Pseudo-processes governed by higher-order fractional differential equations

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Abstract

We study here a heat-type differential equation of order n greater than two, in the case where the time-derivative is supposed to be fractional. The corresponding solution can be described as the transition function of a pseudo-process Ψ_n (coinciding with the one governed by the standard, non-fractional, equation) with a time argument $\mathcal{T}_\alpha$ which is itself random. The distribution of $\mathcal{T}_\alpha$ is presented together with some features of the solution (such as analytic expressions for its moments).

Keywords: Higher-order heat-type equations, Fractional derivatives, Wright functions, Stable laws.

1 Introduction

The study of diffusion equations with a fractional derivative component have been firstly motivated by the analysis of thermal diffusion in fractal media in Nigmatullin (1986) and Saichev, Zaslavsky (1997). This topic has been extensively treated in the probabilistic literature since the end of the Eighties: see, for examples, Wyss (1986), Schneider, Wyss (1989), Mainardi (1996), Angulo et al. (2000). Recently fractional equations of different types have been also studied, such as, for example, the Black and Scholes equation (see Wyss (2000)) and the fractional diffusion equations with stochastic initial conditions (see Anh, Leonenko (2000)).

Our aim will concern the extension, to the case of fractional time-derivative, of a class of equations which is well known in the literature, namely the *higher-order heat-type equations*. Therefore we will be interested in the solution of the following problem, for $0 < \alpha \leq 1,\ n \geq 2$,

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t) & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x) \end{cases} \tag{1}$$

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where $\delta(\cdot)$ is the Dirac delta function, $k_n = (-1)^{q+1}$ for $n = 2q$, $q \in \mathbb{N}$, while $k_n = \pm 1$ for $n = 2q+1$. The fractional derivative appearing in (1) is meant, in the Dzherbashyan-Caputo sense, as

$$(D^\alpha f)(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(z)}{(t-z)^{1+\alpha-m}} dz, & \text{for } m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \text{for } \alpha = m \end{cases},$$

where $m - 1 = \lfloor \alpha \rfloor$ and $f \in C^m$ (see Samko et al. (1993) for a general reference on fractional calculus).

In the non-fractional case (which can be obtained from ours as a particular case, for $\alpha = 1$) the *pseudo-processes* $\Psi_n = \Psi_n(t)$, $t > 0$ driven by n-th order equations like

$$\frac{\partial}{\partial t} p(x,t) = k_n \frac{\partial^n}{\partial x^n} p(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{2}$$

for $n > 2$, have been introduced in the Sixties and studied so far by many authors starting from Krylov (1960), Daletsky (1969) and many others. Moreover the distributions of many functionals of Ψ_n have been obtained: in Hochberg, Orsingher (1994) the distribution of sojourn time on the positive half-line is presented, for n odd, while for an arbitrary n the same topic is analyzed in Lachal (2003). For $n = 3, 4$, the case where the pseudoprocess is constrained to be zero at the end of the time interval is considered in Nikitin, Orsingher (2000) and the corresponding distribution of the sojourn time is evaluated. In Beghin et al. (2000) the distribution of the maximum is obtained under the same circumstances. In the unconditional case the maximal distribution is presented in Orsingher (1991), for n odd, while the joint distribution of the maximum and the process for diffusion of order $n = 3, 4$ is presented in Beghin et al. (2001). Lachal (2003) has extended these results to any order $n > 2$.

Some other functionals, such as the first passage time, are treated in Nishioka (1997) and Lachal (2006). Finally in Beghin and Orsingher (2005) it is proved that the local time in zero possesses a proper probability distribution which coincides with the (folded) solution of a fractional diffusion equation of order $2(n-1)/n$, $n \geq 2$.

In the fractional case under investigation we prove that the fundamental solution of (1) can be expressed in the following form:

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u) \overline{v}_{2\alpha}(u,t) du. \tag{3}$$

where

$$p_n(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixz + k_n t (iz)^n} dz$$

is the fundamental solution to the non-fractional n-th order equation (2) and

$$\overline{v}_{2\alpha}(u,t) = \begin{cases} 2v_{2\alpha}(u,t) & u \geq 0 \\ 0 & u < 0 \end{cases}, \tag{4}$$

where $v_{2\alpha}(u,t)$ is the solution to the fractional diffusion equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}v(u,t)=\frac{\partial^2}{\partial u^2}v(u,t) \qquad u\in\mathbb{R},\ t>0, \tag{5}$$

for $0<\alpha\leq 1$.

Formula (3) proves that the process related to (1) is a pseudo-process Ψ_n evaluated at a random time $\mathcal{T}_\alpha=\mathcal{T}_\alpha(t), t>0$ and that the probability law of the latter is solution to the equation (5), so that we can write it as $\Psi_n(\mathcal{T}_\alpha)$. By using some known results on this kind of equations we can therefore conclude that the density of $\mathcal{T}_\alpha$ can be also written as

$$\overline{v}_{2\alpha}(u,t)=\frac{1}{t^\alpha}W\left(-\frac{u}{t^\alpha};-\alpha,1-\alpha\right), \qquad u\geq 0,\ t>0 \tag{6}$$

where $W(\cdot;\eta,\beta)$ denotes the *Wright function* $W(x;\eta,\beta)=\sum_{k=0}^{\infty}x^k/k!\Gamma\left(\eta k+\beta\right)$.

It is interesting to underline that the introduction of a fractional time-derivative exerts its influence only on the "temporal" argument, while the governing process is not affected and depends only on the degree n of the equation.

If we restrict ourselves to the case $\alpha\in[1/2,1]$, so that $1\leq 2\alpha\leq 2$, it is possible to obtain a more explicit form of the solution. Indeed, in this case, we can take advantage from a well-known result by Fujita (1990): the solution to (5) coincides with

$$v_{2\alpha}(u,t)=\frac{1}{2\alpha}\widetilde{p}_{\frac{1}{\alpha}}(|u|;t), \qquad u\in\mathbb{R},$$

where by $\widetilde{p}_{\frac{1}{\alpha}}(\cdot;t)$ we have denoted the *stable law* of index $1/\alpha$, namely $S_{1/\alpha}(\sigma,-1,0)$, with scale parameter $\sigma=(t\cos(\pi-\frac{\pi}{2\alpha}))^\alpha$ (in the notation by Samorodnitsky, Taqqu (1994)). Therefore the distribution of the random time $\mathcal{T}_\alpha$ coincides with

$$\overline{v}_{2\alpha}(u,t)=\begin{cases}\frac{1}{\alpha}\widetilde{p}_{\frac{1}{\alpha}}(u;t), & u\geq 0\\ 0, & u<0\end{cases} \tag{7}$$

and the fundamental solution of (1) can be expressed in the following form:

$$u_\alpha(x,t)=\frac{1}{\alpha}\int_0^\infty p_n(x,u)\widetilde{p}_{\frac{1}{\alpha}}(u;t)du. \tag{8}$$

As particular cases of the previous result we obtain some known expressions: in the non-fractional case, $\alpha=1$, we easily get $\mathcal{T}_\alpha(t)\overset{a.s.}{=}t$. For $\alpha=1/2$ it can be verified that (7) reduces to the transition function of a reflecting Brownian motion and then (8) coincides with the density of $\Psi_n(|B(t)|), t>0$ where B denotes a standard Brownian motion.

As far as the other interval is concerned (i.e. $\alpha\in(0,1/2]$), an explicit expression of the solution can be evaluated by specifying $\alpha=1/m,\ m\in\mathbb{N}$, $m>2$. In this particular setting equation (5) becomes

$$\frac{\partial^{2/m}}{\partial t^{2/m}}v(u,t)=\frac{\partial^2}{\partial u^2}v(u,t),\ u\in\mathbb{R},\ t>0$$

and then it coincides with a special case of the fractional telegraph equation considered in Beghin, Orsingher (2003). As a consequence of these results it can be shown that our equation (1) is solved by

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u) p_{G(t)}(u) du,$$

where $G(t) = \prod_{j=1}^{m-1} G_j(t), t > 0$ and $G_j(t),\ j = 1, ..., m-1$ are independent random variables with the following probability law

$$p_{G_j(t)}(w) = \frac{1}{m^{\frac{j}{m-1}-1} t^{\frac{j}{m(m-1)}} \Gamma(\frac{j}{m})} \exp\left(- \frac{w^m}{\sqrt[m-1]{m^m t}} \right) w^{j-1}, \quad w > 0 \tag{9}$$

and the corresponding pseudoprocess is represented, in this case, as $\Psi_n(G(t)), t > 0$.

Some interesting results can be obtained by specifying (1) for particular values of n : for example, taking $n = 2$, we can conclude that the process related to

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \qquad x \in \mathbb{R},\ t > 0, \tag{10}$$

is, for $\alpha \in [1/2, 1)$, represented by $B(\mathcal{T}_\alpha)$, with $\mathcal{T}_\alpha = \mathcal{T}_\alpha(t), t > 0$ distributed again with the density (7). This is proved to be in accordance with the results already known on (10). On the other hand, for $\alpha \in (0, 1/2]$ equation (10) turns out to be solved by the density of the process $B(G(t)), t > 0$, where the random variables appearing in the temporal argument possess again distribution (9).

In the special case $n = 3$ the results above coincide with those presented in De Gregorio (2002), while, for $n = 4$, they represent a probabilistic alternative to the analytic approach provided by Agrawal (2000).

2 First expressions for the solution

We start by considering the n-th order fractional equation and the following corresponding initial-value problem, for $0 < \alpha \leq 1,\ n \geq 2$,

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \delta(x) \end{cases} \tag{11}$$

where $k_n = (-1)^{q+1}$ for $n = 2q$, $q \in \mathbb{N}$, while $k_n = \pm 1$ for $n = 2q+1$ and $\delta(\cdot)$ is the Dirac delta function. The first step consists in evaluating the Laplace transform of the solution $u_\alpha(x,t)$, namely

$$U_\alpha(x,s) = \int_0^\infty e^{-st} u_\alpha(x,t) dt, \tag{12}$$

and recognizing that it is related to the Laplace transform of the solution $p_n(x,t)$ of the corresponding non-fractional n-th order equation (which can be derived from (11) for $\alpha = 1$).

Theorem 2.1 Let $\Phi_n(x,s)=\int_0^\infty e^{-st}p_n(x,t)dt$ be the Laplace transform of the solution to (2), then (12) can be expressed as follows

$$U_\alpha(x,s)=s^{\alpha-1}\Phi_n(x,s^\alpha). \tag{13}$$

Proof

By taking the Laplace transform of (11) and considering the initial condition, we get

$$s^\alpha U_\alpha(x,s)-s^{\alpha-1}\delta(x)=k_n\frac{\partial^n}{\partial x^n}U_\alpha(x,s). \tag{14}$$

Then, by integrating (14) with respect to x in $[-\varepsilon,\varepsilon]$ and letting $\varepsilon\to 0$, we have the following condition for the $(n-1)$-th derivative

$$-s^{\alpha-1}=k_n\ \frac{\partial^{n-1}}{\partial x^{n-1}}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+},$$

which must be added to the continuity conditions in zero holding for the j-th derivatives, for $j=0,..,n-2$. Therefore our problem is reduced to the n-th order linear equations

$$\begin{cases} k_n\frac{\partial^n}{\partial x^n}U_\alpha(x,s)=s^\alpha U_\alpha(x,s), & x\neq 0\\ \frac{\partial^j}{\partial x^j}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+}=0, & \text{for } j=0,1,...,n-2\\ \frac{\partial^{n-1}}{\partial x^{n-1}}U_\alpha(x,s)\Big|_{x=0^-}^{x=0^+}=-k_ns^{\alpha-1} & \end{cases}. \tag{15}$$

If we now impose the boundedness condition for $x\to\pm\infty$, we obtain

$$U_\alpha(x,s)=\begin{cases}\sum_{k\in I}c_ke^{\theta_ks^{\alpha/n}x}, & \text{if } x>0\\ \sum_{k\in J}d_ke^{\theta_ks^{\alpha/n}x}, & \text{if } x\leq 0\end{cases}, \tag{16}$$

where θ_k are the n-th roots of k_n, $I=\{k:\mathbb{R}e(\theta_k)<0\}$ and $J=\{k:\mathbb{R}e(\theta_k)>0\}$. The n unknown constants $c_k,k\in I$ and $d_k,k\in J$, appearing in (16) must be determined by taking into account the matching conditions in (15), as follows:

$$\begin{cases}\sum_{k\in I}c_k\theta_k^j-\sum_{k\in J}d_k\theta_k^j=0, & \text{for } j=0,...,n-2\\ \sum_{k\in I}c_k\theta_k^{n-1}-\sum_{k\in J}d_k\theta_k^{n-1}=-k_ns^{\alpha/n-1} & \end{cases}. \tag{17}$$

The linear system in (17) can be rewritten, by defining

$$z_k=\begin{cases}c_k, & \text{if } k\in I\\ -d_k, & \text{if } k\in J\end{cases}, \tag{18}$$

as the following Vandermonde system

$$\sum_{k=0}^{n-1}z_k\theta_k^j=\begin{cases}0, & \text{for } j=0,...,n-2\\ -k_ns^{\alpha/n-1}, & \text{for } j=n-1\end{cases}. \tag{19}$$

By following an argument similar to Beghin and Orsingher (2005) (see p.1024-5) we get

$$\begin{aligned} z_k &= (-1)^n k_n s^{\alpha/n-1} \prod_{\substack{r=0 \\ r\neq k}}^{n-1} \frac{1}{\theta_r - \theta_k} \qquad (20) \\ &= \begin{cases} -\frac{1}{n} s^{\alpha/n-1} e^{\frac{2k\pi i}{n}}, & \text{if } k_n = 1 \\ -\frac{1}{n} s^{\alpha/n-1} e^{\frac{(2k+1)\pi i}{n}}, & \text{if } k_n = -1 \end{cases}, \end{aligned}$$

where, in the last step, we have used formula (2.19) obtained therein. We now substitute into (16) the constants evaluated in (20), taking into account (18) and distinguishing the case of n even from the odd one. Indeed, for $n = 2q + 1$, the roots of k_n are respectively

$$\theta_k = \begin{cases} e^{\frac{2k\pi i}{n}}, & \text{for } k_n = 1 \\ e^{\frac{(2k+1)\pi i}{n}}, & \text{for } k_n = -1 \end{cases} \qquad (21)$$

so that (16) becomes, in this case,

$$U_\alpha(x, s) = \begin{cases} -\frac{1}{n} s^{\alpha/n-1} \sum_{k\in I} \theta_k e^{\theta_k s^{\alpha/n} x}, & \text{for } x > 0 \\ \frac{1}{n} s^{\alpha/n-1} \sum_{k\in J} \theta_k e^{\theta_k s^{\alpha/n} x}, & \text{for } x \leq 0 \end{cases}. \qquad (22)$$

Analogously, for $n = 2q$ and $k_n = (-1)^{q+1}$, the roots are $\theta_k = e^{\frac{(2k+q+1)\pi i}{n}}$ so that we get

$$\theta_k = \begin{cases} e^{\frac{(2k+q+1)\pi i}{n}} = e^{\frac{2k\pi i}{n}}, & \text{for } k_n = 1 \\ e^{\frac{(2k+q+1)\pi i}{n}} = e^{\frac{(2k+1)\pi i}{n}}, & \text{for } k_n = -1 \end{cases}, \qquad (23)$$

where, in the first line, we have used the following relationship

$$e^{(q+1)\pi i} = (-1)^{q+1} = k_n = 1,$$

while, in the second one, we have considered the fact that

$$e^{q\pi i} = (-1)k_n = 1.$$

Since (23) coincides with (21) we obtain even for $n = 2q$ formula (22). Finally we can draw the conclusion of the theorem by comparing it with formula (12) of Lachal (2003), which reads

$$\Phi_n(x, s) = \begin{cases} -\frac{1}{n} s^{1/n-1} \sum_{k\in I} \theta_k e^{\theta_k s^{1/n} x}, & \text{for } x > 0 \\ \frac{1}{n} s^{1/n-1} \sum_{k\in J} \theta_k e^{\theta_k s^{1/n} x}, & \text{for } x \leq 0 \end{cases}.$$

$\square$

By inverting the Laplace transform (13) we can obtain a first expression of the solution in terms of a fractional integral of a particular stable law. We will denote by $S_\alpha(\sigma, \beta, \mu)$ the probability density of a stable random variable X of index α, with characteristic function

$$Ee^{isX} = \exp\left\{-\sigma^\alpha |s|^\alpha (1 - i\beta(sign\, s)\tan\frac{\pi\alpha}{2} + i\mu s\right\}, \qquad \alpha \neq 1,\ s \in \mathbb{R}, \quad (24)$$

and by $I_{(1-\alpha)}$ the Riemann-Liouville fractional integral of order $1-\alpha$, which is defined as $I_{(1-\alpha)}\left[f(w)\right](t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha} f(w)dw$.

Theorem 2.2 Let $\overline{p}_\alpha(\cdot; u)$ be the stable density $S_\alpha(\sigma, 1, 0)$, with parameters $\sigma = (u\cos\pi\alpha/2)^{1/\alpha}, \beta = 1, \mu = 0$, then the fundamental solution to (11) can be expressed, for $0 < \alpha \leq 2$, $\alpha \neq 1$, as

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u) I_{(1-\alpha)}\left[\overline{p}_\alpha(w;u)\right](t)du. \quad (25)$$

Proof

We recall that, for $0 < \alpha \leq 2$ and $\alpha \neq 1$, a stable random variable $X \sim S_\alpha(\sigma, 1, 0)$ has Laplace transform

$$E(e^{-sX}) = e^{-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}s^\alpha}, \qquad s > 0$$

(see Samorodnitsky and Taqqu (1994) for details), so that, in our case (for $\sigma = (u\cos\pi\alpha/2)^{1/\alpha}$), it reduces to $E(e^{-sX}) = e^{-s^\alpha u}$. Therefore we can rewrite (13) as

$$\begin{aligned} U_\alpha(x,s) &= s^{\alpha-1}\int_0^{+\infty} e^{-s^\alpha t} p_n(x,t)dt \quad (26)\\ &= s^{\alpha-1}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-sz}\overline{p}_\alpha(z;u)dz\right)p_n(x,u)du \\ &= s^{\alpha-1}\int_0^{+\infty} e^{-sz}\left(\int_0^{+\infty}\overline{p}_\alpha(z;u)p_n(x,u)du\right)dz. \end{aligned}$$

For $0 < \alpha < 1$ the first term appearing in (26) can be easily inverted by considering that

$$s^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty} e^{-st}t^{-\alpha}dt$$

so that the inverse Laplace transform of (26) can be written as

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha}\left(\int_0^{+\infty}\overline{p}_\alpha(w;u)p_n(x,u)du\right)dw \quad (27)\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty}\left(\int_0^t (t-w)^{-\alpha}\overline{p}_\alpha(w;u)dw\right)p_n(x,u)du. \end{aligned}$$

Finally we recognize in the last expression a fractional Riemann-Liouville integral $I_{(1-\alpha)}$ of order $1-\alpha$ of the stable density (where the integration is intended with respect to the first argument, since the second represents a constants in the scale parameter). $\square$

The previous result suggests that the solution to our problem can be described as the transition function $p_n = p_n(x,u)$ of a pseudoprocess Ψ_n with a time-argument $\mathcal{T}_\alpha$ which is itself random. Only for $\alpha = 1$ we can derive from Theorem 2.1 the obvious result that $\mathcal{T}_\alpha(t) \overset{a.s.}{=} t$, so that the solution to (11) coincides, as expected, with $p_n(x,t)$. In all other cases the governing process coincides with the non-fractional one, while the introduction of a fractional time-derivative exerts its influence only on the time argument (as remarked before).

To check that $\mathcal{T}_\alpha$ possesses a true probability density we can observe that it is non-negative since it coincides with the fractional integral of a stable density $S_\alpha(\sigma,1,0)$ with skewness parameter equal to 1, which, for $0<\alpha<1$, has support restricted to $[0,\infty)$. Moreover it integrates to one, as can be ascertained by the following steps:

$$\begin{aligned}
&\int_0^\infty \frac{du}{\Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} \overline{p}_\alpha(w;u)dw \\
=\;& \frac{1}{\Gamma(1-\alpha)} \int_0^\infty du \int_0^t (t-w)^{-\alpha} dw \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sw} e^{-s^\alpha u} ds \\
=\;& \frac{1}{2\pi i \Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} dw \int_0^\infty du \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sw} e^{-s^\alpha u} ds \\
=\;& \frac{1}{2\pi i \Gamma(1-\alpha)} \int_0^t (t-w)^{-\alpha} dw \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-\alpha} e^{sw} ds \\
=\;& \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t w^{\alpha-1}(t-w)^{-\alpha} dw = \frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} = 1\,.
\end{aligned}$$

Our aim is now to explicit, by means of successive steps, the density $\overline{v}_{2\alpha} = \overline{v}_{2\alpha}(u,t)$ of $\mathcal{T}_\alpha(t), t>0$: we first prove that it satisfies a fractional diffusion equation of order 2α and, as a consequence, can be expressed in terms of a Wright function of appropriate parameters.

Theorem 2.3 The fundamental solution to (11) coincides with

$$u_\alpha(x,t) = \int_0^\infty p_n(x,u)\overline{v}_{2\alpha}(u,t)du, \tag{28}$$

where

$$\overline{v}_{2\alpha}(u,t) = \frac{1}{t^\alpha} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right), \qquad u \geq 0,\ t>0 \tag{29}$$

and $W(\cdot;\alpha,\beta)$ denotes the Wright function.

Proof

It is proved in Orsingher and Beghin (2004) that, for $0 < \alpha < 1$,

$$I_{(1-\alpha)}\left[\overline{p}_\alpha(|w|;u)\right](t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-w)^{-\alpha}\overline{p}_\alpha(|w|;u)dw$$

coincides with the solution $v_{2\alpha}(u,t)$ of the following initial-value problem, for $0 < \alpha < 1$,

$$\begin{cases} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}v(u,t) = \frac{\partial^2}{\partial u^2}v(u,t) & u \in \mathbb{R},\ t > 0 \\ v(u,0) = \delta(u) \\ \frac{\partial}{\partial t}v(u,0) = 0 \\ \lim_{|u|\to\infty} v(u,t) = 0 \end{cases}, \tag{30}$$

where the second initial condition applies only for $\alpha \in (1/2, 1)$. As a consequence, formula (25) can be rewritten as (28) with

$$\overline{v}_{2\alpha}(u,t) = \begin{cases} 2v_{2\alpha}(u,t), & \text{for } u \geq 0 \\ 0, & \text{for } u < 0 \end{cases}. \tag{31}$$

Since it is known (see, among the others, Mainardi (1996)) that the solution to (30) can be expressed as

$$\begin{aligned} v_{2\alpha}(u,t) &= \frac{1}{2t^\alpha}\sum_{k=0}^\infty \frac{(-|u|t^{-\alpha})^k}{k!\Gamma(-\alpha k + 1 - \alpha)} \\ &= \frac{1}{2t^\alpha} W\left(-\frac{|u|}{t^\alpha}; -\alpha, 1-\alpha\right), \qquad u \in \mathbb{R},\ t > 0, \end{aligned}$$

we immediately get (29). □

Remark 2.1

By means of the previous result we can remark again that the random time $\mathcal{T}_\alpha$ possesses a true probability density, which is concentrated on the positive half line and moreover it is possible, thanks to representation (29), to evaluate the moments of any order $\delta \geq 0$ of this distribution. We recall the well known expression of the inverse of the Gamma function as integral on the Hankel contour

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_{Ha} e^\tau \tau^{-x} d\tau,$$

which implies the representation of the Wright function as

$$\begin{aligned} W(x;\eta,\beta) &= \sum_{k=0}^\infty \frac{x^k}{k!\Gamma(\eta k + \beta)} \\ &= \frac{1}{2\pi i}\int_{Ha} e^\tau \sum_{k=0}^\infty \frac{x^k \tau^{-\eta k - \beta}}{k!} d\tau \\ &= \frac{1}{2\pi i}\int_{Ha} \tau^{-\beta} e^{\tau + x\tau^{-\eta}} d\tau. \end{aligned}$$

Therefore we can show that

$$\begin{aligned}
& \int_0^{+\infty} u^\delta \overline{v}_{2\alpha}(u,t)du && (32)\\
= & \int_0^{\infty} \frac{u^\delta}{t^\alpha} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right) du \\
= & \int_0^{\infty} \frac{u^\delta}{t^\alpha} \frac{du}{2\pi i} \int_{Ha} e^{y-\frac{u}{t^\alpha}y^\alpha} y^{\alpha-1} dy \\
= & \frac{1}{2\pi i} \int_{Ha} e^y y^{\alpha-1} dy \frac{1}{t^\alpha} \int_0^{+\infty} e^{-\frac{u}{t^\alpha}y^\alpha} u^\delta du \\
= & \frac{t^{\alpha\delta}}{2\pi i} \int_{Ha} e^y y^{-\alpha\delta-1} dy \int_0^{+\infty} e^{-z} z^\delta dz \\
= & \frac{\Gamma(1+\delta)t^{\alpha\delta}}{\Gamma(1+\alpha\delta)} = \frac{t^{\alpha\delta}\Gamma(\delta)}{\alpha\Gamma(\alpha\delta)} .
\end{aligned}$$

From (32) it is again evident that $\int_0^{+\infty} \overline{v}_{2\alpha}(u,t)du = 1$ by choosing $\delta = 0$.

It is interesting to analyze the particular case obtained for $\alpha = 1/2$: indeed, from the previous results, we can show that the process governed by $\frac{\partial^{1/2}}{\partial t^{1/2}} u(x,t) = k_n \frac{\partial^n}{\partial x^n} u(x,t)$, $x \in \mathbb{R}$, $t > 0$, can be represented as $\Psi_n(|B(t)|), t > 0$, where $B(t), t > 0$ denotes a standard Brownian motion. This can be seen by noting that $S_{1/2}\left(\frac{u^2}{2}, 1, 0\right)$ coincides with the Lévy distribution, so that the fractional integral in (25) reduces to

$$\begin{aligned}
I_{(1/2)}\left[\overline{p}_{1/2}(w;u)\right](t) &= \frac{1}{\Gamma(1/2)} \int_0^t \frac{ue^{-u^2/4w}}{2\sqrt{\pi(t-w)w^3}} dw && (33)\\
&= \frac{e^{-u^2/4t}}{\sqrt{\pi t}}, \qquad u > 0, t > 0,
\end{aligned}$$

where the second step follows by applying formula n.3.471.3, p.384 of Gradshteyn and Rhyzik (1994), for $\mu = 1/2$. Formula (33) represents the density of a Brownian motion with reflecting barrier in $u = 0$. This result is confirmed by noting that equation (30), for $\alpha = 1/2$, reduces to the heat equation $\frac{\partial}{\partial t} v(x,t) = \frac{\partial^2}{\partial x^2} v(x,t)$ and then the corresponding process coincides with a Brownian motion with $\sigma^2 = 2$. Alternatively, from (29), by applying some known properties of the Gamma function, we can write

$$\begin{aligned}
\overline{v}_1(u,t) &= \frac{1}{\sqrt{t}}\sum_{k=0}^{\infty}\frac{(-ut^{-1/2})^k}{k!\Gamma\left(1-\frac{k+1}{2}\right)} \qquad (34)\\
&= \frac{1}{\sqrt{t}}\sum_{\substack{k=0\\ k\text{ even}}}^{\infty}\frac{(-1)^{k/2}(ut^{-1/2})^k\Gamma\left(\frac{k+1}{2}\right)}{\pi k!}\\
&= \frac{1}{\pi\sqrt{t}}\sum_{\substack{k=0\\ k\text{ even}}}^{\infty}\frac{(-1)^{k/2}(ut^{-1/2})^k\Gamma\left(k+1\right)\sqrt{\pi}2^{1-(k+1)}}{k!\Gamma\left(\frac{k}{2}+1\right)}\\
&= \frac{1}{\sqrt{\pi t}}\sum_{j=0}^{\infty}\frac{(-1)^j u^{2j}(4t)^{-j}}{j!} = \frac{e^{-u^2/4t}}{\sqrt{\pi t}}\ .
\end{aligned}$$

3 On the moments of the solution

We are now interested in evaluating the moments of the solution to equation (11), that is the moments of the pseudoprocess $\Psi_n(\mathcal{T}_\alpha(t)), t>0$: as we will see, they can be obtained in two alternative ways.

By using the representation of the solution derived in (28) and thanks to the independence of the leading process from the temporal argument, we can write the r-th order moments as

$$\begin{aligned}
&E\left(\Psi_n^r(\mathcal{T}_\alpha(t))\right) \qquad (35)\\
=& \int_0^{\infty} E\Psi_n^r(s)\overline{v}_{2\alpha}(s,t)ds,
\end{aligned}$$

for $r\in\mathbb{N}, t>0$. The moments of the n-th order pseudoprocess can be evaluated by means of the Fourier transform of the solution to equation (2) which can be expressed as follows

$$\begin{aligned}
E\left(e^{i\beta\Psi_n(t)}\right) &= \int_{-\infty}^{+\infty} e^{i\beta x}p_n(x,t)dx = e^{(-i\beta)^n k_n t} \qquad (36)\\
&= \sum_{j=0}^{\infty}\frac{(i\beta)^{nj}}{(nj)!}\frac{(-1)^{nj}k_n^j t^j (nj)!}{j!}\ .
\end{aligned}$$

Therefore we get

$$E\Psi_n^r(t) = \begin{cases} \frac{(-1)^r (k_n t)^{r/n} r!}{(r/n)!} & r=nj,\ j=1,2,... \\ 0 \qquad r\neq nj \end{cases},$$

which, inserted together with (32) into (35), gives, for $r = nj$, $j = 1, 2, ...$

$$\begin{aligned} & E\left(\Psi_n^r(\mathcal{T}_\alpha(t))\right) \qquad (37) \\ = \; & \frac{(-1)^{nj} k_n^j (nj)!}{j!} \int_0^\infty s^j \overline{v}_{2\alpha}(s,t) ds \\ = \; & (-1)^{nj} k_n^j t^{\alpha j} \frac{\Gamma(nj+1)}{\Gamma(\alpha j+1)} , \end{aligned}$$

while it is equal to zero for $r \neq nj$.

We can alternatively derive the moments of the pseudoprocesses by evaluating them directly from the characteristic function of the solution. The latter can be obtained by performing successively the Fourier and Laplace transforms of equation (11) as follows: let us denote by $\widetilde{u_\alpha}(\beta, t)$ the Fourier transform of the solution, i.e.

$$\widetilde{u}_\alpha(\beta, t) = \int_{-\infty}^{+\infty} e^{i\beta x} u_\alpha(x,t) dx, \qquad \beta, t > 0,$$

then we get form (11)

$$\frac{\partial^\alpha \widetilde{u}_\alpha}{\partial t^\alpha}(\beta, t) = k_n(-i\beta)^n \widetilde{u}_\alpha(\beta, t). \qquad (38)$$

By applying now the Laplace transform to (38) we get

$$s^\alpha \widetilde{U}_\alpha(\beta, s) - s^{\alpha-1} = k_n(-i\beta)^n \widetilde{U}_\alpha(\beta, s),$$

so that the Fourier-Laplace transform of the solution can be written as

$$\widetilde{U}_\alpha(\beta, s) = \frac{s^{\alpha-1}}{s^\alpha - k_n(-i\beta)^n} . \qquad (39)$$

Now recall that for the *Mittag-Leffler function*

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}$$

the Laplace transform (for $\beta = 1$) is equal to

$$\int_0^\infty e^{-sz} \mathrm{E}_{\alpha,1}(cz^\alpha) dz = \frac{s^{\alpha-1}}{s^\alpha - c}$$

(see Podlubny (1999), formula (1.80) p. 21, for $k = 0$, $\beta = 1$); hence from (39) we get the following expression for the characteristic function of the solution

$$\widetilde{u}_\alpha(\beta, t) = \mathrm{E}_{\alpha,1}(k_n(-i\beta)^n t^\alpha). \qquad (40)$$

In the particular case $\alpha = 1$ the Mittag-Leffler function reduces to the exponential so that (40) coincides with the Fourier transform of the solution to the n-th order equation, reported in (36), as it should be in the non-fractional case.

Finally we can evaluate the moments of the solution by rewriting formula (40) as

$$\widetilde{u}_\alpha(\beta,t) = \sum_{j=0}^{\infty} \frac{(i\beta)^{nj}}{(nj)!} \frac{(-1)^{nj} k_n^j t^{\alpha j}}{\Gamma(\alpha j+1)} \Gamma(nj+1),$$

so that we get again expression (37).

4 More explicit forms of the solution

In order to obtain a more explicit form of the solution to (11), in terms of known densities, we need to distinguish between two intervals of values for α.

(i) Case $1/2 \leq \alpha < 1$

If we restrict ourselves to the case $\alpha \in [1/2, 1)$, so that $1 \leq 2\alpha < 2$, it is possible to apply a result obtained in Fujita (1990), which expresses the solution to a time-fractional diffusion equation in terms of a stable density of appropriate index. By adapting that result to our case, we can conclude that the solution to (30) coincides with

$$v_{2\alpha}(u,t) = \frac{1}{2\alpha} \widetilde{p}_{1/\alpha}(|u|;t), \qquad u \in \mathbb{R},\, t > 0,$$

where $\widetilde{p}_{1/\alpha}(\cdot;t)$ denotes a stable density of index $1/\alpha \in [1,2)$ with parameters $\sigma = (t\cos(\pi - \frac{\pi}{2\alpha}))^\alpha$, $\beta = -1$, $\mu = 0$ (for brevity $S_{1/\alpha}(\sigma,-1,0)$).

The density of $\mathcal{T}_\alpha(t), t > 0$ is then proved to be proportional to the positive branch of a stable density, as the following expression shows:

$$\overline{v}_{2\alpha}(u,t) = \frac{1}{\alpha} \widetilde{p}_{1/\alpha}(u;t), \qquad u > 0,\, t > 0. \tag{41}$$

Remark 4.1

It is possible to recognize, in the previous expression, a known density, by resorting to results on the supremum of stable processes (see, for example, Bingham (1973)). More precisely, let us define $Y(t) = \sup_{0 \leq s \leq t} X_{1/\alpha}(s)$ where $X_{1/\alpha}(t), t > 0$ is a stable process of index $1/\alpha$ and with characteristic function

$$E(e^{isX_{1/\alpha}(t)}) = \exp\left\{ -t|s|^{1/\alpha} \left(1 + i \tan \frac{\pi}{2\alpha} \frac{s}{|s|} \right) \right\}, \qquad t, s > 0.$$

It corresponds, for any fixed t, to the stable law $\widetilde{p}_{1/\alpha}(\cdot;t)$ defined above and, for t varying, to a spectrally negative process, which has no positive jumps (since, for $\beta = -1$, the Lévy-Khinchine measure assigns no mass to $(0,\infty)$, see Samorodnitsky and Taqqu (1994), p.6). Under these circumstances and for $1/\alpha \in [1,2)$, it is known that the Laplace transform of $Y(t)$ is equal, for any $s, t > 0$, to

$$E(e^{-sY(t)}) = \mathrm{E}_{\alpha,1}(-st^\alpha),$$

where $\mathrm{E}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined above. Since it is also well-known that

$$\int_0^\infty e^{-su}\widetilde{p}_{1/\alpha}(u;t)du = \alpha\mathrm{E}_{\alpha,1}(-st^\alpha), \qquad t,s>0,$$

we can conclude that

$$E(e^{-sY(t)}) = \int_0^\infty e^{-su}\frac{1}{\alpha}\widetilde{p}_{1/\alpha}(u;t)du.$$

Alternatively it can be shown, by adapting the result of Bingham (1973), that the density of $Y(t)$ can be written as

$$\begin{aligned} P\{Y(t)\in du\} &= \frac{t^{-\alpha}}{\alpha\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!}\sin(\pi n\alpha)\Gamma(1+n\alpha)\left(\frac{u}{t^\alpha}\right)^{n-1}du \\ &= \frac{1}{\alpha}\widetilde{p}_{1/\alpha}(u;t)du, \qquad u>0,\ t>0 \end{aligned}$$

which coincides with (41).

From the previous results we can conclude that, for $1/2\leq\alpha<1$,

$$I_{(1-\alpha)}\left[\overline{p}_\alpha(w;u)\right](t) = \frac{1}{\alpha}\widetilde{p}_{1/\alpha}(u;t), \qquad u>0,\ t>0.$$

Then, as a result of the fractional integration of the stable density $\overline{p}_\alpha(\cdot;t)$, which is totally skewed to the right (with support $[0,\infty)$), we obtain the positive (normalized) branch of a new stable density $\widetilde{p}_{1/\alpha}(\cdot;t)$ (defined on the whole real axes, since it is $1/\alpha\in(1,2]$), which represents the distribution of the maximum of a stable process of index $1/\alpha$.

(ii) *Case* $0<\alpha\leq 1/2$

We turn now to the other interval of values for α, i.e. $[1/2,1)$, so that, in this case, it is $0<2\alpha\leq 1$. An explicit expression of the solution can be evaluated by specifying $\alpha=1/m$, $m\in\mathbb{N}$, $m>2$. In this particular setting, problem (30) becomes, for $0<\alpha<1$,

$$\begin{cases} \frac{\partial^{2/m}}{\partial t^{2/m}}v(u,t) = \frac{\partial^2}{\partial u^2}v(u,t), & u\in\mathbb{R},\ t>0 \\ v(u,0)=\delta(u) \\ \lim_{|u|\to\infty}v(u,t)=0 \end{cases} \tag{42}$$

so that it can be considered as a special case of the fractional telegraph equation studied in Beghin, Orsingher (2003), for $\lambda=0$ and $c=1$. By applying formula (2.11) of the paper mentioned above, the solution to (42) can be expressed as

$$\begin{aligned} & v_{2/m}(u,t) \\ =\ & \frac{m^{\frac{m+1}{2}}}{(2\pi)^{\frac{m-1}{2}}}\frac{1}{2t^{\frac{2-n}{2n}}}\int_0^\infty dw_1 ... \int_0^\infty dw_{m-1}\,\cdot \\ & \cdot e^{-\frac{w_1^m+...+w_{m-1}^m}{\sqrt[m-1]{m^m t}}}\, w_2\cdots w_{m-1}^{m-2}\left[\delta(u-w_1\cdots w_{m-1})+\delta(u+w_1\cdots w_{m-1}\right]. \end{aligned}$$

Therefore the solution to our problem (11) can be expressed, in this case as

$$u_{1/m}(x,t)=\int_0^\infty p_n(x,u)p_{G(t)}(u)du,$$

where $G(t)=\prod_{j=1}^{m-1}G_j(t), t>0$ and $G_j(t),\ j=1,...,m-1$ are independent random variables with the following probability law

$$p_{G_j(t)}(w)=\frac{1}{m^{\frac{j}{m-1}-1}t^{\frac{j}{m(m-1)}}\Gamma(\frac{j}{m})}\exp\left(-\frac{w^m}{\sqrt[m-1]{m^m t}}\right)w^{j-1}\quad w>0. \tag{43}$$

Indeed we can check that

$$\begin{aligned}
&\prod_{j=1}^{m-1}p_{G_j(t)}(w_j) &(44)\\
=&\prod_{j=1}^{m-1}\frac{1}{m^{\frac{j}{m-1}-1}t^{\frac{j}{m(m-1)}}\Gamma(\frac{j}{m})}\exp\left(-\frac{w_j^m}{\sqrt[m-1]{m^m t}}\right)w_j^{j-1}\\
=&\frac{m^{\frac{m+1}{2}}}{t^{\frac{1}{2}-\frac{1}{m}}(2\pi)^{\frac{m-1}{2}}}\exp\left(-\frac{\sum_{j=1}^{m-1}w_j^m}{\sqrt[m-1]{m^m t}}\right)\prod_{j=1}^{m-1}w_j^{j-1},
\end{aligned}$$

where, in the second step we have applied the multiplication formula of the Gamma function

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)...\Gamma\left(z+\frac{m-1}{m}\right)=(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz),$$

for $z=1/m$.

The corresponding pseudoprocess is represented, in this case, as $\Psi_n(G(t)), t>0$.

Remark 4.2

We can check the previous results, obtained in the two cases above, by choosing $\alpha=1/2$. From both cases we obtain again that the pseudoprocess governed by our equation can be represented by $\Psi_n(|B(t)|), t>0$.

Indeed from the case $1/2\le\alpha<1$ we get, by means of (41), that the density of $\mathcal{T}_\alpha(t), t>0$, for $\alpha=1/2$, coincides with the folded normal. More precisely, in this case, $S_2(\sqrt{t},-1,0)$ coincides with $N(0,2t)$ and then

$$\overline{v}_1(u,t)=2\widetilde{p}_2(u;t)=\frac{e^{-u^2/4t}}{\sqrt{\pi t}} \tag{45}$$

for $u>0$, $t>0$.

On the other hand, if we consider the expression of the density of $\mathcal{T}_\alpha$ obtained for $0<\alpha\le 1/2$, we get, for $\alpha=1/2$ and $m=2$, from (44) that again it is

$$p_{G_1(t)}(u)=\frac{e^{-u^2/4t}}{\sqrt{\pi t}}. \tag{46}$$

Moreover both (45) and (46) coincides with (33) obtained above, as expected.

Remark 4.3
Finally some interesting results can be obtained by specifying (11) for particular values of n. In the special case $n = 3$ the results above coincide with those presented in De Gregorio (2002), while, for $n = 4$, they represent a probabilistic alternative to the analytic approach provided by Agrawal (2000).

By taking $n = 2$, we can conclude that the process related to

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \qquad x \in \mathbb{R},\ t > 0, \tag{47}$$

for $0 < \alpha \leq 1$, is represented by $B(\mathcal{T}_\alpha)$, with $\mathcal{T}_\alpha(t), t > 0$ possessing again density (29). We can prove that this is in accordance with what is already known on (47): for $n = 2$ we can substitute in (28) the transition function of the Brownian motion, so that we get:

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{t^\alpha} \int_0^\infty \frac{e^{-x^2/4u} du}{\sqrt{4\pi u}} W\left(-\frac{u}{t^\alpha}; -\alpha, 1-\alpha\right) \\ &= \frac{1}{t^\alpha} \int_0^\infty \frac{e^{-x^2/4u} du}{\sqrt{4\pi u}} \frac{1}{2\pi i} \int_{Ha} \frac{e^{y - \frac{u}{t^\alpha} y^\alpha}}{y^{1-\alpha}} dy \\ &= \frac{1}{4 i t^\alpha \sqrt{\pi^3}} \int_{Ha} \frac{e^y}{y^{1-\alpha}} dy \int_0^\infty \frac{e^{-\frac{x^2}{4u} - \frac{u}{t^\alpha} y^\alpha}}{\sqrt{u}} du. \end{aligned} \tag{48}$$

If we prove now that

$$\int_0^\infty \frac{e^{-\frac{x^2}{4u} - \frac{u}{t^\alpha} y^\alpha}}{\sqrt{u}} du = \sqrt{\pi} t^{\alpha/2} y^{-\alpha/2} e^{-\frac{|x|}{t^{\alpha/2}} y^{\alpha/2}} \tag{49}$$

and substitute (49) into (48), we finally get the known result

$$\begin{aligned} u_\alpha(x,t) &= \frac{1}{2t^{\alpha/2}} \frac{1}{2\pi i} \int_{Ha} \frac{e^{y - \frac{|x|}{t^{\alpha/2}} y^{\alpha/2}}}{y^{1-\alpha/2}} dy \\ &= \frac{1}{2t^{\alpha/2}} W\left(-\frac{|x|}{t^{\alpha/2}}; -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right). \end{aligned}$$

In order to verify formula (49) we use the following relationship, known for the Laplace transform of the first-passage time of the Brownian motion,

$$e^{-|x|\sqrt{s}} = \int_0^\infty e^{-su} \frac{|x|}{2\sqrt{\pi}\sqrt{u^3}} e^{-\frac{|x|^2}{4u}} du,$$

which, integrated with respect to x gives (49), for $s = y^\alpha / t^\alpha$. Alternatively, we can apply formula n.3.471.9, p.384 of Gradshteyn and Ryzhik (1994), for $\beta = x^2/4$, $\gamma = y^\alpha/t^\alpha$, $\nu = 1/2$ (noting that $K_{1/2}(z) = \sqrt{\pi/2z} e^{-z}$, see Gradhseyn and Ryzhik (1994), n.8469.3, p.978).

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