Poisson process with different Brownian clocks

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Abstract

In this paper different types of Poisson processes N subordinated to random time processes X, depending on Brownian motion, are analyzed. In particular the processes X considered here are the elastic Brownian motion B^{el}, the Brownian sojourn time on the positive half-line Γ_t^+, the first-passage time T_t (through the level t) of a Brownian motion, with or witout drift, and the γ-Bessel process ${}_\gamma R$, for $\gamma > 0$.

In all these cases we obtain the explicit state probability distributions $p_k(t) = \Pr\{N(X(t)) = k\}$, $k \geq 0$, $t > 0$, their governing difference-differential equations and some moments. The connections among different models and, in particular, of $N({}_\gamma R(t))$ with birth and death processes are obtained and discussed.

Key words: Fractional difference-differential equations; Generalized Mittag-Leffler functions; Fractional Poisson processes; Processes with random time; Elastic Brownian motion; Birth and death process; Confluent hypergeometric functions.

AMS classification: 60K99; 33E12; 26A33.

1 Introduction

In a series of previous papers fractional extensions of the Poisson process have been analyzed by different authors (Jumarie [7], Laskin [8], Beghin and Orsingher [2]-[3]). The idea underlying these papers is to construct the fractional Poisson process by introducing a fractional time-derivative in the difference-differential equation governing the state probabilities $p_k^\nu(t), t > 0$, that is, for $0 < \nu < 1$,

$$\frac{d^\nu p_k}{dt^\nu} = -\lambda \left[p_k(t) - p_{k-1}(t)\right], \qquad k \geq 0,\ t > 0,\ \lambda > 0 \tag{1.1}$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}. \tag{1.2}$$

The derivative appearing in (1.1) is intended in the following sense:

$$\frac{d^\nu}{dt^\nu} u(t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{1}{(t-s)^{1+\nu-m}} \frac{d^m}{ds^m} u(s) ds, & \text{for } m < \nu < m-1 \\ \frac{d^m}{dt^m} u(t), & \text{for } \nu = m \end{cases}, \tag{1.3}$$

where $m = \lfloor \nu \rfloor + 1$.

Cahoy [4] has shown that the fractional Poisson process exhibits a long-memory behavior with intermittency (which means clustering of events). This feature makes

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the process more suitable for several applications, for example, in queueing systems (Saji and Pillai [11]) and in financial analysis (Mainardi et al. [9]).

In Beghin and Orsingher [2] it is proved that the fractional Poisson process $N_\nu(t), t > 0$, with state probabilities p_k^ν can be represented as

$$N_\nu(t) \overset{i.d.}{=} N(\mathcal{T}_{2\nu}(t)), \qquad t > 0, \tag{1.4}$$

where N is the homogeneous Poisson process with rate λ (which is obtained in the particular case $\nu = 1$). The time process $\mathcal{T}_{2\nu}(t), t > 0$ appearing in (1.4) is independent from N and possesses a probability density obtained by folding the solution to the following fractional diffusion equation:

$$\begin{cases} \frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial x^2}, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = \delta(x) \end{cases}, \tag{1.5}$$

for $0 < \nu < 1$, with the additional condition $v_t(y,0) = 0$, for $1/2 < \nu < 1$. In particular, for $\nu = 1/2$, the process (1.4) becomes

$$N_{1/2}(t) = N(|B(t)|), \qquad t > 0, \tag{1.6}$$

where B is a standard Brownian motion with volatility parameter equal to 2 (whose density is governed by (1.5) for $\nu = 1/2$).

In the next section we treat a process of the form (1.6), where B is replaced by the elastic Brownian motion $B_\alpha^{el}(t), t > 0$, with absorbing rate $\alpha > 0$ (see Ito and McKean [6]), defined as

$$B_\alpha^{el}(t) = \begin{cases} |B(t)|, & t < T_\alpha \\ 0, & t \geq T_\alpha \end{cases}, \tag{1.7}$$

where T_α is a random time with distribution

$$\Pr\{T_\alpha > t | \mathcal{B}_t\} = e^{-\alpha L(0,t)}, \qquad \alpha > 0, \tag{1.8}$$

$\mathcal{B}_t = \sigma\{B(s), s \leq t\}$ is the natural filtration and $L(0,t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} meas\{s \leq t : |B(t)| < \varepsilon\}$ is the local time in the origin of B. We show that the process

$$\widehat{N}^{el}(t) = N(B_\alpha^{el}(t)), \qquad t > 0, \alpha > 0$$

has state probabilities $\widehat{p}_k^{el}$, $k \geq 0$, which can be expressed by generalized Mittag-Leffler functions (see Saxena and Mathai [12]) or in terms of the survival probabilities of B^{el}. This distribution coincides with that of process (1.6) for $\alpha = 0$. Finally we prove that the state probabilities of $\widehat{N}^{el}$ are solutions to difference-differential equations of the form (1.1) for $\nu = 1/2$.

The remaining part of the paper concerns different compositions of the Poisson process with randomly varying times, leading to higher-order governing equations, instead of fractional ones.

In section 3 we analyze the process obtained by composing the standard Poisson process with the first-passage time of a Brownian motion through the level t. It is defined as $\widehat{N}(t) = N(T_t),\ t > 0$, where

$$T_t = \inf\{s > 0 : B(s) = t\}$$

and B is a standard Brownian motion independent from N.

We obtain the explicit distribution of $\widehat{N}$, i.e. $\widehat{p}_k(t) = \Pr\left\{\widehat{N}(t) = k\right\}$, $k \geq 0$, as follows

$$\widehat{p}_k(t) = \frac{2^{\frac{3}{4} - \frac{k}{2}} \lambda^{\frac{k}{2} + \frac{1}{4}} t^{k + \frac{1}{2}}}{k! \sqrt{\pi}} K_{k - \frac{1}{2}}(t\sqrt{2\lambda}), \tag{1.9}$$

where $K_\nu(z)$ is the modified Bessel function of order ν (see definition (3.7) below). We show that the probability generating function has the following simple structure

$$\widehat{G}(u,t) = \sum_{k=0}^{\infty} u^k \widehat{p}_k(t) = e^{-t\sqrt{2\lambda(1-u)}}, \qquad |u| \le 1, t > 0. \tag{1.10}$$

Since the expected number of events turns out to be infinite, we consider also the Poisson process with clock $T_t^\mu = \inf\{s > 0 : B^\mu(s) = t\}$, where B^μ is a Brownian motion with drift μ. For its distribution $\widehat{p}_k^\mu(t) = \Pr\{N(T_t^\mu) = k\}$, $k \ge 0$, we obtain the second-order governing equation

$$\frac{d^2}{dt^2} p_k - 2\mu \frac{d}{dt} p_k = 2\lambda[p_k - p_{k-1}], \qquad k \ge 0. \tag{1.11}$$

The corresponding probability generating function $\widehat{G}^\mu$ takes the form

$$\widehat{G}^\mu(u,t) = e^{\mu t - t\sqrt{\mu^2 + 2\lambda(1-u)}}, \qquad |u| \le 1$$

and solves the following equation:

$$\frac{\partial^2}{\partial t^2} G - 2\mu \frac{\partial}{\partial t} G = 2\lambda(1-u)G. \tag{1.12}$$

For the Poisson process stopped at the n-times iterated first-passage instant

$$\widehat{N}^n(t) = N(T_1(T_2 ... (T_{n-1}(T_n(t)))...)), \qquad t > 0, \tag{1.13}$$

where

$$T_j(t) = \inf\{s > 0 : B_j(s) = t\} \tag{1.14}$$

and $B_j(t)$, for $j = 1, ..., n$, are Brownian motions independent among themselves and from N, we obtain the 2^n-th order equation

$$\frac{d^{2^n}}{dt^{2^n}} p_k(t) = 2^{2^n - 1}\lambda[p_k(t) - p_{k-1}(t)], \qquad t > 0, k \ge 0, \tag{1.15}$$

governing the state probabilities $\widehat{p}_k^n(t), t > 0$. For the version of the process (1.13) where the Brownian motion figuring in (1.14) is endowed with drift $\mu > 0$, we have derived the probability generating function, which reads

$$\widehat{G}_\mu^n(u,t) = e^{\mu t - 2^{\frac{1}{2}} t \sqrt{\frac{\mu^2}{2} - \mu + 2^{\frac{1}{2}} \sqrt{\frac{\mu^2}{2} - \mu + 2^{\frac{1}{2}} \sqrt{ .....2^{\frac{1}{2}} \sqrt{\frac{\mu^2}{2} + \lambda(1-u)}}}}}, \qquad |u| \le 1 \tag{1.16}$$

and from which we extract

$$\mathbb{E}\widehat{N}_\mu^n(t) = \frac{\lambda t}{\mu^n}, \qquad n \ge 1. \tag{1.17}$$

In section 4 we examine the Poisson process with subordinator represented by the Brownian sojourn time on the positive half-line, i.e. $\Gamma_t^+ = meas\{s < t : B(s) > 0\}$. This process is defined as

$$\overline{N}(t) = N(\Gamma_t^+), \qquad t > 0 \tag{1.18}$$

and displays a slowing down behavior of the time flow (with respect to the natural time t). This fact is reflected by the relation

$$\mathbb{E}\overline{N}(t) = \frac{\lambda t}{2} = \frac{1}{2}\mathbb{E}N(t).$$

The state probabilities of $\overline{N}$ can be expressed in terms of confluent hypergeometric functions ${}_1F_1(\alpha, \beta; x)$ and are related to the distribution p_k, $k \geq 0$ of the homogeneous Poisson process by means of the following formula

$$\overline{p}_k(t) = p_k(t) \binom{2k-1}{k} 2^{1-2k} \, {}_1F_1 \left(\frac{1}{2}, k+1; \lambda t \right). \tag{1.19}$$

We show that the distribution (1.19) satisfies the equations

$$\frac{d}{dt} p_k(t) = \frac{k}{t} p_k(t) - \frac{k+1}{t} p_{k+1}(t), \qquad k \geq 0, \tag{1.20}$$

with time-depending coefficients.

In the last section we derive a surprising connection between the process

$$\widetilde{N}_\gamma(t) = N(R_\gamma^2(t)), \qquad t > 0,$$

where $R_\gamma(t), t > 0$ is a γ-Bessel process starting at zero (defined in (5.1) and (5.2) below) and the birth and death process $M(t), t > 0$ (with equal birth and death rates).

We show that the distribution ${}_\gamma\widetilde{p}_k = \Pr\left\{ \widetilde{N}_\gamma(t) = k \right\}$, $k \geq 0$ can be written as

$${}_\gamma\widetilde{p}_k(t) = \frac{(2\lambda t)^k}{(2\lambda t + 1)^{k+\frac{\gamma}{2}}} \frac{\Gamma\left(k + \frac{\gamma}{2}\right)}{k! \Gamma\left(\frac{\gamma}{2}\right)} \tag{1.21}$$

and simplifies substantially, when $\gamma = 2$, and in this case takes the form

$${}_2\widetilde{p}_k(t) = 2\lambda t \frac{(2\lambda t)^{k-1}}{(2\lambda t + 1)^{k+1}} = 2\lambda t \Pr\left\{ M(t) = k \right\}. \tag{1.22}$$

The equation governing the distribution (1.22) coincides with (1.20), which is related to the previous process $\overline{N}(t) = N(\Gamma_t^+)$.

2 Poisson processes at elastic Brownian times

We consider now the process $\widehat{N}^{el}(t) = N(B_\alpha^{el}(t)), t > 0$ obtained by means of the composition of the Poisson process with the elastic Brownian motion $B_\alpha^{el} = B_\alpha^{el}(t), t > 0$, with absorbing rate $\alpha > 0$. See Ito and McKean [6], p. 45, for details on elastic Brownian motion. It is defined in (1.7) -(1.8) and possesses transition function given by

$$q^{el}(s,t) = 2e^{\alpha s} \int_s^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw + q_\alpha(t)\delta(s), \tag{2.1}$$

where $\delta(s)$ is the Dirac's Delta function with pole in the origin and

$$q_\alpha(t) = 1 - \Pr\left\{ B_\alpha^{el}(t) > 0 \right\} = 1 - 2e^{\frac{\alpha^2 t}{2}} \int_{\alpha\sqrt{t}}^{+\infty} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$$

is the probability that the process is absorbed by the barrier in zero up to time t.

Then the probability distribution of $\widehat{N}^{el}$ is defined, for any $k \geq 0$, by

$$\begin{aligned} \widehat{p}_k^{el}(t) &= \Pr\left\{ N(B_\alpha^{el}(t)) = k \right\} = \int_0^{+\infty} p_k(s) q^{el}(s,t) ds \qquad (2.2) \\ &= 2\int_0^{+\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda s} e^{\alpha s} \int_s^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw ds + q_\alpha(t) \int_0^{+\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda s} \delta(s) ds \end{aligned}$$

and is explicitly evaluated in the next theorem.

Theorem 2.1 *For $k \geq 1$ and for any $\lambda \neq \alpha$ the state probabilities of $\widehat{N}^{el}$ are given by*

$$\begin{aligned} \widehat{p}_k^{el}(t) &= \Pr\left\{N(B_\alpha^{el}(t)) = k\right\} \qquad (2.3)\\ &= \frac{\lambda^k}{(\lambda-\alpha)^{k+1}} \Pr\left\{B_\alpha^{el}(t) > 0\right\} - \frac{\lambda^k}{(\lambda-\alpha)^{k+1}} \sum_{j=0}^{k} \frac{(\alpha-\lambda)^j}{j!} \frac{d^j}{d\lambda^j} \Pr\left\{B_\lambda^{el}(t) > 0\right\}, \end{aligned}$$

while, for $k = 0$, we have instead

$$\begin{aligned} \widehat{p}_0^{el}(t) &= \Pr\left\{N(B^{el}(t)) = 0\right\} \qquad (2.4)\\ &= 1 - \frac{\lambda-\alpha-1}{\lambda-\alpha} \Pr\left\{B_\alpha^{el}(t) > 0\right\} - \frac{1}{\lambda-\alpha} \Pr\left\{B_\lambda^{el}(t) > 0\right\}. \end{aligned}$$

Proof From (2.2), we have that

$$\begin{aligned} &\widehat{p}_k^{el}(t) \qquad (2.5)\\ =\ & 2\int_0^{+\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda s} e^{\alpha s} \int_s^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw ds\\ =\ & \frac{2\lambda^k}{k!} \int_0^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw \int_0^w s^k e^{-s(\lambda-\alpha)} ds\\ =\ & \text{[by successive integrations by parts]}\\ =\ & 2\lambda^k \int_0^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} \left[-\frac{w^k e^{-w(\lambda-\alpha)}}{(\lambda-\alpha)k!} - \frac{w^{k-1} e^{-w(\lambda-\alpha)}}{(\lambda-\alpha)^2(k-1)!} - \dots - \frac{e^{-w(\lambda-\alpha)}-1}{(\lambda-\alpha)^{k+1}} \right] dw\\ =\ & -\frac{2\lambda^k}{(\lambda-\alpha)^{k+1}} \int_0^{+\infty} w e^{-\alpha w} e^{-w(\lambda-\alpha)} \sum_{j=0}^{k} \frac{w^j(\lambda-\alpha)^j}{j!} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw + \frac{2\lambda^k}{(\lambda-\alpha)^{k+1}} \int_0^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw\\ =\ & \frac{\lambda^k}{(\lambda-\alpha)^{k+1}} \Pr\left\{B_\alpha^{el}(t) > 0\right\} - \frac{2\lambda^k}{(\lambda-\alpha)^{k+1}} \sum_{j=0}^{k} \frac{(\lambda-\alpha)^j}{j!} \int_0^{+\infty} w^{j+1} e^{-w\lambda} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw\\ =\ & \frac{\lambda^k}{(\lambda-\alpha)^{k+1}} \Pr\left\{B_\alpha^{el}(t) > 0\right\} - \frac{2\lambda^k}{(\lambda-\alpha)^{k+1}} \sum_{j=0}^{k} \frac{(\lambda-\alpha)^j}{j!} (-1)^j \frac{d^j}{d\lambda^j} \left(\int_0^{+\infty} w e^{-w\lambda} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw \right), \end{aligned}$$

which coincides with (2.3). For $k = 0$, formula (2.2) becomes instead

$$\begin{aligned} \widehat{p}_0^{el}(t) &= \Pr\left\{N(B^{el}(t)) = 0\right\} = 2\int_0^{+\infty} e^{-\lambda s} e^{\alpha s} \int_s^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw ds + q_\alpha(t)\\ &= 1 - \Pr\left\{B_\alpha^{el}(t) > 0\right\} + 2\int_0^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw \int_0^w e^{-s(\lambda-\alpha)} ds \qquad (2.6)\\ &= 1 - \Pr\left\{B_\alpha^{el}(t) > 0\right\} + \frac{2}{\lambda-\alpha} \int_0^{+\infty} w e^{-\alpha w} \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t^3}} dw \left[1 - e^{-w(\lambda-\alpha)}\right]\\ &= 1 - \Pr\left\{B_\alpha^{el}(t) > 0\right\} + \frac{1}{\lambda-\alpha} \left[\Pr\left\{B_\alpha^{el}(t) > 0\right\} - \Pr\left\{B_\lambda^{el}(t) > 0\right\}\right]. \end{aligned}$$

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An alternative way of studying the probability distribution of this process is by evaluating the Laplace transform of (2.2). The implied results, which are valid for any $\alpha, \lambda > 0$, are expressed in terms of generalized Mittag-Leffler functions, defined as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_r \, z^r}{r!\Gamma(\alpha r + \beta)}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \ \ \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0, \tag{2.7}$$

where $(\gamma)_r = \gamma(\gamma+1)...(\gamma+r-1)$ (for $r = 1, 2, ...,$ and $\gamma \neq 0$) and $(\gamma)_0 = 1$.

Theorem 2.2 *The state probabilities of* $\widehat{N}(t) = N(B^{el}(t)), t > 0$ *are given by*

$$\widehat{p}^{el}_k(t) = \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} \sum_{l=0}^{\infty} \left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^l E^{k+1}_{\frac{1}{2},\frac{l+k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \tag{2.8}$$

for $k \geq 1$, *while, for* $k = 0$, *we get*

$$\widehat{p}^{el}_0(t) = 1 - E_{\frac{1}{2},1}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) + \sum_{j=0}^{\infty} \left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^j E_{\frac{1}{2},\frac{j}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right). \tag{2.9}$$

Proof For any $\alpha, \lambda > 0$ and $k \geq 1$, we evaluate the Laplace transform of the first line of (2.5):

$$\begin{aligned} \mathcal{L}\left\{\widehat{p}^{el}_k; \eta\right\} &= 2\int_0^{+\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda s} e^{\alpha s} \int_s^{+\infty} e^{-\alpha w - w\sqrt{2\eta}} dw ds \qquad (2.10) \\ &= 2\int_0^{+\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda s} e^{\alpha s} \left[\frac{e^{-\alpha w - w\sqrt{2\eta}}}{\alpha + \sqrt{2\eta}}\right]_{w=s}^{w=+\infty} ds \\ &= 2\int_0^{+\infty} \frac{(\lambda s)^k}{k!} \frac{e^{-s(\lambda+\sqrt{2\eta})}}{\alpha+\sqrt{2\eta}} ds \\ &= \frac{2}{\alpha+\sqrt{2\eta}} \frac{\lambda^k}{(\lambda+\sqrt{2\eta})^{k+1}} \\ &= \frac{1}{\sqrt{2^k}} \frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}} \frac{\lambda^k}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^{k+1}}. \end{aligned}$$

In order to invert (2.10) we recall the following formula (see Prabhakar [10]):

$$\mathcal{L}\left\{t^{\gamma-1} E^{\delta}_{\beta,\gamma}(\omega t^{\beta}); \eta\right\} = \frac{\eta^{\beta\delta-\gamma}}{(\eta^{\beta}-\omega)^{\delta}}. \tag{2.11}$$

Therefore we invert the first term in (2.10) by taking in (2.7) $\delta = 1$, $\beta = \frac{1}{2}$, $\omega = -\frac{\alpha}{\sqrt{2}}$ and $\gamma = \frac{1}{2}$, while, for the second term we put $\delta = k+1$, $\beta = \frac{1}{2}$, $\omega = -\frac{\lambda}{\sqrt{2}}$ and $\gamma = \frac{k+1}{2}$, so that we get

$$\begin{aligned} &\widehat{p}^{el}_k(t) \qquad (2.12) \\ = \; & \frac{\lambda^k}{\sqrt{2^k}} \int_0^t E^{k+1}_{\frac{1}{2},\frac{k+1}{2}}\left(-\frac{\lambda}{\sqrt{2}}\sqrt{s}\right) s^{\frac{k-1}{2}} E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t-s}\right)(t-s)^{-\frac{1}{2}} ds \\ = \; & \frac{\lambda^k}{\sqrt{2^k} k!} \sum_{j=0}^{\infty} \frac{(k+j)!(-\frac{\lambda}{\sqrt{2}})^j}{j!\Gamma(\frac{j}{2}+\frac{k+1}{2})} \sum_{l=0}^{\infty} \frac{(-\frac{\alpha}{\sqrt{2}})^l}{\Gamma(\frac{l}{2}+\frac{1}{2})} \int_0^t s^{\frac{k-1}{2}+\frac{j}{2}} (t-s)^{\frac{l}{2}-\frac{1}{2}} ds \\ = \; & \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k} k!} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j \left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^l \frac{1}{\Gamma(\frac{j+l+k}{2}+1)} \frac{(k+j)!}{j!}, \end{aligned}$$

which coincides with (2.8). Analogously, for $k=0$, the Laplace transform of the first expression in (2.6) reads

$$\begin{aligned}
&\mathcal{L}\left\{\widehat{p}_0^{el};\eta\right\} \qquad (2.13)\\
=\;& 2\int_0^{+\infty} e^{-\lambda s+\alpha s}\int_s^{+\infty} e^{-\alpha w - w\sqrt{2\eta}}dwds + \int_0^{+\infty} e^{-\eta t}q_\alpha(t)dt\\
=\;& 2\int_0^{+\infty} \frac{e^{-\lambda s - s\sqrt{2\eta}}}{\alpha+\sqrt{2\eta}}ds + \int_0^{+\infty} e^{-\eta t}\left(1-2e^{\frac{\alpha^2 t}{2}}\int_{\alpha\sqrt{t}}^{+\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dw\right)dt\\
=\;& \frac{2}{\alpha+\sqrt{2\eta}}\frac{1}{\lambda+\sqrt{2\eta}}+\frac{1}{\eta}-2\int_0^{+\infty} e^{-\eta t+\frac{\alpha^2 t}{2}}\int_{\alpha\sqrt{t}}^{+\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}dwdt\\
=\;& \frac{2}{\alpha+\sqrt{2\eta}}\frac{1}{\lambda+\sqrt{2\eta}}+\frac{1}{\eta}-2\int_0^{+\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}\int_0^{w^2/\alpha^2} e^{-\eta t+\frac{\alpha^2 t}{2}}dtdw\\
=\;& \frac{2}{\alpha+\sqrt{2\eta}}\frac{1}{\lambda+\sqrt{2\eta}}+\frac{1}{\eta}-2\int_0^{+\infty}\frac{1}{\sqrt{\frac{2\pi\alpha^2}{2\eta}}}\frac{e^{-\eta\frac{w^2}{\alpha^2}}}{\frac{\alpha^2}{2}-\eta}\frac{\alpha}{\sqrt{2\eta}}dw+\frac{2}{\alpha^2-2\eta}\\
=\;& \frac{2}{\alpha+\sqrt{2\eta}}\frac{1}{\lambda+\sqrt{2\eta}}+\frac{1}{\eta}-\frac{2}{\alpha^2-2\eta}\frac{\alpha}{\sqrt{2\eta}}+\frac{2}{\alpha^2-2\eta}\\
=\;& \frac{2}{\alpha+\sqrt{2\eta}}\frac{1}{\lambda+\sqrt{2\eta}}+\frac{1}{\eta}-\frac{2}{\alpha^2-2\eta}\frac{\alpha-\sqrt{2\eta}}{\sqrt{2\eta}}\\
=\;& \frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{1}{\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}}+\frac{1}{\eta}-\frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{1}{\sqrt{\eta}},
\end{aligned}$$

so that, by inverting (2.13), we get

$$\begin{aligned}
&\widehat{p}_0^{el}(t)\\
=\;& 1-\frac{1}{\sqrt{\pi}}\int_0^t E_{\frac{1}{2},\frac{1}{2}}(-\frac{\alpha}{\sqrt{2}}\sqrt{s})s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}ds+\\
&+\int_0^t E_{\frac{1}{2},\frac{1}{2}}(-\frac{\alpha}{\sqrt{2}}\sqrt{s})s^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-\frac{\lambda}{\sqrt{2}}\sqrt{t-s})(t-s)^{-\frac{1}{2}}ds\\
=\;& 1-\frac{1}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{1}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}\left(-\frac{\alpha}{\sqrt{2}}\right)^j\int_0^t s^{\frac{j}{2}-\frac{1}{2}}(t-s)^{-\frac{1}{2}}ds+\\
&+\sum_{j=0}^{\infty}\frac{1}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}\left(-\frac{\alpha}{\sqrt{2}}\right)^j\sum_{l=0}^{\infty}\frac{1}{\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)}\left(-\frac{\lambda}{\sqrt{2}}\right)^l\int_0^t s^{\frac{j}{2}-\frac{1}{2}}(t-s)^{\frac{l}{2}-\frac{1}{2}}ds\\
=\;& 1-\sum_{j=0}^{\infty}\frac{1}{\Gamma\left(\frac{j}{2}+1\right)}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^j+\sum_{j=0}^{\infty}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)^j\sum_{l=0}^{\infty}\frac{1}{\Gamma\left(\frac{j+l}{2}+1\right)}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^l.
\end{aligned}$$

■

Remark 2.1 By means of (2.10) we can evaluate the mean value of $\widehat{N}^{el}$, as follows

$$\begin{aligned}\mathcal{L}\left\{\mathbb{E}\widehat{N}^{el};\eta\right\} &= \sum_{k=0}^{\infty}\frac{k}{\sqrt{2^k}}\frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{\lambda^k}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^{k+1}} \\ &= \frac{\lambda}{\sqrt{2}}\frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{1}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^2}\sum_{k=0}^{\infty}k\left(\frac{\frac{\lambda}{\sqrt{2}}}{\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}}\right)^{k-1} \\ &= \frac{\lambda}{\sqrt{2}}\frac{\eta^{-1}}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}.\end{aligned}$$

Therefore we get that

$$\mathbb{E}\widehat{N}^{el}(t)=\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right)=\frac{\lambda}{\alpha}\left\{1-E_{\frac{1}{2},1}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right)\right\}, \tag{2.14}$$

where, in the last step, we have used the following relation

$$E_{\frac{1}{2},\frac{3}{2}}(x)=x^{-1}\left[E_{\frac{1}{2},1}(x)-1\right].$$

The structure of the elastic Brownian motion is the reason of the fading behavior of $\mathbb{E}\widehat{N}^{el}$. This is intuitively explained by the fact that the elastic barrier at the origin makes the time length shorter and shorter as t increases and thus the mean number of Poisson events is constrained to decrease. Moreover we establish an interesting relation between the expected number of events for the process $\widehat{N}^{el}$ and the corresponding quantity for the process $N(|B(t)|), t>0$, which reads

$$\mathbb{E}N(|B(t)|)=\int_0^{+\infty}\lambda s\Pr\{|B(t)|\in ds\}=\frac{\lambda\sqrt{2t}}{\sqrt{\pi}}. \tag{2.15}$$

Indeed by comparing (2.14) with (2.15) we can write that

$$\mathbb{E}\widehat{N}^{el}(t)=\mathbb{E}(N(B^{el}(t)))=\frac{\sqrt{\pi}}{2}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right)\mathbb{E}N(|B(t)|). \tag{2.16}$$

We note that the elastic Brownian motion with absorbing rate α reduces to the reflected Brownian motion for $\alpha=0$ and then, in this particular case, the constant in (2.16) becomes equal to one, as it should be.

By analogous steps we can evaluate the variance of the process: the Laplace transform of the second-order factorial moment is equal to

$$\sum_{k=0}^{\infty}k(k-1)\frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{\left(\frac{\lambda}{\sqrt{2}}\right)^k}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^{k+1}} \tag{2.17}$$

$$\begin{aligned} &= \frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{\left(\frac{\lambda}{\sqrt{2}}\right)^2}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^3}\sum_{k=0}^{\infty}k(k-1)\left(\frac{\frac{\lambda}{\sqrt{2}}}{\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}}\right)^{k-2} \\ &= \frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{\left(\frac{\lambda}{\sqrt{2}}\right)^2}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^3}\frac{2}{\left(1-\frac{\frac{\lambda}{\sqrt{2}}}{\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}}\right)^3} \\ &= \frac{1}{\frac{\alpha}{\sqrt{2}}+\sqrt{\eta}}\frac{\lambda^2}{\sqrt{\eta}^3}.\end{aligned}$$

The Laplace transform (2.17) can be inverted by applying formula (2.11), for $\gamma = 2$, thus giving

$$\mathbb{E}\left[\widehat{N}^{el}(t)\left(\widehat{N}^{el}(t)-1\right)\right] = \lambda^2 t E_{\frac{1}{2},2}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right). \tag{2.18}$$

Therefore the variance is obtained as follows

$$\mathbb{V}ar\left(\widehat{N}^{el}(t)\right) = \lambda^2 t E_{\frac{1}{2},2}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right) + \frac{\lambda\sqrt{t}}{\sqrt{2}} E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right) - \frac{\lambda^2 t}{2}\left(E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\alpha}{\sqrt{2}}\sqrt{t}\right)\right)^2. \tag{2.19}$$

It can be checked that (2.18) and (2.19) for $\alpha = 0$ coincide with $\mathbb{E}\left[N(|B(t)|)\left(N(|B(t)|)(t)-1\right)\right]$ and $\mathbb{V}ar(N(|B(t)|))$, respectively.

We analyze now the particular case where $\alpha = \lambda$, since the previous results are considerably simplified and thus it is possible to evaluate the equation governing the distribution of $\widehat{N}^{el}$, as we did for the other processes in the previous sections.

Theorem 2.3 *For $\alpha = \lambda$ the state probabilities of $\widehat{N}^{el}$ read*

$$\widehat{p}_k^{el}(t) = \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+2}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \qquad k \geq 1 \tag{2.20}$$

and, for $k = 0$,

$$\begin{aligned}\widehat{p}_0^{el}(t) &= E^2_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) + \frac{\lambda\sqrt{t}}{\sqrt{2}} E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\ &= 1 - \frac{\lambda}{\sqrt{2}}\sqrt{t} E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda}{\sqrt{2}}\sqrt{t}\right).\end{aligned} \tag{2.21}$$

Proof The Laplace transform (2.10) can be immediately inverted, for $\alpha = \lambda$, as follows

$$\begin{aligned}\widehat{p}_k^{el}(t) &= \frac{\lambda^k}{\sqrt{2^k}} \mathcal{L}^{-1}\left\{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^{-k-2}; t\right\} \\ &= \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+2}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \qquad k \geq 1,\end{aligned} \tag{2.22}$$

by applying again (2.11). For $k = 0$, if we put $\alpha = \lambda$ the Laplace transform (2.13) reduces to

$$\mathcal{L}\left\{\widehat{p}_0^{el}; \eta\right\} = \frac{1}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^2} + \frac{\lambda}{\sqrt{2}} \frac{1}{\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}} \frac{1}{\eta},$$

which gives the first expression in (2.21). An alternative expression for $\widehat{p}_0^{el}$ can be obtained by rewriting (2.13), for $\alpha = \lambda$, as follows

$$\mathcal{L}\left\{\widehat{p}_0^{el}; \eta\right\} = \frac{1}{\eta} - \frac{1}{\sqrt{2}} \frac{\lambda\eta^{-\frac{1}{2}}}{\left(\frac{\lambda}{\sqrt{2}}+\sqrt{\eta}\right)^2}. \tag{2.23}$$

The Laplace transform (2.23) can be inverted by applying (2.11) for $\delta = 2$, $\beta = \frac{1}{2}$, $\omega = -\frac{\lambda}{\sqrt{2}}$ and $\gamma = \frac{3}{2}$, thus obtaining the second form of (2.21). We check that the two expressions of (2.21) coincide, by applying the relation holding for generalized Mittag-Leffler functions proved in Beghin and Orsingher [3] (see formula (3.8), for $n = 0$, $m = 2$, $z = 1$, $\nu = \frac{1}{2}$):

$$
\begin{aligned}
& E^2_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)+\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)-1+\frac{\lambda}{\sqrt{2}}\sqrt{t}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda}{\sqrt{2}}\sqrt{t}\right) \\
= & E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)+\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)-1 \\
= & \sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+1\right)}+\frac{\lambda\sqrt{t}}{\sqrt{2}}\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{1}{2}+1\right)}-1 \\
= & \sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+1\right)}+\frac{\lambda\sqrt{t}}{\sqrt{2}}\sum_{l=1}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^{l-1}}{\Gamma\left(\frac{l}{2}+1\right)}-1 \\
= & \sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+1\right)}-\sum_{l=1}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^{l}}{\Gamma\left(\frac{l}{2}+1\right)}-1=0.
\end{aligned}
$$

■

Remark 2.2 By comparing (2.20) with (2.8) for $\alpha=\lambda$, we extract the following interesting relation holding for generalized Mittag-Leffler functions:

$$
\sum_{l=0}^{\infty} x^l E^{k+1}_{\alpha,\alpha(l+k)+z}(x)=E^{k+2}_{\alpha,\alpha k+z}(x), \qquad x\in\mathbb{R},\ z\geq 0,\ k\geq 1,\ \alpha>0. \tag{2.24}
$$

Formula (2.24) can be directly verified as follows

$$
\begin{aligned}
\sum_{l=0}^{\infty} x^l E^{k+1}_{\alpha,\alpha(l+k)+z}(x) &= \frac{1}{k!}\sum_{l=0}^{\infty}x^l\sum_{m=0}^{\infty}\frac{x^m(m+k)!}{m!\Gamma\left(\alpha(m+k+l)+z\right)} \qquad (2.25)\\
&= [m'=m+l] \\
&= \frac{1}{k!}\sum_{l=0}^{\infty}x^l\sum_{m'=l}^{\infty}\frac{x^{m'-l}(m'-l+k)!}{(m'-l)!\Gamma\left(\alpha(m'+k)+z\right)} \\
&= \frac{1}{k!}\sum_{m=0}^{\infty}\frac{x^m}{\Gamma\left(\alpha(m+k)+z\right)}\sum_{l=0}^{m}\frac{(m-l+k)!}{(m-l)!} \\
&= \frac{1}{(k+1)!}\sum_{m=0}^{\infty}\frac{x^m(k+m+1)!}{\Gamma\left(\alpha(m+k)+z\right)m!},
\end{aligned}
$$

which gives (2.24), by noting that the following result holds

$$
\begin{aligned}
& \sum_{l=0}^{m}\binom{m-l+k}{k} \\
= & 1+\binom{k+1}{k}+...+\binom{m+k}{k} \\
= & 1+(k+1)+\frac{(k+1)(k+2)}{2}+\frac{(k+1)(k+2)(k+3)}{3!}+...+\frac{(k+1)(k+2)...(k+m)}{m!} \\
= & (k+2)\left[1+\frac{(k+1)}{2}+\frac{(k+1)(k+3)}{3!}+...+\frac{(k+1)(k+3)...(k+m)}{m!}\right] \\
= & \frac{(k+2)(k+3)}{2}\left[\frac{(k+1)}{3}+...+\frac{(k+1)(k+4)...(k+m)}{3\cdot 4\cdot ...m}\right] \\
= & \frac{(k+2)(k+3)...(k+m+1)}{m!}=\frac{(k+m+1)!}{m!(k+1)!}=\binom{m+k+1}{m}.
\end{aligned}
$$

Remark 2.3 We check that the state probabilities sum up to one. Indeed we can rewrite the distribution (2.22) as follows, for $k \geq 1$, by using again formula (3.8) cited above,

$$\widehat{p}_k^{el}(t) = \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) - \frac{(\lambda\sqrt{t})^{k+1}}{\sqrt{2^{k+1}}} E^{k+2}_{\frac{1}{2},\frac{k+1}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \tag{2.26}$$

and consider it together with (2.21) so that we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \widehat{p}_k^{el}(t) \qquad (2.27)\\
= \;& 1 - \frac{\lambda}{\sqrt{2}}\sqrt{t}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda}{\sqrt{2}}\sqrt{t}\right) + \sum_{k=1}^{\infty} \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) - \sum_{k=1}^{\infty} \frac{(\lambda\sqrt{t})^{k+1}}{\sqrt{2^{k+1}}} E^{k+2}_{\frac{1}{2},\frac{k+1}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\\
= \;& 1 - \sum_{k=0}^{\infty} \frac{(\lambda\sqrt{t})^{k+1}}{\sqrt{2^{k+1}}} E^{k+2}_{\frac{1}{2},\frac{k+1}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) + \sum_{k=1}^{\infty} \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) = 1.
\end{aligned}$$

In order to obtain the recursive differential equation governing the distribution of the process $\widehat{N}^{el}(t), t > 0$, we note that $\widehat{p}_k^{el}(t)$ (in the form given in (2.26)) can be rewritten in terms of the probability distribution $p_k^{1/2}$, $k \geq 1$, of the fractional Poisson process $\mathcal{N}_\nu(t), t > 0$, with parameters $\nu = \frac{1}{2}$ and $\frac{\lambda}{\sqrt{2}}$ (see Beghin and Orsingher [3]). We recall that

$$p_k^{1/2}(t) = \Pr\left\{\mathcal{N}_{\frac{1}{2}}(t) = k\right\} = \frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}} E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right), \qquad t > 0$$

solves the fractional recursive differential equation

$$\frac{d^{1/2}p_k}{dt^{1/2}} = -\frac{\lambda}{\sqrt{2}}\left[p_k^{1/2}(t) - p_{k-1}^{1/2}(t)\right], \qquad k \geq 0 \tag{2.28}$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}$$

and $p_{-1}(t) = 0$ (see Theorem 2.1 in Beghin and Orsingher [3]). The process $\mathcal{N}_{\frac{1}{2}}$ analyzed there is equal in distribution to $N(|B(t)|)$, where B is a Brownian motion with variance equal to $2t$ (and this is the reason of the appearance here of $\frac{\lambda}{\sqrt{2}}$ instead of λ). Therefore we can write, in view of formula (2.12) of Beghin and Orsingher [3], that

$$\begin{aligned}
\widehat{p}_k^{el}(t) &= \Pr\left\{\mathcal{N}_{1/2}(t) = k\right\} - \Pr\left\{\mathcal{N}_{1/2}(t) = k+1\right\} \qquad (2.29)\\
&= \Pr\left\{N(|B(t)|) = k\right\} - \Pr\left\{N(|B(t)|) = k+1\right\}\\
&= p_k^{1/2}(t) - p_{k+1}^{1/2}(t), \qquad \text{for } k \geq 1.
\end{aligned}$$

Analogously, for $k = 0$, we get, in view of (2.27), that

$$\widehat{p}_0^{el}(t) = 1 - \sum_{k=1}^{\infty} \widehat{p}_k^{el}(t) = 1 - p_1^{1/2}(t). \tag{2.30}$$

Theorem 2.4 *For $\alpha=\lambda$, the state probabilities $\widehat{p}_k^{el}$ of $\widehat{N}^{el}$, given in Theorem 2.3, are solutions to the following recursive differential equation*

$$\frac{d^{1/2}}{dt^{1/2}}p_k(t)=-\frac{\lambda}{\sqrt{2}}\left[p_k(t)-p_{k-1}(t)\right], \qquad k>1, \tag{2.31}$$

with initial condition $\widehat{p}_k^{el}(0)=0$; for $k=1$, the governing equation is given by

$$\frac{d^{1/2}}{dt^{1/2}}p_1(t)=-\frac{\lambda}{\sqrt{2}}\left[p_1(t)-p_0(t)+\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right], \tag{2.32}$$

with $\widehat{p}_1^{el}(0)=0$, while, for $k=0$, it reads

$$\frac{d^{1/2}}{dt^{1/2}}p_0(t)=-\frac{\lambda}{\sqrt{2}}\left[p_0(t)-\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right], \tag{2.33}$$

with initial condition $\widehat{p}_0^{el}(0)=1$.

Proof By (2.28) and (2.29) we can write, for $k\geq 2$,

$$\begin{aligned}\frac{d^{1/2}}{dt^{1/2}}\widehat{p}_k^{el}(t) &= \frac{d^{1/2}}{dt^{1/2}}p_k^{1/2}(t)-\frac{d^{1/2}}{dt^{1/2}}p_{k+1}^{1/2}(t)\\ &= -\frac{\lambda}{\sqrt{2}}\left[p_k^{1/2}(t)-p_{k-1}^{1/2}(t)\right]+\frac{\lambda}{\sqrt{2}}\left[p_{k+1}^{1/2}(t)-p_k^{1/2}(t)\right],\end{aligned}$$

which gives (2.31). For $k=1$, we have instead

$$\begin{aligned}&\frac{d^{1/2}}{dt^{1/2}}\widehat{p}_1^{el}(t)=\frac{d^{1/2}}{dt^{1/2}}p_1^{1/2}(t)-\frac{d^{1/2}}{dt^{1/2}}p_2^{1/2}(t)\\ =\ & -\frac{\lambda}{\sqrt{2}}\left[p_1^{1/2}(t)-p_0^{1/2}(t)\right]+\frac{\lambda}{\sqrt{2}}\left[p_2^{1/2}(t)-p_1^{1/2}(t)\right]\\ =\ & [\text{by (2.30)}]\\ =\ & -\frac{\lambda}{\sqrt{2}}\left\{\widehat{p}_1^{el}(t)-\widehat{p}_0^{el}(t)-\frac{\lambda}{\sqrt{2}}\left[1-p_0^{1/2}(t)\right]\right\}\\ =\ & -\frac{\lambda}{\sqrt{2}}\left\{\widehat{p}_1^{el}(t)-\widehat{p}_0^{el}(t)-\frac{\lambda}{\sqrt{2}}\left[1-E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right]\right\}\\ =\ & -\frac{\lambda}{\sqrt{2}}\left[\widehat{p}_1^{el}(t)-\widehat{p}_0^{el}(t)+\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right].\end{aligned}$$

The presence of the term $\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)$ in (2.32) is explained by the fact that $\widehat{p}_0^{el}(t)$ (given in (2.21)) can be obtained from the general formula (2.20) by putting $k=0$ and adding the term produced by the absorbing probability q_α. The same is true for $k=0$, so that we get (2.33) by similar steps, as follows:

$$\begin{aligned}\frac{d^{1/2}}{dt^{1/2}}\widehat{p}_0^{el}(t) &= [\text{by (2.30)}]\\ &= -\frac{d^{1/2}}{dt^{1/2}}p_1^{1/2}(t)\\ &= \frac{\lambda}{\sqrt{2}}\left[p_1^{1/2}(t)-p_0^{1/2}(t)\right]\\ &= -\frac{\lambda}{\sqrt{2}}\left[\widehat{p}_0^{el}(t)-1+E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right]\\ &= -\frac{\lambda}{\sqrt{2}}\left[\widehat{p}_0^{el}(t)-\frac{\lambda\sqrt{t}}{\sqrt{2}}E_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right].\end{aligned}$$

Equation (2.33) can be checked directly by taking the fractional derivative of $\widehat{p}_0^{el}(t)$ in the form (2.21):

$$\begin{aligned}
\frac{d^{1/2}}{dt^{1/2}}\widehat{p}_0^{el}(t) = -\frac{\lambda}{\sqrt{2}}\frac{d^{1/2}}{dt^{1/2}}\left[\sqrt{t}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda}{\sqrt{2}}\sqrt{t}\right)\right]& \\
= \quad -\frac{\lambda}{2\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{(j+1)\left(-\frac{\lambda}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{3}{2}\right)}\int_0^t (t-s)^{-\frac{1}{2}}s^{\frac{j}{2}-\frac{1}{2}}ds +& \\
-\frac{\lambda}{2\sqrt{2\pi}}\sum_{j=0}^{\infty}\frac{j(j+1)\left(-\frac{\lambda}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{3}{2}\right)}\int_0^t (t-s)^{-\frac{1}{2}}s^{\frac{j}{2}-\frac{1}{2}}ds& \\
= \quad -\frac{\lambda}{\sqrt{2}}\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} - \frac{\lambda}{\sqrt{2}}\sum_{j=0}^{\infty}\frac{j\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)}& \\
= \quad -\frac{\lambda}{\sqrt{2}}E^2_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right),&
\end{aligned}$$

which yields (2.33). ■

Remark 2.4 We evaluate now the probability generating function, by using the expressions of the probabilities given in (2.26) and (2.21):

$$\widehat{G}^{el}(u,t) = \sum_{k=0}^{\infty}u^k\widehat{p}_k^{el}(t) = \widehat{p}_0^{el}(t) + \sum_{k=1}^{\infty}u^k\widehat{p}_k^{el}(t) \tag{2.34}$$

$$\begin{aligned}
= \quad & 1 - \frac{\lambda\sqrt{t}}{\sqrt{2}}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) + \sum_{k=1}^{\infty}u^k\frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}}E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) - \frac{1}{u}\sum_{k=1}^{\infty}u^{k+1}\frac{(\lambda\sqrt{t})^{k+1}}{\sqrt{2^{k+1}}}E^{k+2}_{\frac{1}{2},\frac{k+1}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\
= \quad & 1 + (u-1)\frac{\lambda\sqrt{t}}{\sqrt{2}}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) + \sum_{k=2}^{\infty}u^k\frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}}E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\
& -\frac{1}{u}\sum_{k=1}^{\infty}u^{k+1}\frac{(\lambda\sqrt{t})^{k+1}}{\sqrt{2^{k+1}}}E^{k+2}_{\frac{1}{2},\frac{k+1}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\
= \quad & 1 + (u-1)\frac{\lambda\sqrt{t}}{\sqrt{2}}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) + \frac{u-1}{u}\sum_{k=2}^{\infty}u^k\frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}}E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\
= \quad & 1 + \frac{u-1}{u}\sum_{k=1}^{\infty}u^k\frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}}E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) \\
= \quad & 1 + \frac{u-1}{u}\left[\sum_{k=0}^{\infty}u^k\frac{(\lambda\sqrt{t})^k}{\sqrt{2^k}}E^{k+1}_{\frac{1}{2},\frac{k}{2}+1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) - E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right] \\
= \quad & 1 + \frac{u-1}{u}\left[E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}(1-u)\right) - E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right],
\end{aligned}$$

where, in the last step, we have applied formula (2.47) of Beghin and Orsingher [3]. We note that, for $u=1$, formula (2.34) reduces to one, while for $u=0$ it gives $\widehat{p}_0^{el}(t)$, since it is

$$\lim_{u\to 0}\widehat{G}^{el}(u,t) = 1-\lim_{u\to 0}\frac{E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}(1-u)\right)-E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)}{u} \tag{2.35}$$

$$= 1-\frac{d}{du}\left[E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}(1-u)\right)-E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right]_{u=0}$$

$$= 1+\sum_{m=1}^{\infty}\frac{m\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^m(1-u)^{m-1}}{\Gamma\left(\frac{m}{2}+1\right)}\Bigg|_{u=0}$$

$$= 1+2\sum_{m=1}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^m(1-u)^{m-1}}{\Gamma\left(\frac{m}{2}\right)}\Bigg|_{u=0} = [j=m-1]$$

$$= 1-2\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^{j+1}(1-u)^{j}}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}\Bigg|_{u=0} = 1-\sqrt{2}\lambda\sqrt{t}E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right).$$

We can check that (2.35) coincides with $\widehat{p}_0^{el}(t)$ by showing that

$$\sqrt{2}\lambda\sqrt{t}E_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right) = \sqrt{2}\lambda\sqrt{t}\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{1}{2}\right)}$$

$$= \frac{\lambda\sqrt{t}}{\sqrt{2}}\sum_{j=0}^{\infty}\frac{(j+1)\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)^j}{\Gamma\left(\frac{j}{2}+\frac{3}{2}\right)} = \frac{\lambda\sqrt{t}}{\sqrt{2}}E^2_{\frac{1}{2},\frac{3}{2}}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right).$$

By taking the first derivative of $\widehat{G}^{el}$ we can evaluate the expected value of $\widehat{N}^{el}$, in the case $\alpha=\lambda$:

$$\mathbb{E}\widehat{N}^{el}(t) = \frac{d}{du}\widehat{G}^{el}(u,t)\Big|_{u=1}$$

$$= \frac{1}{u^2}\left[E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}(1-u)\right)-E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right)\right]+\frac{u-1}{u}\frac{d}{du}E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}(1-u)\right)\Bigg|_{u=1}$$

$$= 1-E_{\frac{1}{2},1}\left(-\frac{\lambda\sqrt{t}}{\sqrt{2}}\right),$$

which coincides with (2.14) for $\alpha=\lambda$.

3 Poisson processes at Brownian first-passage times

In this section we analyze the Poisson process stopped at the random time $T_t = \inf\{s>0 : B(s)=t\}$, where B is a standard Brownian motion. Clearly T_t is the first passage time of B through level t. The probability density of T_t reads

$$\Pr\{T_t\in ds\}/ds = q(t,s) = \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}, \qquad s,t>0 \tag{3.1}$$

and satisfies the partial differential equation

$$\frac{\partial^2 q}{\partial t^2} = 2\frac{\partial q}{\partial s}. \tag{3.2}$$

We consider the process

$$\widehat{N}(t) = N(T_t),$$

with state probabilities defined, for $k = 0, 1, ..,$ as

$$\begin{aligned} \widehat{p}_k(t) &= \Pr\left\{\widehat{N}(t) = k\right\} = \int_0^{+\infty} p_k(s) q(t,s) ds \qquad (3.3) \\ &= \frac{\lambda^k}{k!}\int_0^{+\infty} s^k e^{-\lambda s} \frac{t e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds, \end{aligned}$$

where $p_k(t), t > 0$ represents the standard Poisson distribution which solves the equation

$$\frac{d}{dt} p_k(t) = \lambda[p_k(t) - p_{k-1}(t)]. \qquad (3.4)$$

Theorem 3.1 *The state probabilities* $\widehat{p}_k(t)$, $k \geq 0$, $t > 0$, *given in (3.3) satisfy the following difference-differential equation*

$$\frac{d^2}{dt^2} p_k(t) = 2\lambda[p_k(t) - p_{k-1}(t)]. \qquad (3.5)$$

Proof By taking the derivatives of (3.3) and considering (3.2), we get that

$$\begin{aligned} \frac{d^2}{dt^2}\widehat{p}_k(t) &= \int_0^{+\infty} p_k(s) \frac{\partial^2 q}{\partial t^2}(t,s) ds \qquad (3.6) \\ &= 2\int_0^{+\infty} p_k(s) \frac{\partial q}{\partial s} ds \\ &= 2 p_k(s) q(t,s)|_0^{\infty} - 2\int_0^{+\infty} \frac{dp_k(s)}{ds} q(t,s) ds \\ &= [\text{by } (3.4)] \\ &= 2\lambda[\widehat{p}_k(s) - \widehat{p}_{k-1}(s)]. \end{aligned}$$

The first term in the third line of (3.6) is zero for $k = 0$, because $\lim_{s\to 0^+} q(t,s) = 0$, while, for $k \geq 1$, this is implied by the form of the Poisson distribution ■

The explicit distribution of $\widehat{N}(t), t > 0$, is given in the next theorem.

Theorem 3.2 *The state probabilities* $\widehat{p}_k(t)$, $k \geq 0$, $t > 0$, *given in (3.3) are given by*

$$\widehat{p}_k(t) = \frac{2^{\frac{3}{4}-\frac{k}{2}} \lambda^{\frac{k}{2}+\frac{1}{4}} t^{k+\frac{1}{2}}}{k!\sqrt{\pi}} K_{k-\frac{1}{2}}(t\sqrt{2\lambda}),$$

where

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} \int_0^{+\infty} \frac{e^{-t-\frac{z^2}{4t}}}{t^{\nu+1}} dt \qquad (3.7)$$

is the modified Bessel function of index ν.

Proof We rewrite (3.3) as follows:

$$\begin{aligned} \widehat{p}_k(t) &= \frac{\lambda^k t}{k!\sqrt{2\pi}} \int_0^{+\infty} s^{k-\frac{3}{2}} e^{-\lambda s} e^{-\frac{t^2}{2s}} ds \qquad (3.8) \\ &= \frac{2\lambda^k t}{k!\sqrt{2\pi}} \left(\frac{t^2}{2\lambda}\right)^{\frac{k}{2}-\frac{1}{4}} K_{k-\frac{1}{2}}(t\sqrt{2\lambda}) \\ &= \frac{2^{\frac{3}{4}-\frac{k}{2}} \lambda^{\frac{k}{2}+\frac{1}{4}} t^{k+\frac{1}{2}}}{k!\sqrt{\pi}} K_{k-\frac{1}{2}}(t\sqrt{2\lambda}), \end{aligned}$$

by applying formula 3.471.9, p.340 of Gradshteyn and Ryzhik [5] for $\nu = k - \frac{1}{2}$, $\beta = \frac{t^2}{2}$ and $\gamma = \lambda$. ■

Remark 3.1 We evaluate $\widehat{p}_k(t)$ for some values of k, directly from (3.3). First of all, we note that

$$\widehat{p}_0(t) = \int_0^{+\infty} e^{-\lambda s} \frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}} ds = e^{-t\sqrt{2\lambda}},$$

by a well-known result on the Laplace transform of T_t. This result can be checked by considering that

$$K_{-\frac{1}{2}}(x) = K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

(see formula 8.469.3, p.967 of Gradshteyn and Ryzhik [5]), so that we get, from (3.8),

$$\begin{aligned} \widehat{p}_0(t) &= \frac{2^{\frac{3}{4}} \lambda^{\frac{1}{4}} t^{\frac{1}{2}}}{\sqrt{\pi}} K_{-\frac{1}{2}}(t\sqrt{2\lambda}) \\ &= \frac{2^{\frac{3}{4}} \lambda^{\frac{1}{4}} t^{\frac{1}{2}}}{\sqrt{\pi}} \sqrt{\frac{\pi}{2t\sqrt{2\lambda}}} e^{-t\sqrt{2\lambda}} = e^{-t\sqrt{2\lambda}}. \end{aligned} \tag{3.9}$$

The probability (3.9) coincides with the density of the waiting-time of the first event of the process $\widehat{N}(t), t > 0$.

Analogously, we obtain, for $k = 1, 2$, that

$$\widehat{p}_1(t) = \lambda t \int_0^{+\infty} \frac{1}{\sqrt{2\pi s}} e^{-\lambda s} e^{-\frac{t^2}{2s}} ds = \lambda t \frac{e^{-t\sqrt{2\lambda}}}{\sqrt{2\lambda}} \tag{3.10}$$

and

$$\begin{aligned} \widehat{p}_2(t) &= \int_0^{+\infty} \frac{(\lambda s)^2}{2\sqrt{2\pi s^3}} t e^{-\lambda s} e^{-\frac{t^2}{2s}} ds \\ &= \frac{\lambda^2 t}{2\sqrt{2\pi}} \int_0^{+\infty} \sqrt{s} e^{-\lambda s} e^{-\frac{t^2}{2s}} ds \\ &= \frac{\lambda^2 t}{2\sqrt{2\pi}} \left(-\frac{e^{-\lambda s}}{\lambda} \sqrt{s} e^{-\frac{t^2}{2s}} \Big|_0^{\infty} \right) + \frac{\lambda t}{2\sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-\lambda s}}{2\sqrt{s}} e^{-\frac{t^2}{2s}} ds + \\ &\quad + \frac{\lambda t^3}{4\sqrt{2\pi}} \int_0^{+\infty} e^{-\lambda s} \sqrt{s} \frac{e^{-\frac{t^2}{2s}}}{s^2} ds \\ &= \frac{\lambda t}{4} \frac{e^{-t\sqrt{2\lambda}}}{\sqrt{2\lambda}} + \frac{\lambda t^2 e^{-t\sqrt{2\lambda}}}{4}. \end{aligned} \tag{3.11}$$

Theorem 3.3 *The probability generating function of $\widehat{N}$ is given by*

$$\widehat{G}(u, t) = \sum_{k=0}^{\infty} u^k \widehat{p}_k(t) = e^{-t\sqrt{2\lambda(1-u)}}, \qquad |u| \le 1, \tag{3.12}$$

which gives the following alternative expression for the state probabilities

$$\widehat{p}_k(t) = \sum_{m=0}^{\infty} \frac{(-1)^{k+m} (t\sqrt{2\lambda})^m}{m!} \binom{\frac{m}{2}}{k}. \tag{3.13}$$

Proof From (3.3) we get

$$\begin{aligned}\widehat{G}(u,t) &= t\int_0^{+\infty} e^{-\lambda s}\frac{e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}\sum_{k=0}^{\infty}\frac{(\lambda s u)^k}{k!}ds \qquad (3.14)\\ &= t\int_0^{+\infty} e^{-\lambda(1-u)s}\frac{e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}ds = e^{-t\sqrt{2\lambda(1-u)}},\end{aligned}$$

which coincides with (3.12). If we now consider its series expansion we get

$$\begin{aligned}\widehat{G}(u,t) &= \sum_{m=0}^{\infty}\frac{(-t)^m}{m!}\left(2\lambda(1-u)\right)^{m/2}\\ &= \sum_{m=0}^{\infty}\frac{(-t\sqrt{2\lambda})^m}{m!}\sum_{k=0}^{\infty}\binom{\frac{m}{2}}{k}(-u)^k\\ &= \sum_{k=0}^{\infty}u^k\sum_{m=0}^{\infty}\frac{(-1)^{k+m}(t\sqrt{2\lambda})^m}{m!}\binom{\frac{m}{2}}{k},\end{aligned}$$

from which (3.13) follows. Moreover, simple calculation suffices to check that the probabilities (3.13) yield (3.9), (3.10) and (3.11) for $k = 0, 1, 2$, respectively, by rewriting

$$\binom{\frac{m}{2}}{k} = \frac{\frac{m}{2}(\frac{m}{2}-1)...(\frac{m}{2}-k+1)}{k!}.$$

■

Remark 3.2 By taking the first derivative of (3.12), for $u = 1$, it is easy to check that the first moment of $N(T_t)$ is infinite:

$$\begin{aligned}\mathbb{E}N(T_t) &= \frac{\partial}{\partial u}\widehat{G}(u,t)\Big|_{u=1} = \frac{\partial}{\partial u}e^{-\lambda t\sqrt{2\lambda(1-u)}}\Big|_{u=1}\\ &= \frac{\lambda t\sqrt{2\lambda}}{2\sqrt{(1-u)}}\Big|_{u=1} = \infty.\end{aligned}$$

For this reason we consider a different time-argument instead of T_t: we define $T_t^\mu = \inf\{s > 0 : B^\mu(s) = t\}$, where $B^\mu = B^\mu(t), t > 0$ denotes a Brownian motion with drift μ. Therefore the composition of a standard Poisson process with the first passage-time of a Brownian motion with drift T_t^μ corresponds to considering the following process

$$\widehat{N}^\mu(t) = N(T_t^\mu), \qquad t > 0,$$

with probability distribution given by

$$\begin{aligned}\widehat{p}_k^\mu(t) &= \int_0^{+\infty} p_k(s)q^\mu(t,s)ds \qquad (3.15)\\ &= \frac{\lambda^k t}{k!}\int_0^{+\infty} s^k e^{-\lambda s}\frac{e^{-\frac{(t-\mu s)^2}{2s}}}{\sqrt{2\pi s^3}}ds\end{aligned}$$

where

$$q^\mu(t,s) = \frac{te^{-\frac{(t-\mu s)^2}{2s}}}{\sqrt{2\pi s^3}}, \qquad s,t > 0,\ \mu \in \mathbb{R}, \qquad (3.16)$$

denotes the density of the first-passage time of B^μ through the level t. We note that, for $\mu<0$, density (3.16) does not integrate to unity; indeed it is, in this case,

$$\Pr\left\{T_t^\mu<\infty\right\}=e^{-2|\mu|t}$$

and thus $\Pr\left\{T_t^\mu=\infty\right\}=1-e^{-2|\mu|t}$. This result is intuitively justified because the negative drift drives the sample paths away from the threshold t.

Theorem 3.4 *The state probabilities $\widehat{p}_k^\mu(t)$, $k\geq 0$, $t>0$, given in (3.15) are solutions to the difference-differential equations*

$$\frac{d^2}{dt^2}p_k-2\mu\frac{d}{dt}p_k=2\lambda[p_k-p_{k-1}], \tag{3.17}$$

with initial conditions

$$\widehat{p}_k^\mu(0)=\begin{cases}1, & k=0\\ 0, & k\geq 1\end{cases}.$$

Proof We first show that the density q^μ, defined in (3.16) satisfies the partial diffrential equation

$$\frac{\partial^2}{\partial t^2}q(t,s)-2\mu\frac{\partial}{\partial t}q(t,s)=2\frac{\partial}{\partial s}q(t,s). \tag{3.18}$$

Indeed, by taking the derivative of (3.16) with respect to s we get

$$\begin{aligned}\frac{\partial}{\partial s}q^\mu(t,s) &= e^{\mu t}\frac{\partial}{\partial s}\left\{t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}\right\}\\ &= e^{\mu t}\left\{\frac{e^{-\frac{\mu^2 s}{2}}}{2}\frac{\partial^2}{\partial t^2}\left(t\frac{e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}\right)+t\frac{e^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}\left(-\frac{\mu^2}{2}\right)e^{-\frac{\mu^2 s}{2}}\right\}.\end{aligned}$$

Taking the derivatives with respect to t we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}q^\mu(t,s) &= \frac{\partial}{\partial t}\left\{\mu e^{\mu t}t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}+e^{\mu t}\frac{\partial}{\partial t}\left(t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}\right)\right\}\\ &= \mu^2e^{\mu t}t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}+2e^{\mu t}\frac{\partial}{\partial t}\left(t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}\right)+\\ &\quad +e^{\mu t}\frac{\partial^2}{\partial t^2}\left(t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}\right)\\ &= 2\frac{\partial q^\mu}{\partial s}+2\mu^2e^{\mu t}t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}+2\mu e^{\mu t}\frac{\partial}{\partial t}\left(t\frac{e^{-\frac{t^2}{2s}}e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s^3}}\right)\\ &= 2\frac{\partial q^\mu}{\partial s}+2\mu\frac{\partial q^\mu}{\partial t},\end{aligned}$$

which gives equation (3.18). As a consequence we can derive the equation solved by

(3.15):

$$
\begin{aligned}
\frac{d^2}{dt^2}\widehat{p}_k^{\mu}(t) &= \int_0^{+\infty} p_k(s)\frac{d^2}{dt^2}q^{\mu}(t,s)ds \\
&= [\text{by (3.18)}] \\
&= 2\int_0^{+\infty} p_k(s)\left(\frac{\partial q^{\mu}}{\partial s}+\mu\frac{\partial q^{\mu}}{\partial t}\right)ds \\
&= 2p_k(s)q^{\mu}(t,s)|_{s=0}^{s=+\infty} - 2\int_0^{+\infty}\frac{d}{ds}p_k(s)q^{\mu}(t,s)ds + 2\mu\frac{d}{dt}\widehat{p}_k(t) \\
&= [\text{by (3.4}] \\
&= 2\lambda\int_0^{+\infty}[p_k(s)-p_{k-1}(s)]ds + 2\mu\frac{d}{dt}\widehat{p}_k(t) \\
&= 2\lambda[\widehat{p}_k^{\mu}(t)-\widehat{p}_{k-1}^{\mu}(t)] + 2\mu\frac{d}{dt}\widehat{p}_k^{\mu}(t).
\end{aligned}
$$

■

Remark 3.3 As a consequence of the previous result the probability generating function $\widehat{G}^{\mu}(u,t)$ solves the following equation:

$$
\frac{\partial^2}{\partial t^2}G - 2\mu\frac{\partial}{\partial t}G = 2\lambda(1-u)G, \tag{3.19}
$$

subject to $\widehat{G}^{\mu}(u,0)=1$. From (3.15) the solution to (3.19) can be evaluated as follows:

$$
\begin{aligned}
\widehat{G}^{\mu}(u,t) &= \sum_{k=0}^{\infty} u^k\widehat{p}_k^{\mu}(t) = t\int_0^{+\infty} e^{-\lambda s}\frac{e^{-\frac{(t-\mu s)^2}{2s}}}{\sqrt{2\pi s^3}}\sum_{k=0}^{\infty}\frac{(\lambda s u)^k}{k!}ds \qquad (3.20)\\
&= e^{\mu t}\int_0^{+\infty} e^{\lambda s(u-1)-\frac{\mu^2 s}{2}}\frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}ds \\
&= e^{\mu t - t\sqrt{\mu^2+2\lambda(1-u)}}.
\end{aligned}
$$

For $\mu=0$, (3.20) reduces to (3.12). By taking the first derivative of (3.20), for $u=1$, we derive the first moment of $N(T_t^{\mu})$ and show that it is finite in this case

$$
\begin{aligned}
\mathbb{E}N(T_t^{\mu}) &= \left.\frac{\partial}{\partial u}\widehat{G}^{\mu}(u,t)\right|_{u=1} = \left.\frac{\partial}{\partial u}e^{\mu t - t\sqrt{\mu^2+2\lambda(1-u)}}\right|_{u=1} \\
&= \left.\frac{\lambda t e^{\mu t - t\sqrt{\mu^2+2\lambda(1-u)}}}{\sqrt{\mu^2+2\lambda(1-u)}}\right|_{u=1} = \frac{\lambda t e^{-t(|\mu|-\mu)}}{|\mu|}.
\end{aligned}
$$

Therefore we get

$$
\mathbb{E}N(T_t^{\mu}) = \begin{cases} \frac{\lambda t e^{-2t|\mu|}}{|\mu|}, & \mu<0 \\ \infty, & \mu=0 \\ \frac{\lambda t}{\mu}, & \mu>0 \end{cases}.
$$

The variance can be obtained analogously, as follows:

$$\begin{aligned}
& \mathbb{E}\left\{N(T_t^\mu)\left[N(T_t^\mu)-1\right]\right\} \\
= & \left.\frac{\partial^2}{\partial u^2}\widehat{G}^\mu(u,t)\right|_{u=1} \\
= & \left.\frac{(\lambda t)^2 e^{\mu t - t\sqrt{\mu^2+2\lambda(1-u)}}}{\mu^2+2\lambda(1-u)}\right|_{u=1} + \left.\frac{\lambda^2 t e^{\mu t - t\sqrt{\mu^2+2\lambda(1-u)}}}{\sqrt{\left[\mu^2+2\lambda(1-u)\right]^3}}\right|_{u=1} \\
= & \left[\frac{(\lambda t)^2}{\mu^2}+\frac{\lambda^2 t}{|\mu|^3}\right] e^{-(|\mu|-\mu)t},
\end{aligned}$$

so that

$$Var(N(T_t^\mu)) = \frac{\lambda t}{|\mu|}\left(1+\frac{\lambda}{\mu^2}\right)e^{-(|\mu|-\mu)t} = \mathbb{E}N(T_t^\mu)\left(1+\frac{\lambda}{\mu^2}\right).$$

For the process $N(T_t^\mu)$ the variance is proportional to the mean value and this distinguishes this model from the classical one.

Remark 3.4 We derive now the probability distribution of $N(T_t^\mu), t>0$:

$$\begin{aligned}
\widehat{p}_k^\mu(t) &= \frac{\lambda^k t e^{\mu t}}{k!\sqrt{2\pi}}\int_0^{+\infty} s^{k-\frac{3}{2}} e^{-(\lambda+\frac{\mu^2}{2})s} e^{-\frac{t^2}{2s}} ds \qquad (3.21)\\
&= \frac{2\lambda^k t e^{\mu t}}{k!\sqrt{2\pi}}\left(\frac{t^2}{2\lambda+\mu^2}\right)^{\frac{k}{2}-\frac{1}{4}} K_{k-\frac{1}{2}}(t\sqrt{2\lambda+\mu^2}).
\end{aligned}$$

For $k=0$ we obtain the probability density of the waiting time of the first event of $N(T_t^\mu)$:

$$\begin{aligned}
\widehat{p}_0^\mu(t) &= \frac{2te^{\mu t}}{\sqrt{2\pi}}\left(\frac{t^2}{2\lambda+\mu^2}\right)^{-\frac{1}{4}} K_{-\frac{1}{2}}(t\sqrt{2\lambda+\mu^2}) \\
&= \frac{\sqrt{2}t^{\frac{1}{2}}e^{\mu t}\sqrt[4]{2\lambda+\mu^2}}{\sqrt{\pi}}\sqrt{\frac{\pi}{t\sqrt{2\lambda+\mu^2}}}e^{-t\sqrt{2\lambda+\mu^2}} = e^{\mu t - t\sqrt{2\lambda+\mu^2}},
\end{aligned}$$

which coincides with (3.20) for $u=0$.

We generalize the results obtained so far to the case of n successive iterations: let us denote by

$$T_j(t) = \inf\{s>0 : B_j(s)=t\}$$

the first-passage time through the level t of a Brownian motion $B_j(t)$, for $j=1,...,n$, and let us assume that B_j is independent from any other B_i, $i\neq j$ and from N. The process defined as

$$\widehat{N}^n(t) = N(T_1(T_2...(T_{n-1}(T_n(t)))...)), \qquad t>0 \qquad (3.22)$$

possesses distribution given by

$$\begin{aligned}
& \widehat{p}_k^n(t) \qquad (3.23)\\
= & \int_0^{+\infty}\int_0^{+\infty}...\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q(w_2,w_1)...q(w_n,w_{n-1})q(t,w_n)dw_1dw_2...dw_{n-1}dw_n \\
= & \frac{\lambda^k}{k!}\int_0^{+\infty}\int_0^{+\infty}...\int_0^{+\infty}\int_0^{+\infty} w_1^k e^{-\lambda w_1} w_2\frac{e^{-\frac{w_2^2}{2w_1}}}{\sqrt{2\pi w_1^3}}...w_n\frac{e^{-\frac{w_n^2}{2w_{n-1}}}}{\sqrt{2\pi w_{n-1}^3}}t\frac{e^{-\frac{t^2}{2w_n}}}{\sqrt{2\pi w_n^3}}dw_1dw_2...dw_{n-1}dw_n.
\end{aligned}$$

We state the following result.

Theorem 3.5 *The state distributions $\widehat{p}_k^n$ of the n-times iterated Poisson process $\widehat{N}^n(t), t > 0$, given in (3.23), are solutions to the following equations*

$$\frac{d^{2^n}}{dt^{2^n}} p_k(t) = 2^{2^n-1}\lambda[p_k(t) - p_{k-1}(t)], \qquad t > 0,\ k \geq 0, \tag{3.24}$$

with initial conditions

$$\widehat{p}_k^n(0) = \begin{cases} 1, & k = 0 \\ 0, & k \geq 1 \end{cases}.$$

Proof For $n = 1$ equations (3.24) reduce to (3.5). We prove this result in the special case $n = 2$:

$$\begin{aligned}
\frac{d^4}{dt^4}\widehat{p}_k^2(t) &= \int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q(w_2,w_1)\frac{\partial^4}{\partial t^4}q(t,w_2)dw_1dw_2 \qquad (3.25)\\
&= [\text{by } (3.2)]\\
&= 2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q(w_2,w_1)\frac{\partial^2}{\partial w_2^2}q(t,w_2)dw_1dw_2\\
&= 2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial^2}{\partial w_2^2}q(w_2,w_1)q(t,w_2)dw_1dw_2\\
&= 2^2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_1}q(w_2,w_1)q(t,w_2)dw_1dw_2\\
&= -2^2\int_0^{+\infty}\int_0^{+\infty} \frac{d}{dw_1}p_k(w_1)q(w_2,w_1)q(t,w_2)dw_1dw_2\\
&= 2^2\lambda\left[\widehat{p}_k^2(t) - \widehat{p}_{k-1}^2(t)\right].
\end{aligned}$$

By induction it can be checked that (3.24) holds for any $n \geq 1$. ■

Remark 3.5 We derive the probability generating function that, in this case, is equal to

$$\widehat{G}^n(u,t) = \sum_{k=0}^{\infty} u^k \widehat{p}_k^n(t) = e^{-2^{\left(1-\frac{1}{2^n}\right)}\lambda^{\frac{1}{2^n}}(1-u)^{\frac{1}{2^n}}t}. \tag{3.26}$$

By taking the first derivative of (3.26) it is easy to see that the expected value of the process is infinite:

$$\begin{aligned}
\mathbb{E}\widehat{N}^n(t) &= \left.\frac{d}{du}\widehat{G}^n(u,t)\right|_{u=1}\\
&= \left. 2^{\left(1-\frac{1}{2^n}\right)}\frac{\lambda^{\frac{1}{2^n}}}{2^n}(1-u)^{\frac{1}{2^n}-1}t\, e^{-2^{\left(1-\frac{1}{2^n}\right)}\lambda^{\frac{1}{2^n}}(1-u)^{\frac{1}{2^n}}t}\right|_{u=1} = \infty.
\end{aligned}$$

Remark 3.6 In the case where each Brownian motion is endowed by a drift μ, the process is defined as

$$\widehat{N}_\mu^n(t) = N(T_1^\mu(T_2^\mu ...(T_{n-1}^\mu(T_n^\mu(t)))...)), \qquad t > 0.$$

For the sake of simplicity we will assume hereafter that $\mu > 0$. We start again by considering the case where $n = 2$: the probability distribution is, in this case,

$$\begin{aligned} & \widehat{p}_k^n(t) \\ = & \int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 \\ = & \frac{\lambda^k}{k!}\int_0^{+\infty}\int_0^{+\infty} w_1^k e^{-\lambda w_1} w_2 \frac{e^{-\frac{(w_2-\mu w_1)^2}{2w_1}}}{\sqrt{2\pi w_1^3}} t \frac{e^{-\frac{(t-\mu w_2)^2}{2w_2}}}{\sqrt{2\pi w_2^3}} dw_1dw_2, \end{aligned} \tag{3.27}$$

for $k \geq 0$. We start by taking the second-order derivative with respect to t of (3.18):

$$\begin{aligned} & \frac{\partial^4}{\partial t^4}q^\mu(t,w) \\ = & \frac{\partial^2}{\partial t^2}\left[2\frac{\partial}{\partial w}q^\mu(t,w) + 2\mu\frac{\partial}{\partial t}q^\mu(t,w)\right] \\ = & 2\left[2\frac{\partial^2}{\partial w^2}q^\mu(t,w) + 2\mu\frac{\partial^2}{\partial t\partial w}q^\mu(t,w)\right] + 2\mu\frac{\partial^3}{\partial t^3}q^\mu(t,w). \end{aligned}$$

Therefore, by taking the fourth-order derivative of (3.27) we get

$$\begin{aligned} & \frac{d^4}{dt^4}\widehat{p}_k^n(t) = \int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)\frac{\partial^4}{\partial t^4}q^\mu(t,w_2)dw_1dw_2 \\ = & 2^2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)\frac{\partial^2}{\partial w_2^2}q^\mu(t,w_2)dw_1dw_2 + \\ & +2^2\mu\frac{\partial}{\partial t}\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)\frac{\partial}{\partial w_2}q^\mu(t,w_2)dw_1dw_2 + \\ & +2\mu\frac{\partial^3}{\partial t^3}\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 \\ = & -2^2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_2}q^\mu(w_2,w_1)\frac{\partial}{\partial w_2}q^\mu(t,w_2)dw_1dw_2 + \\ & -2^2\mu\frac{\partial}{\partial t}\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 + \\ & +2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t) \\ = & 2^2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial^2}{\partial^2 w_2^2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 \\ & -2^2\mu\frac{\partial}{\partial t}\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 + \\ & +2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t). \end{aligned} \tag{3.28}$$

By considering that for the second-order derivative of (3.27) the following result holds

$$\begin{aligned} \frac{d^2}{dt^2}\widehat{p}_k^n(t) = & \int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)\frac{\partial^2}{\partial t^2}q^\mu(t,w_2)dw_1dw_2 \\ = & 2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)q^\mu(w_2,w_1)\left[\frac{\partial}{\partial w_2}q^\mu(t,w_2) + \mu\frac{\partial}{\partial t}q^\mu(t,w_2)\right]dw_1dw_2 \\ = & -2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 + \\ & +2\mu\frac{d}{dt}\widehat{p}_k^n(t), \end{aligned} \tag{3.29}$$

we get, from (3.29),

$$
\begin{aligned}
& \int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2 \\
= & -\frac{1}{2}\frac{d^2}{dt^2}\widehat{p}_k^n(t)+\mu\frac{d}{dt}\widehat{p}_k^n(t).
\end{aligned}
$$

Therefore formula (3.28) can be rewritten as

$$
\begin{aligned}
& \frac{d^4}{dt^4}\widehat{p}_k^n(t) \\
= & \; 2^2\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial^2}{\partial^2 w_2^2}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2+ \\
& -2^2\mu\frac{d}{dt}\left[-\frac{1}{2}\frac{d^2}{dt^2}\widehat{p}_k^n(t)+\mu\frac{d}{dt}\widehat{p}_k^n(t)\right]+2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t) \\
= & \; 2^3\int_0^{+\infty}\int_0^{+\infty} p_k(w_1)\frac{\partial}{\partial w_1}q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2+ \\
& +2^3\mu\left[-\frac{1}{2}\frac{d^2}{dt^2}\widehat{p}_k^n(t)+\mu\frac{d}{dt}\widehat{p}_k^n(t)\right]+ \\
& +2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t)-2^2\mu^2\frac{d^2}{dt^2}\widehat{p}_k^n(t)+2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t) \\
= & -2^3\int_0^{+\infty}\int_0^{+\infty} \frac{d}{dw_1}p_k(w_1)q^\mu(w_2,w_1)q^\mu(t,w_2)dw_1dw_2+ \\
& -2^2\mu\frac{d^2}{dt^2}\widehat{p}_k^n(t)+2^3\mu^2\frac{d}{dt}\widehat{p}_k^n(t)+2^2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t)-2^2\mu^2\frac{d^2}{dt^2}\widehat{p}_k^n(t) \\
= & -2^3\lambda\left[\widehat{p}_{k-1}^n(t)-\widehat{p}_k^n(t)\right]+2^3\mu^2\frac{d}{dt}\widehat{p}_k^n(t)+2^2\mu\frac{d^3}{dt^3}\widehat{p}_k^n(t)+ \\
& -2^2\mu(1+\mu)\frac{d^2}{dt^2}\widehat{p}_k^n(t).
\end{aligned}
$$

Finally we get that, for $n=2$, the state probabilities (3.27) satisfy

$$
\frac{d^4}{dt^4}p_k(t)-2^2\mu\frac{d^3}{dt^3}p_k(t)+2^2\mu(1+\mu)\frac{d^2}{dt^2}p_k(t)-2^3\mu^2\frac{d}{dt}p_k(t)=2^3\lambda\left[p_k(t)-p_{k-1}(t)\right].
$$

The expression of the probability generating function is much more complicated in this case, due to the presence of the drift.

Theorem 3.6 *The probability generating function of the process* $\widehat{N}_\mu^n(t), t>0$ *is given, for any* $n\geq 1$, *by*

$$
\widehat{G}_\mu^n(u,t)=e^{\mu t-2^{\frac{1}{2}}t\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\dots 2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}}}, \qquad |u|\leq 1 \tag{3.30}
$$

and the expected value is equal to

$$
\mathbb{E}\widehat{N}_\mu^n(t)=\frac{\lambda t}{\mu^n}, \qquad n\geq 1. \tag{3.31}
$$

Proof We give the details of the calculations in the case where $n=2$:

$$\widehat{G}^n_\mu(u,t) = \sum_{k=0}^{\infty} u^k \widehat{p}^n_k(t) \tag{3.32}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} u^k \frac{\lambda^k t}{k!} \int_0^{+\infty}\int_0^{+\infty} w_1^k e^{-\lambda w_1} w_2 \frac{e^{-\frac{(w_2-\mu w_1)^2}{2w_1}}}{\sqrt{2\pi w_1^3}} \frac{e^{-\frac{(t-\mu w_2)^2}{2w_2}}}{\sqrt{2\pi w_2^3}} dw_1 dw_2 \\
&= te^{\mu t}\int_0^{+\infty} \frac{1}{\sqrt{2\pi w_2^3}} e^{-\frac{t^2}{2w_2}-\frac{\mu^2 w_2}{2}+\mu w_2} \int_0^{+\infty} e^{-\lambda(1-u)w_1-\frac{\mu^2 w_1}{2}} \frac{w_2 e^{-\frac{w_2^2}{2w_1}}}{\sqrt{2\pi w_1^3}} dw_1 dw_2 \\
&= te^{\mu t}\int_0^{+\infty} \frac{1}{\sqrt{2\pi w_2^3}} e^{-\frac{t^2}{2w_2}-\frac{\mu^2 w_2}{2}+\mu w_2} e^{-w_2\sqrt{2\lambda(1-u)+\mu^2}} dw_2 \\
&= e^{\mu t}\int_0^{+\infty} \frac{t}{\sqrt{2\pi w_2^3}} e^{-\frac{t^2}{2w_2}-w_2\left(\frac{\mu^2}{2}-\mu+\sqrt{2\lambda(1-u)+\mu^2}\right)} dw_2 \\
&= e^{\mu t - t 2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}.
\end{aligned}$$

By taking the first derivative of (3.32), it is easy to see that the expected value of the process is finite:

$$\begin{aligned}
\mathbb{E}\widehat{N}^n_\mu(t) &= \left.\frac{d}{du}\widehat{G}^n_\mu(u,t)\right|_{u=1} \\
&= \left.\frac{2^{\frac{1}{2}-2+\frac{1}{2}}\lambda t e^{\mu t - t2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}}{\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}\right|_{u=1} = \frac{\lambda t}{\mu^2}.
\end{aligned}$$

For $n=3$ the probability generating function can be obtained in an analogous way:

$$\widehat{G}^n_\mu(u,t) \tag{3.33}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} u^k \frac{\lambda^k t}{k!}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} w_1^k e^{-\lambda w_1} \frac{w_2 e^{-\frac{(w_2-\mu w_1)^2}{2w_1}}}{\sqrt{2\pi w_1^3}} \frac{w_3 e^{-\frac{(w_3-\mu w_2)^2}{2w_2}}}{\sqrt{2\pi w_2^3}} \frac{e^{-\frac{(t-\mu w_3)^2}{2w_3}}}{\sqrt{2\pi w_3^3}} dw_1 dw_2 dw_3 \\
&= t\int_0^{+\infty}\int_0^{+\infty} e^{-\frac{\mu^2 w_2}{2}+\mu w_2+\mu w_3} \frac{w_3 e^{-\frac{(w_3-\mu w_2)^2}{2w_2}}}{\sqrt{2\pi w_2^3}} \frac{e^{-\frac{(t-\mu w_3)^2}{2w_3}}}{\sqrt{2\pi w_3^3}} e^{-w_2\sqrt{2\left[\frac{\mu^2}{2}+\lambda(1-u)\right]}} dw_2 dw_3 \\
&= te^{\mu t}\int_0^{+\infty} e^{-\frac{\mu^2 w_3}{2}+\mu w_3} \frac{e^{-\frac{t^2}{2w_3}}}{\sqrt{2\pi w_3^3}} e^{-w_3\sqrt{2\left[\frac{\mu^2}{2}-\mu+\sqrt{2\left[\frac{\mu^2}{2}+\lambda(1-u)\right]}\right]}} dw_3 \\
&= e^{\mu t - t2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}}
\end{aligned}$$

and then the expected value reads, for $n = 3$,

$$\mathbb{E}\widehat{N}^n_\mu(t)$$

$$= \frac{2^{\frac{3}{2}-3}\lambda t e^{\mu t - t 2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}}}{\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}}} \cdot$$

$$\cdot \frac{1}{\sqrt{\frac{\mu^2}{2}-\mu+2^{\frac{1}{2}}\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}} \frac{1}{\sqrt{\frac{\mu^2}{2}+\lambda(1-u)}}\Bigg|_{u=1} = \frac{\lambda t}{\mu^3}.$$

By the same reasoning we arrive at formulas (3.30) and (3.31) for any $n \geq 1$. For $\mu = 0$ formula (3.30) coincides with (3.26). ∎

Remark 3.7 By considering (3.24) it is easy to check that (3.30) satisfies the following recursive differential equation

$$\begin{aligned} \frac{d^{2^n}}{dt^{2^n}}\widehat{G}^n(u,t) &= 2^{2^n-1}\lambda\sum_{k=0}^{\infty}u^k[\widehat{p}^n_k(t)-\widehat{p}^n_{k-1}(t)] \\ &= 2^{2^n-1}\lambda(u-1)\widehat{G}^n(u,t). \end{aligned}$$

Indeed by taking the derivatives of (3.30) we get

$$\begin{aligned} \frac{d^{2^n}}{dt^{2^n}}\widehat{G}^n_\mu(u,t) &= \left(2^{1-\frac{1}{2^n}}\left(\lambda(u-1)\right)^{\frac{1}{2^n}}\right)^{2^n} e^{\mu t - 2^{1-\frac{1}{2^n}}t(\lambda(1-u))^{\frac{1}{2^n}}} \\ &= 2^{2^n-1}\lambda(u-1)\widehat{G}^n(u,t). \end{aligned}$$

4 Poisson processes at Brownian sojourn times

We consider the composition of a homogeneous Poisson process with a random process, distributed as the sojourn time on the positive half-line of a standard Brownian motion $\Gamma^+_t = meas\{s < t : B(s) > 0\}$, i.e.

$$\overline{N}(t) = N(\Gamma^+_t), \qquad t > 0. \tag{4.1}$$

Since the density function of Γ^+_t is equal to

$$\Pr\left\{\Gamma^+_t \in ds\right\} = \frac{ds}{\pi\sqrt{s(t-s)}}, \quad 0 < s < t, \tag{4.2}$$

the probability distribution of $\overline{N}(t), t > 0$ is given by

$$\overline{p}_k(t) = \Pr\left\{\overline{N}(t) = k\right\} = \frac{1}{\pi k!}\int_0^t \frac{(\lambda s)^k e^{-\lambda s}}{\sqrt{s(t-s)}}ds, \quad k \geq 0, t > 0. \tag{4.3}$$

An explicit expression for (4.3) is obtained in the following result.

Theorem 4.1 *The state probabilities of the process $\overline{N}$ can be expressed as follows:*

$$\overline{p}_k(t) = p_k(t)\binom{2k-1}{k}2^{1-2k}\,{}_1F_1\left(\frac{1}{2}, k+1; \lambda t\right), \tag{4.4}$$

where p_k, $k = 0, 1, ...$ is the probability distribution of the homogeneous Poisson process and ${}_1F_1(\alpha, \beta; x)$ denotes the confluent hypergeometric function defined as

$$\begin{aligned} {}_1F_1(\alpha;\beta;x) &= 1 + \sum_{j=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+j-1)}{\gamma(\gamma+1)...(\gamma+j-1)} \frac{z^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} \frac{z^j}{j!} \end{aligned}$$

where $(\gamma)_r = \gamma(\gamma+1)...(\gamma+r-1)$ (for $r = 1, 2, ...,$ and $\gamma \neq 0$) and $(\gamma)_0 = 1$.(see Gradshteyn and Ryzhik [5], p.1085).

Proof We can recognize in the integral (4.3) formula 3.383.1, p.365 of Gradshteyn and Ryzhik [5],i.e.

$$\int_0^u x^{\mu-1}(u-x)^{\nu-1}e^{\beta x}dx = B(\mu,\nu)\, u^{\mu+\nu-1}\, {}_1F_1(\mu, \mu+\nu; \beta u), \tag{4.5}$$

so that we get

$$\begin{aligned} \overline{p}_k(t) &= \frac{(\lambda t)^k}{\pi k!} B\left(k+\frac{1}{2}, \frac{1}{2}\right) {}_1F_1\left(\frac{1}{2}, k+1; -\lambda t\right) \\ &= \text{[by 9.212.1, p.1086 of Gradshteyn and Ryzhik [5]]} \\ &= \frac{(\lambda t)^k e^{-\lambda t}}{\pi k!} B\left(k+\frac{1}{2}, \frac{1}{2}\right) {}_1F_1\left(\frac{1}{2}, k+1; \lambda t\right) \\ &= \text{[by the duplication formula of Gamma function]} \\ &= p_k(t)\binom{2k-1}{k} 2^{1-2k}\, {}_1F_1\left(\frac{1}{2}, k+1; \lambda t\right). \end{aligned}$$

■

Remark 4.1 We can interpret the process (4.1) in some distributionally equivalent forms. Since it is well-known that

$$\mathcal{T}_0(t) = \sup\{s < t : B(s) = 0\}$$

and

$$\Theta(t) = \inf\left\{s < t : B(s) = \max_{0 \le z \le t} B(z)\right\}$$

possess the same distribution (4.2) as Γ_t^+, we can interpret the results of this section as pertaining to the following compositions

$$N(\mathcal{T}_0(t)) \quad \text{and} \quad N(\Theta(t)), \quad t > 0.$$

Theorem 4.2 *The state probabilities $\overline{p}_k$ given in (4.4) solve the following recursive differential equations:*

$$\frac{d}{dt}p_k(t) = \frac{k}{t}p_k(t) - \frac{k+1}{t}p_{k+1}(t), \qquad k \ge 0,\ t > 0 \tag{4.6}$$

with initial conditions

$$\overline{p}_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \ge 1 \end{cases}.$$

Proof We rewrite (4.3) as follows

$$\overline{p}_k(t) = \frac{(\lambda t)^k}{\pi k!} \int_0^1 e^{-\lambda t z} z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} dz \tag{4.7}$$

and take the first order derivative with respect to t, so that we get

$$\begin{aligned}
&\frac{d}{dt}\overline{p}_k(t) = \frac{\lambda^k}{\pi k!} \int_0^1 z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} \frac{d}{dt}\left(t^k e^{-\lambda t z}\right) dz \\
= \; & \frac{\lambda^k}{\pi k!} \left[k t^{k-1} \int_0^1 z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} e^{-\lambda t z} dz - \lambda t^k \int_0^1 z^{k+\frac{1}{2}} (1-z)^{-\frac{1}{2}} e^{-\lambda t z} dz \right] \\
= \; & \frac{k}{t}\overline{p}_k(t) - \frac{k+1}{t}\overline{p}_{k+1}(t).
\end{aligned}$$

■

Remark 4.2 We evaluate the Laplace transform of (4.7) which reads

$$\begin{aligned}
\mathcal{L}\{\overline{p}_k(t), \eta\} &= \int_0^\infty e^{-\eta t} \overline{p}_k(t) dt \qquad (4.8) \\
&= \int_0^\infty e^{-\eta t} \frac{\lambda^k t^k}{\pi k!} \int_0^1 z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} e^{-(\lambda z)t} dz dt \\
&= \frac{1}{\pi} \int_0^1 z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} \int_0^\infty \frac{\lambda^k t^k}{k!} e^{-(\eta + \lambda z)t} dt dz \\
&= \frac{1}{\pi} \int_0^1 z^{k-\frac{1}{2}} (1-z)^{-\frac{1}{2}} \frac{\lambda^k \Gamma(k+1)}{k!(\eta + \lambda z)^{k+1}} dz \\
&= \frac{1}{\pi} \int_0^1 z^{-\frac{1}{2}} (1-z)^{-\frac{1}{2}} \frac{(\lambda z)^k}{(\eta + \lambda z)^{k+1}} dz.
\end{aligned}$$

The last expression in (4.8) permits us to interpret the process $\overline{N}(t), t > 0$ as the standard homogeneous Poisson process with random rate Λ distributed as a Beta random variable of parameters $\frac{1}{2}, \frac{1}{2}$. Indeed the Laplace transform of a standard Poisson process is given by

$$\mathcal{L}\{p_k(t), \eta\} = \frac{(\lambda z)^k}{(\eta + \lambda z)^{k+1}}.$$

The same conclusion can be drawn directly from (4.7).

Remark 4.3 The probability generating function can be evaluated as follows:

$$\begin{aligned}
\overline{G}(u,t) &= \sum_{k=0}^\infty u^k \overline{p}_k(t) = \int_0^t \frac{e^{-\lambda s}}{\pi\sqrt{s(t-s)}} \sum_{k=0}^\infty \frac{(\lambda s u)^k}{k!} ds \qquad (4.9) \\
&= \int_0^t \frac{e^{-\lambda s(1-u)}}{\pi\sqrt{s(t-s)}} ds \\
&= [\text{by } (4.5)] \\
&= {}_1F_1\left(\frac{1}{2}, 1; \lambda t(u-1)\right)
\end{aligned}$$

$$
\begin{aligned}
&= [\text{by 9.215.2, p.1086 of Gradshteyn and Ryzhik [5]}]\\
&= e^{-\frac{\lambda t(1-u)}{2}} J_0(-\frac{\lambda t}{2}(1-u)e^{\frac{i\pi}{2}})\\
&= e^{-\frac{\lambda t(1-u)}{2}} \sum_{k=0}^{\infty} \frac{(-e^{i\pi})^k}{(k!)^2} \frac{(-\lambda t(1-u))^{2k}}{2^{4k}}\\
&= e^{-\frac{\lambda t(1-u)}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{(\lambda t(1-u))^{2k}}{2^{4k}}\\
&= G(u,t) I_0(\frac{\lambda t(1-u)}{2}),
\end{aligned}
$$

where $G(u,t)$ denotes the probability generating function of the homogeneous Poisson process with rate $\lambda/2$.

We can derive the same result by evaluating the integral in (4.9) directly, as follows,

$$
\begin{aligned}
& \int_0^t \frac{e^{-\lambda s(1-u)} s^{-\frac{1}{2}}}{\pi\sqrt{t-s}} ds\\
&= [\text{by putting } s = t\sin^2\phi]\\
&= \frac{2\sqrt{t}}{\pi} \int_0^{\frac{\pi}{2}} e^{-\lambda(1-u)t\sin^2\phi} \frac{(t\sin^2\phi)^{-\frac{1}{2}}}{\cos\phi} \sin\phi\cos\phi d\phi\\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-\lambda(1-u)t\sin^2\phi} d\phi\\
&= [\sin^2\phi = \frac{1-\cos 2\phi}{2}]\\
&= \frac{2e^{-\frac{\lambda(1-u)t}{2}}}{\pi} \int_0^{\frac{\pi}{2}} e^{\frac{\lambda(1-u)t\cos 2\phi}{2}} d\phi\\
&= \frac{e^{-\frac{\lambda(1-u)t}{2}}}{\pi} \int_0^{\pi} e^{\frac{\lambda(1-u)t\cos\theta}{2}} d\theta\\
&= e^{-\frac{\lambda(1-u)t}{2}} I_0\left(\frac{\lambda t(1-u)}{2}\right).
\end{aligned}
$$

For the factorial moments, we get from (4.9)

$$
\begin{aligned}
& \mathbb{E}\left[\overline{N}(t)(\overline{N}(t)-1)...(\overline{N}(t)-r+1)\right] \qquad (4.10)\\
&= \frac{d^r}{du^r}\overline{G}(u,t)\bigg|_{u=1}\\
&= \lambda^r \int_0^t \frac{s^r e^{-\lambda s(1-u)}}{\pi\sqrt{s(t-s)}} ds\bigg|_{u=1} = \frac{(\lambda t)^r}{\pi} B\left(r+\frac{1}{2},\frac{1}{2}\right)\\
&= \frac{(\lambda t)^r}{r!\sqrt{\pi}}\Gamma\left(r+\frac{1}{2}\right) = p_r(t)e^{\lambda t}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}}.
\end{aligned}
$$

From (4.10) it is easy to derive

$$\mathbb{E}\overline{N}(t) = \frac{\lambda t}{2}$$

and

$$Var(\overline{N}(t)) = \frac{\lambda^2 t^2}{8} + \frac{\lambda t}{2}.$$

Remark 4.4 We can give an alternative representation to the distribution (4.4) and the factorial moments (4.10), in terms of the time T^0 of the first return in zero of a coin tossing random walk, whose distribution is given by

$$\Pr\left\{T^0 = 2k+2\right\} = \binom{2k}{k}\frac{1}{k+1}\frac{1}{2^{2k+1}}, \qquad k = 0, 1, ... \tag{4.11}$$

Indeed the distribution (4.4) can be written as

$$\overline{p}_k(t) = 2(k+1)p_k(t)\Pr\left\{T^0 = 2k+2\right\} {}_1F_1\left(\frac{1}{2}, k+1; \lambda t\right).$$

The factorial moments, instead, read

$$\mathbb{E}\left[\overline{N}(t)(\overline{N}(t)-1)...(\overline{N}(t)-r+1)\right] = 2(r+1)p_r(t)\Pr\left\{T^0 = 2r+2\right\}.$$

5 Poisson processes at Bessel times

Let us denote by $R_\gamma(t), t > 0$ the γ-Bessel process, starting at zero, with transition function given by

$$p_\gamma(s,t) = \frac{2s^{\gamma-1}e^{-\frac{s^2}{2t}}}{(2t)^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \tag{5.1}$$

for $s, t, \gamma > 0$, and with generator

$$\mathcal{A} = \frac{1}{2}\left\{\frac{\partial^2}{\partial s^2} + \frac{\gamma-1}{s}\frac{\partial}{\partial s}\right\}. \tag{5.2}$$

We study now the composition of a homogeneous Poisson process with a process defined as the square of $R_\gamma(t), t > 0$, which will be denoted by $R_\gamma^2 = (R_\gamma(t))^2, t > 0$. We derive the transition density of this second process, as follows:

$$\begin{aligned} p_\gamma^2(s,t) &= \frac{d}{ds}\Pr\left\{R_\gamma^2(t) < s\right\} = \frac{d}{ds}\int_0^{\sqrt{s}} \frac{2w^{\gamma-1}e^{-\frac{w^2}{2t}}}{(2t)^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)}dw \\ &= \frac{s^{\frac{\gamma}{2}-1}e^{-\frac{s}{2t}}}{(2t)^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)}, \qquad s, t > 0. \end{aligned}$$

Therefore are interested in deriving the probability distribution of the following process

$$\widetilde{N}_\gamma(t) = N(R_\gamma^2(t)), \qquad t > 0,$$

and its governing equation.

Theorem 5.1 *The state probabilities ${}_\gamma\widetilde{p}_k$ of the process $\widetilde{N}_\gamma(t), t > 0$ are given, for any $k \geq 0$, by*

$$ {}_\gamma\widetilde{p}_k(t) = \Pr\left\{\widetilde{N}_\gamma(t) = k\right\} = \frac{(2\lambda t)^k}{(2\lambda t+1)^{k+\frac{\gamma}{2}}}\frac{\Gamma\left(k+\frac{\gamma}{2}\right)}{k!\Gamma\left(\frac{\gamma}{2}\right)}. \tag{5.3}$$

The probability generating function of the distribution (5.3) has the following form

$$\widetilde{G}_\gamma(u,t) = \frac{1}{(2\lambda t(1-u)+1)^{\gamma/2}}, \qquad |u| \leq 1. \tag{5.4}$$

Proof The distribution is obtained directly as follows

$$\begin{aligned}
{}_{\gamma}\widetilde{p}_k(t) &= \int_0^{+\infty} \frac{\lambda^k}{k!} s^k e^{-\lambda s} p(s,t) ds \\
&= \frac{\lambda^k}{k!(2t)^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^{+\infty} e^{-\lambda s} s^{k+\frac{\gamma}{2}-1} e^{-\frac{s}{2t}} ds \\
&= \frac{\lambda^k}{k!(2t)^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \left(\frac{2t}{2\lambda t+1}\right)^{k+\frac{\gamma}{2}} \Gamma\left(k+\frac{\gamma}{2}\right),
\end{aligned}$$

which coincides with (5.3). We derive the probability generating function as follows:

$$\begin{aligned}
\widetilde{G}_{\gamma}(u,t) &= \sum_{k=0}^{\infty} u^k \, {}_{\gamma}\widetilde{p}_k(t) = \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \int_0^{+\infty} e^{-z} z^{\frac{\gamma}{2}-1} e^{\frac{2\lambda t u z}{2\lambda t+1}} dz \\
&= \frac{1}{(2\lambda t+1)^{\gamma/2}} \frac{1}{\left(1-\frac{2\lambda t u}{2\lambda t+1}\right)^{\frac{\gamma}{2}}} \\
&= \frac{1}{(2\lambda t(1-u)+1)^{\gamma/2}}.
\end{aligned}$$

■

Remark 5.1 An alternative expression for the probabilities (5.3) can be obtained by rewriting it as follows:

$$\begin{aligned}
{}_{\gamma}\widetilde{p}_k(t) &= \frac{(2\lambda t)^k}{(2\lambda t+1)^{k+\frac{\gamma}{2}}} \frac{1}{k!\Gamma\left(\frac{\gamma}{2}\right)} \int_0^{+\infty} e^{-w} w^{k+\frac{\gamma}{2}-1} dw \qquad (5.5) \\
&= \frac{(2\lambda t)^k}{(2\lambda t+1)^{k+\frac{\gamma}{2}}} \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)} \int_0^{+\infty} \Pr\left\{N(w)=k\right\} w^{\frac{\gamma}{2}-1} dw.
\end{aligned}$$

Formula (5.5) possesses an interesting interpretation for $k \geq 1$, since, in this case, we can recognize the probability distribution of a birth-death linear process $M(t), t > 0$ with birth and death rates equal to 2λ, which reads

$$\Pr\left\{M(t)=k\right\} = \frac{(2\lambda t)^{k-1}}{(2\lambda t+1)^{k+1}} \qquad k \geq 1$$

and

$$\Pr\left\{M(t)=0\right\} = \frac{2\lambda t}{2\lambda t+1}.$$

(see, for example, Bailey [1]). Therefore we get

$${}_{\gamma}\widetilde{p}_k(t) = \frac{2\lambda t}{(2\lambda t+1)^{\frac{\gamma}{2}-1}} \frac{\Pr\left\{M(t)=k\right\}}{\Gamma\left(\frac{\gamma}{2}\right)} \int_0^{+\infty} \Pr\left\{N(w)=k\right\} w^{\frac{\gamma}{2}-1} dw. \qquad (5.6)$$

In the special case where $\gamma = 2$, formula (5.6) reduces to

$${}_{2}\widetilde{p}_k(t) = 2\lambda t \frac{(2\lambda t)^{k-1}}{(2\lambda t+1)^{k+1}} = 2\lambda t \Pr\left\{M(t)=k\right\}. \qquad (5.7)$$

The presence of the factor $2\lambda t$ can be explained by considering that, for the Poisson process, the extinction probability must be equal to zero.

Remark 5.2 It is easy to check that (5.7) represents, for $k \geq 0$, a genuine probability distribution:

$$\sum_{k=0}^{\infty} {}_2\widetilde{p}_k(t) = \sum_{k=0}^{\infty} \frac{(2\lambda t)^k}{(2\lambda t+1)^{k+1}} = 1.$$

In the general case $\gamma > 0$, this check is a bit more complicated:

$$\sum_{k=0}^{\infty} {}_\gamma\widetilde{p}_k(t) = \sum_{k=0}^{\infty} \frac{(2\lambda t)^k}{(2\lambda t+1)^k} \frac{\Gamma\left(k+\frac{\gamma}{2}\right)}{k!\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \tag{5.8}$$

$$= \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \sum_{k=0}^{\infty} \frac{(2\lambda t)^k}{k!\,(2\lambda t+1)^k} \int_0^{+\infty} e^{-z} z^{k+\frac{\gamma}{2}-1} dz$$

$$= \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \int_0^{+\infty} e^{-z} z^{\frac{\gamma}{2}-1} \sum_{k=0}^{\infty} \frac{(2\lambda t z)^k}{k!\,(2\lambda t+1)^k} dz$$

$$= \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \int_0^{+\infty} e^{-z} z^{\frac{\gamma}{2}-1} e^{\frac{2\lambda t z}{2\lambda t+1}} dz$$

$$= \frac{1}{\Gamma\left(\frac{\gamma}{2}\right)(2\lambda t+1)^{\gamma/2}} \frac{1}{\left(1-\frac{2\lambda t}{2\lambda t+1}\right)^{\frac{\gamma}{2}}} \Gamma\left(\frac{\gamma}{2}\right) = 1.$$

Remark 5.3 By taking the first derivative of (5.4) we get that the first moment is equal to

$$\mathbb{E}\widetilde{N}_\gamma(t) = \left. \frac{2\lambda t\gamma\,(2\lambda t(1-u)+1)^{\frac{\gamma}{2}-1}}{(2\lambda t(1-u)+1)^{\gamma}} \right|_{u=1} = \lambda t\gamma, \tag{5.9}$$

while its variance can be obtained as follows

$$\mathbb{E}\left[\widetilde{N}_\gamma(t)\left(\widetilde{N}_\gamma(t)-1\right)\right]$$

$$= \left. \frac{(2\lambda t)^2\,\gamma\left(\frac{\gamma}{2}+1\right)}{2\,(2\lambda t(1-u)+1)^{\frac{\gamma}{2}+2}} \right|_{u=1} = (\lambda t)^2\,\gamma\,(\gamma+2)\,,$$

so that we get

$$\mathbb{V}ar\left(\widetilde{N}_\gamma(t)\right) = \lambda t\gamma\,[2\lambda t+1]\,. \tag{5.10}$$

Results (5.9) and (5.10) can be checked, in the case $\gamma = 2$, by using (5.7) and considering that

$$\mathbb{E}M(t) = 1, \qquad \mathbb{V}arM(t) = 2\lambda t.$$

Finally we derive the differential equations satisfied by (5.3) and (5.4).

Theorem 5.2 *The state probabilities $\widetilde{p}_k$, given in (5.3), are solutions to the following difference-differential equations*

$$\frac{d}{dt}p_k(t) = \frac{k}{t}p_k(t) - \frac{k+1}{t}p_{k+1}(t), \qquad t>0,\ k\geq 0 \tag{5.11}$$

subject to the initial conditions

$$\widetilde{p}_k(0) = \begin{cases} 1, & k=0 \\ 0, & k\geq 1 \end{cases},$$

while the probability generating function $\widetilde{G}_\gamma(u,t)$ is solution to

$$\frac{\partial G}{\partial t}(u,t) = -\frac{1-u}{t}\frac{\partial G}{\partial u}(u,t), \qquad t>0,\ |u|\leq 1, \tag{5.12}$$

with $\widetilde{G}_\gamma(u,0) = 1$.

Proof We can check (5.11), directly, by taking the derivatives of (5.3)

$$\begin{aligned}
\frac{d}{dt}\,{}_\gamma\widetilde{p}_k(t) &= \frac{\Gamma\left(k+\frac{\gamma}{2}\right)}{k!\Gamma\left(\frac{\gamma}{2}\right)}\frac{d}{dt}\frac{(2\lambda t)^k}{(2\lambda t+1)^{\frac{\gamma}{2}+k}} \\
&= 2\lambda\frac{\Gamma\left(k+\frac{\gamma}{2}\right)}{k!\Gamma\left(\frac{\gamma}{2}\right)}\frac{k(2\lambda t)^{k-1}\,(2\lambda t+1)^{\frac{\gamma}{2}+k} - \left(\frac{\gamma}{2}+k\right)(2\lambda t)^k\,(2\lambda t+1)^{\frac{\gamma}{2}+k-1}}{(2\lambda t+1)^{\gamma+2k}} \\
&= 2\lambda(2\lambda t)^{k-1}\frac{\Gamma\left(k+\frac{\gamma}{2}\right)}{k!\Gamma\left(\frac{\gamma}{2}\right)}\left[\frac{k}{(2\lambda t+1)^{\frac{\gamma}{2}+k}} - \frac{\left(\frac{\gamma}{2}+k\right)2\lambda t}{(2\lambda t+1)^{\frac{\gamma}{2}+k+1}}\right] \\
&= \frac{k}{t}\,{}_\gamma\widetilde{p}_k(t) - \frac{k+1}{t}\,{}_\gamma\widetilde{p}_{k+1}(t).
\end{aligned}$$

Since the partial derivatives of $\widetilde{G}_\gamma$ are equal to

$$\frac{\partial\widetilde{G}_\gamma}{\partial t}(u,t) = \sum_{k=0}^{\infty} u^k\frac{d}{dt}\,{}_\gamma\widetilde{p}_k(t) \tag{5.13}$$

and

$$\frac{\partial\widetilde{G}_\gamma}{\partial u}(u,t) = \sum_{k=0}^{\infty} ku^{k-1}\,{}_\gamma\widetilde{p}_k(t), \tag{5.14}$$

we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} u^k\frac{d}{dt}\,{}_\gamma\widetilde{p}_k(t) \qquad (5.15)\\
&= -\frac{1-u}{t}\sum_{k=0}^{\infty} ku^{k-1}\,{}_\gamma\widetilde{p}_k(t) \\
&= -\frac{1}{t}\sum_{k=0}^{\infty} ku^{k-1}\,{}_\gamma\widetilde{p}_k(t) + \frac{1}{t}\sum_{k=0}^{\infty} ku^k\,{}_\gamma\widetilde{p}_k(t) \\
&= -\frac{1}{t}\sum_{k=1}^{\infty} ku^{k-1}\,{}_\gamma\widetilde{p}_k(t) + \frac{1}{t}\sum_{k=0}^{\infty} ku^k\,{}_\gamma\widetilde{p}_k(t) \\
&= [\text{ for } k-1=l \text{ in the first sum}] \\
&= -\frac{1}{t}\sum_{l=0}^{\infty}(l+1)u^l\,{}_\gamma\widetilde{p}_{l+1}(t) + \frac{1}{t}\sum_{k=0}^{\infty} ku^k\,{}_\gamma\widetilde{p}_k(t),
\end{aligned}$$

which coincides with (5.12). ■

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