\[
Z(w) = \sum_{i=1}^{d} w_i X_i , \ 0 \leq w_i \leq 1
\]

**Optimal Asset allocation**: find the allocation \( w^* \) which minimizes the risk of \( Z(w) \), defining as risk measures RM (either the \( \text{VaR}_\alpha \) or the expected shortfall \( S_\alpha \)) with confidence level \( \alpha \) (design parameter)

\[
w^* = \arg\min_w RM_\alpha(Z(w)), \text{Constraints: } \sum_{i=1}^{d} w_i = 1, w_i \geq 0
\]

\( \text{VaR}_\alpha \) and \( S_\alpha \) estimation

The univariate EVT approach called the **structure variable method** (SVM)
VaR_{\alpha} \text{ and } S_{\alpha} \text{ estimation}

**Hp:** Suppose $Z_1, Z_2, \ldots$ are i.i.d. with distribution function $F$ representing the loss (negative return) of the portfolio $Z(w)$.

We apply the *threshold excess method* to approximate the tail of the structure variable $Z(w)$ distribution:

$$F_u(z) = P\{Z - u \leq z | Z > u\} = \frac{F(z + u) - F(u)}{1 - F(u)} = 1 - \left[1 + \xi \left(\frac{z}{\sigma}\right)\right]^{-\frac{1}{\xi}}$$

We obtain for $z > u$ and $u$ large

$$F(z) = 1 - \lambda_u \left[1 + \xi \left(\frac{z - u}{\sigma}\right)\right]^{-\frac{1}{\xi}}, \lambda_u = P\{Z > u\}$$

Setting $F(z_{\alpha}) = \alpha$ and solving for $z_{\alpha}$

$$\text{VaR}_{\alpha}(Z) = u + \frac{\sigma}{\xi} \left\{\frac{1}{\lambda_u} (1 - \alpha)\right\}^{-\frac{1}{\xi}}$$

Provided $\text{VaR}_{\alpha}(Z) > u$

$$S_{\alpha}(Z) = E(Z | Z > \text{VaR}_{\alpha}(Z)) = \frac{\text{VaR}_{\alpha}(Z)}{1 - \xi} + \frac{\sigma - \xi u}{1 - \xi}.$$
Model estimation – step procedure

Optimal Asset allocation – fixed confidence level $\alpha$

$$w^* = \arg\min_w RM_\alpha(Z(w)), \text{Constraints: } \sum_{i=1}^d w_i = 1, w_i \geq 0$$

where $RM_\alpha(Z(w))$ is $VaR_\alpha(Z)$ or $S_\alpha(Z)$

Varying $w = (w_1, w_2, \ldots, w_d), \forall w$:

1. Sample of negative returns: $X_i = (X_{i1}, X_{i2}, \ldots, X_{iT}), i=1,\ldots,d \rightarrow PTF$ negative returns $Z_j(w) = \sum_{i=1}^d w_i X_{ij}, j = 1, \ldots, T$

2. We estimate the parameters $(\lambda, \sigma, \xi)$ of the tail portfolio distribution (GPD) using maximum likelihood.

3. Risk measure: $VaR_\alpha$ or $S_\alpha$ calculation using previous parametric formulas

Identifying the $w$ that minimize the PTF risk measure calculated in step 3.

Threshold choice: trade off between bias-variance. According with literature, we take $u$ as the 95% quantile of the empirical distribution of $Z(w)$. 
Optimal minimum risk portfolios

**OPTIMUM SEARCH ALGORITHM**

Increasing the number of assets, the optimum search becomes **computationally hard to manage**

**Two stage methodology:**

- **Sampling algorithm of Bensalah** to pick a **starting point**:
  - randomly sample \( n_s \) portfolio weights from uniform distributions, picking the one that leads to minimal risk.

- **Incremental trade algorithm** to step away from the starting allocation:
  - The algorithm takes **steps of size** \( \delta_w \) in each market away from its current position (buying and selling) and it picks the trade which is most risk reducing.
Extremal dependence

We use two dependence measures from EVT proposed by Coles et al for bivariate r.v.
• Influence of the marginals removed by standardizing to have unit Frèchet distribution

\[
\chi = \lim_{y \to \infty} P\{Y_2 > y|Y_1 > y\}, P\{Y_2 > y|Y_1 > y\} \sim L(y)y^{1-1/\eta} \\
\begin{cases} 
\chi > 0 \text{ asymptotically dependent} \\
\chi = 0 \text{ asymptotically independent}
\end{cases}
\]

\[L(y)\] slowly varying function, \(\eta \in (0, 1] \) coefficient of tail dependence.

\[
\bar{\chi} = \lim_{y \to \infty} \frac{2\log P\{Y_1 > y\}}{\log P\{Y_1 > y, Y_2 > y\}} - 1 = 2\eta - 1
\]

**Interpretation**

1) \(\chi = 0\) \(\bar{\chi}\) provides a measure of the strength of dependence

2) \(\bar{\chi} = 1\) \(\chi\) measures the strength of asymptotic dependence
Data and results
SAMPLE FINANCIAL DATA

- **Daily simple return:** calculated from total return equity indices
- **Source:** Datastream
- **Period:** 02-Jan-1980 to 3-Sept-2015
- **Assets:** \( d=12, \) 12 international equity indices representative of 12 markets
<table>
<thead>
<tr>
<th>Market</th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>stdev</th>
<th>skew</th>
<th>kurt</th>
<th>MV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hong Kong (HK)</td>
<td>34,724</td>
<td>16,832</td>
<td>0.043</td>
<td>1,648</td>
<td>-</td>
<td>1,454</td>
<td>31,767</td>
</tr>
<tr>
<td>Japan (JP)</td>
<td>16,000</td>
<td>12,000</td>
<td>0.031</td>
<td>1,352</td>
<td>-</td>
<td>0.010</td>
<td>6,215</td>
</tr>
<tr>
<td>Australia (AU)</td>
<td>26,157</td>
<td>8,739</td>
<td>0.031</td>
<td>1,383</td>
<td>-</td>
<td>1,191</td>
<td>20,756</td>
</tr>
<tr>
<td>Belgium (BG)</td>
<td>11,146</td>
<td>10,204</td>
<td>0.034</td>
<td>1,180</td>
<td>-</td>
<td>0.090</td>
<td>6,316</td>
</tr>
<tr>
<td>Canada (CN)</td>
<td>12,660</td>
<td>9,988</td>
<td>0.031</td>
<td>1,110</td>
<td>-</td>
<td>0,611</td>
<td>12,607</td>
</tr>
<tr>
<td>France (FR)</td>
<td>10,142</td>
<td>11,234</td>
<td>0.035</td>
<td>1,328</td>
<td>-</td>
<td>0,085</td>
<td>5,921</td>
</tr>
<tr>
<td>Germany (BD)</td>
<td>11,733</td>
<td>17,659</td>
<td>0.033</td>
<td>1,279</td>
<td>-</td>
<td>0,077</td>
<td>8,838</td>
</tr>
<tr>
<td>Italy (IT)</td>
<td>10,328</td>
<td>11,914</td>
<td>0.036</td>
<td>1,529</td>
<td>-</td>
<td>0,048</td>
<td>4,891</td>
</tr>
<tr>
<td>Netherlands (NL)</td>
<td>10,852</td>
<td>10,727</td>
<td>0.036</td>
<td>1,243</td>
<td>-</td>
<td>0,125</td>
<td>7,514</td>
</tr>
<tr>
<td>Switzerland (SW)</td>
<td>10,497</td>
<td>9,465</td>
<td>0.040</td>
<td>1,083</td>
<td>-</td>
<td>0,200</td>
<td>6,125</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>13,528</td>
<td>12,543</td>
<td>0.033</td>
<td>1,205</td>
<td>-</td>
<td>0,194</td>
<td>8,865</td>
</tr>
<tr>
<td>United States (US)</td>
<td>18,705</td>
<td>11,518</td>
<td>0.039</td>
<td>1,084</td>
<td>-</td>
<td>0,649</td>
<td>17,879</td>
</tr>
</tbody>
</table>

Table 1. Descriptive statistics based on 9,307 observations of daily simple periodic returns determined from the price index. MV in millions of US dollars.
Results – Extremal dependence analysis

We considered 66 different pairs of market and studied the dependence analysing both the left and the right tail:

- **Japan**: only asymptotic independence market
- **US**: lowest asymptotic dependence strength
- **France**: market with the highest dependence

<table>
<thead>
<tr>
<th>Country Pair</th>
<th>$\bar{\chi}$</th>
<th>$\sigma(\bar{\chi})$</th>
<th>$\bar{\chi} \geq 1 + 1.96\sigma(\bar{\chi})$</th>
<th>$\chi$</th>
<th>$\sigma(\chi)$</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>JP vs FR</td>
<td>0.4529</td>
<td>0.0674</td>
<td>FALSE</td>
<td></td>
<td></td>
<td>0.249</td>
</tr>
<tr>
<td>JP vs BD</td>
<td>0.4051</td>
<td>0.0652</td>
<td>FALSE</td>
<td></td>
<td></td>
<td>0.257</td>
</tr>
<tr>
<td>JP vs UK</td>
<td>0.4270</td>
<td>0.0662</td>
<td>FALSE</td>
<td></td>
<td></td>
<td>0.265</td>
</tr>
<tr>
<td>JP vs US</td>
<td>0.3639</td>
<td>0.0633</td>
<td>FALSE</td>
<td></td>
<td></td>
<td>0.062</td>
</tr>
<tr>
<td>FR vs BD</td>
<td>1.0000</td>
<td>0.0972</td>
<td>TRUE</td>
<td>0.568</td>
<td>0.026</td>
<td>0.762</td>
</tr>
<tr>
<td>FR vs UK</td>
<td>1.0000</td>
<td>0.0937</td>
<td>TRUE</td>
<td>0.524</td>
<td>0.024</td>
<td>0.721</td>
</tr>
<tr>
<td>FR vs US</td>
<td>0.9184</td>
<td>0.0890</td>
<td>TRUE</td>
<td>0.334</td>
<td>0.015</td>
<td>0.402</td>
</tr>
<tr>
<td>BD vs UK</td>
<td>1.0000</td>
<td>0.0954</td>
<td>TRUE</td>
<td>0.463</td>
<td>0.021</td>
<td>0.663</td>
</tr>
<tr>
<td>BD vs US</td>
<td>0.9803</td>
<td>0.0918</td>
<td>TRUE</td>
<td>0.340</td>
<td>0.015</td>
<td>0.424</td>
</tr>
<tr>
<td>UK vs US</td>
<td>0.8749</td>
<td>0.0870</td>
<td>TRUE</td>
<td>0.345</td>
<td>0.016</td>
<td>0.412</td>
</tr>
</tbody>
</table>

Dependence between G5 countries (Loss tail)
Results - Asset allocation

Risk Measure: **Expected Shortfall**

Assets: equity indices of **G5 countries**

Optimal $S_\alpha$ allocations of the two stage incremental trade algorithm for the G5 markets as a function of high confidence level $\alpha$.

<table>
<thead>
<tr>
<th>$S_\alpha$</th>
<th>$w^N$</th>
<th>$\alpha_1=0.975$</th>
<th>$\alpha_2=0.99$</th>
<th>$\alpha_3=0.999$</th>
<th>$\alpha_4=0.9999$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$w_{\alpha_1^*}$</td>
<td>$S_{\alpha_1}$</td>
<td>$w_{\alpha_2^*}$</td>
<td>$S_{\alpha_2}$</td>
</tr>
<tr>
<td>US</td>
<td>0.453</td>
<td>0.429</td>
<td>3.229</td>
<td>0.434</td>
<td>4.375</td>
</tr>
<tr>
<td>JP</td>
<td>0.303</td>
<td>0.357</td>
<td>3.682</td>
<td>0.362</td>
<td>4.700</td>
</tr>
<tr>
<td>UK</td>
<td>0.163</td>
<td>0.182</td>
<td>3.484</td>
<td>0.172</td>
<td>4.562</td>
</tr>
<tr>
<td>FR</td>
<td>0.012</td>
<td>0.004</td>
<td>3.865</td>
<td>0.004</td>
<td>4.982</td>
</tr>
<tr>
<td>BD</td>
<td>0.070</td>
<td>0.028</td>
<td>3.665</td>
<td>0.028</td>
<td>4.718</td>
</tr>
<tr>
<td>$S_\alpha(N)$</td>
<td></td>
<td>2,42888</td>
<td>3,25481</td>
<td>6,41318</td>
<td>12,07465</td>
</tr>
<tr>
<td>$S_\alpha(O)$</td>
<td></td>
<td>2,41145</td>
<td>3,20894</td>
<td>5,67920</td>
<td>8,83772</td>
</tr>
<tr>
<td>$S_\alpha(N)/S_\alpha(O)$</td>
<td></td>
<td>1,007</td>
<td>1,014</td>
<td>1,129</td>
<td>1,366</td>
</tr>
</tbody>
</table>
Conclusions and future developments

• The **optimal allocation is quantile-based**, i.e. depends on $\alpha$, confirming the findings of Bensalah (2002);

• Using the EVT dependence measures we find that almost half pairs of the twelve equity markets examined here are **asymptotically independent**.

• A surprising result is the **robustness of the assumption of normality** on the allocation problem at standard confidence levels ($\alpha = 0.975, 0.99$).

• When moving to more extreme quantiles ($\alpha = 0.999, 0.9999$), the difference between the two approaches can no longer be ignored.

**Future developments**

• Insert a **bond component**: move to a more realistic portfolio;

• **Comparing** performance of different methodologies:
  - Block maxima method vs Excess over threshold method;
  - parametric vs non parametric approach.

• Insert **constraint** about the equity-bond portfolio composition;

• Take into account a **target return** in the asset allocation problem.
Main references


Thank you!

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