ON THE SOLUTIONS OF LINEAR ODD-ORDER HEAT-TYPE EQUATIONS WITH RANDOM INITIAL CONDITIONS

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Abstract. In this paper odd-order heat-type equations with different random initial conditions are examined. In particular, we give rigorous conditions for the existence of the solutions in the case where the initial condition is represented by a strictly φ-subGaussian harmonized process $\eta = \eta(x)$. Also the case where η is represented by a stochastic integral with respect to a process with independent increment is studied.

1. Introduction

Third-order heat-type equations have been considered either as linear approximations of the Korteweg-de Vries equation (see [4]) or in connection with certain chemical reactions ([7], p.299). By means of the solutions of these equations some pseudoprocesses have been constructed and some of the related relevant functionals (sojourn time and maximum) have been investigated by means of extensions of the Feynman-Kac functional in [19]. In [3] the case where the pseudoprocess is constrained to be zero at the end of the time interval is considered; the distribution of the maximum is then obtained under these circumstances. In the unconditional case, the joint distribution of the maximum and of the process for this higher-order diffusion is presented in [2].

Odd-order heat-type equations of the form

$$\frac{\partial u}{\partial t} = c_n \frac{\partial^{2n+1} u}{\partial x^{2n+1}} , \qquad n = 1, 2, ... \tag{1.1}$$

(where $c_n = \pm 1$), subject to the initial condition $u(x, 0) = \delta(x)$, have also been examined by many authors: in [9] the Laplace transforms of the sojourn times have been obtained while their inverse, and thus the explicit distributions, have been derived by Lachal [16].

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In [1] the analysis of the local time in zero of the pseudoprocesses related to 1.1 is performed and the connection of its distribution with a fractional diffusion equation is established and discussed

While in all the investigations mentioned above the key tool is the Feynman-Kac functional, the approach of Lachal [17] is somewhat different and consists in some approximation of the underlying pseudoprocesses by means of generalized random walks and the application of a generalization of the Spitzer identity.

The idea of studying equations of the form 1.1 subject to random initial conditions (represented by stationary processes) is presented in [4]. In the spirit of the last work we analyze here more general odd-order equations of the following form

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u}{\partial x^{2k+1}}, \qquad N = 1, 2, ..., \tag{1.2}$$

subject to the random condition

$$u(0, x) = \eta(x), \tag{1.3}$$

where

$$\eta(x) = \int_R e^{iux} dy(u)$$

and y is a complex-valued process. We remark that, in the special case where η is a stationary process, y is a white noise. We present the exact expression for the solution of the problem (1.2)-(1.3) and formulate rigorous conditions on the initial data which guarantee that the process representing the solution satisfies the equation with probability one (Section 3).

We concentrate our attention, in particular, to the case where the initial condition is represented by a strictly φ-subGaussian harmonized process. The general conditions of Section 3 are reduced to a more convenient and tractable form (see our main result in Section 6). We consider also the problem where the initial data is represented by a stochastic integral with respect to a process with independent increments (Section 8).

Many random processes relevant for applications (as numerous recent studies confirm) display a non-Gaussian behaviour, possess heavy tails and have non-symmetric densities. However, some of these processes can be considered as φ-subGaussian because they display the corresponding properties. φ-subGaussian random variables and processes, which are generalizations of sub-Gaussian and Gaussian random variables and processes, were introduced in the papers [11],[13]. The theory of φ-subGaussian random variables and processes is presented in the book [5]. In the paper [8] a more general definition of φ-subGaussian random variables is presented.

In order to make the paper self-contained, a certain digression on sub-Gaussian and φ-subGaussian processes is presented in Sections 4 and 5, as well as some auxiliary results in Sections 7, 8 needed to treat the case of initial condition represented by stochastic integrals with respect to processes with independent increments.

Note that our study can be specialized in order to include the case of Gaussian initial conditions.

2. Harmonized random processes

We now present the definitions of integrals in the mean square sense and also of the harmonized random processes (see, for example, Loeve [18]).

Let $y = \{y(t), t \in I\}$ be a complex-valued, centered random process of second order (that is $E|y(t)|^2 < \infty,\ t \in I$), $I = [a, b]$ a finite or infinite interval and $\Gamma_y(t, s) = Ey(t)\overline{y(s)}$ the covariance function of $y(t)$.

Definition 2.1. ([18]) Let D and D' be the following partitions of the interval $[a, b]$:

$D = \{t_j, j = 1, ..., n+1 : a = t_1 < t_2 < \ldots < t_{n+1} = b\}$;

$D' = \left\{t'_j, j = 1, ..., m+1 : a = t'_1 < t'_2 < \ldots < t'_{m+1} = b\right\}$.

Let also

$\Delta\Delta'\Gamma_y(t_k, t'_k) = \Gamma_y\left(t_{k+1}, t'_{k+1}\right) - \Gamma_y(t_{k+1}, t'_k) - \Gamma_y\left(t_k, t'_{k+1}\right) + \Gamma_y(t_k, t'_k)$.

The covariance function $\Gamma_y(t, s)$ has finite variation on the finite interval $I = [a, b]$ if there exists a number $0 < C_I < \infty$ such that, for all D and D', the following inequality holds

$$\sum_{t\in D}\sum_{t\in D'} |\Delta\Delta'\Gamma_y(t, t')| < C_I.$$

The covariance function $\Gamma_y(t, s)$ has finite variation on the infinite interval I if there exists a number $C < \infty$ such that $C_{I'} < C$ for all finite I' such that $I' \subset I$.

Definition 2.2. ([18]) Let $f = \{f(t), t \in I\}$ be a measurable function (where $I = [a, b]$ is a finite interval), $y = \{y(t), t \in I\}$ a centered second-order random process and $\Gamma_y(t, s) = Ey(t)\overline{y(s)}$ the covariance function of y. The integral $\int_I f(t)\, dy(t)$ is defined as the mean square limit of the Riemann sums $\sum_k f(t'_k)(y(t_{k+1}) - y(t_k))$, $t_k \le t'_k \le t_{k+1}$.

The integral $\int_R f(t)\, dy(t)$ is defined as the mean square limit of the integrals $\int_{-a}^{b} f(t)\, dy(t)$ as $a \to \infty, b \to \infty$.

The integral $\int_I f(t)\, dy(t)$ exists iff the integral $\int_I \int_I f(t) f(s)\, d\Gamma_y(t,s)$ exists.

Definition 2.3.([18]) The second-order random function $X = \{X(t), t \in R\}$ is called harmonized if there exists a second-order random function $y = \{y(t), t \in R\}$ such that the covariance $\Gamma_y(t,s) = Ey(t)\overline{y(s)}$ has finite variation and $X(t) = \int_R e^{itu} dy(u)$.

Theorem 2.1.([18]) *The second-order random function* $X = \{X(t), t \in R\}$ *is harmonized iff there exists a covariance function* $\Gamma_y(t,s)$ *with finite variation such that*

$$\Gamma_x(u,v) = EX(u)\overline{X(v)} = \int_R \int_R e^{i(tu-t'v)} d\Gamma_y(t,t').$$

Example. Let $X = \{X(t), t \in R\}$ be a second-order centered stationary random process and let its covariance function $B(\tau) = EX(t+\tau)\overline{X(t)}$ be mean square continuous. In this case $B(\tau) = \int_R e^{iu\tau} dF(u)$, where $F(u)$ is a non-decreasing left continuous function such that $F(-\infty) = 0$, $F(+\infty) = B(0)$ and $X(t) = \int_R e^{itu} dy(u)$, where $y = \{y(t), t \in R\}$ is a second-order random process with uncorrelated increments such that $E|y(t) - y(s)|^2 = F(t) - F(s)$ as $t > s$.

3. A general theorem on the solution of odd-order heat-type equations

Let us consider the linear equation

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t,x)}{\partial x^{2k+1}} = \frac{\partial u(t,x)}{\partial t}, \quad t > 0,\ x \in R^1 \tag{3.1}$$

subject to the random initial condition

$$u(0,x) = \eta(x),\ x \in R^1, \tag{3.2}$$

where $a_k, k = 1, \ldots, N$ are some constants.

Let $\eta(x)$, $x \in R^1$, be a harmonized process

$$\eta(x) = \int_R e^{iux} dy(u),$$

where

$$Ey(t)\overline{y(s)} = \Gamma_y(t,s), \tag{3.3}$$

with covariance function

$$\Gamma_\eta(x,x') = E\eta(x)\overline{\eta(x')} = \int_R\int_R e^{i(xu-x'v)}d\Gamma_y(u,v). \tag{3.4}$$

Theorem 3.1. *Let*

$$I(t,x,\lambda) = \exp\left\{i\left(\lambda x + t\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right)\right\}, \tag{3.5}$$

and

$$U(t,x) = \int_R I(t,x,\lambda)\,dy(\lambda). \tag{3.6}$$

If the following integrals exists

$$\int_R \lambda^s I(t,x,\lambda)\,dy(\lambda),\ \ s = 0,1,2,\ldots,2N+1, \tag{3.7}$$

and if there is a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that for all $A > 0$ and $T > 0$ the sequence of the related integrals $\int_{-a_n}^{a_n} \lambda^s I(t,x,\lambda)\,dy(\lambda)$ converges in probability, uniformly for $|x| \le A$, $0 \le t \le T$, then $U(t,x)$ is the classical solution to the problem (3.1)-(3.2) (that is $U(t,x)$ satisfies equation (3.1) with probability one and $U(0,x) = \eta(x)$).

Proof. Since $\int_{-a_n}^{a_n} \lambda^s I(t,x,\lambda)\,dy(\lambda)$ converges in probability uniformly for $|x| \le A$, $0 \le t \le T$, then there exists a subsequence $b_n > 0$, $b_n \to \infty$ as $n \to \infty$, such that $\int_{-b_n}^{b_n} \lambda^s I(t,x,\lambda)\,dy(\lambda)$ converges with probability one to $\int_R \lambda^s I(t,x,\lambda)\,dy(\lambda)$, uniformly for $|x| \le A$, $0 \le t \le T$.

Let

$$U_{b_n}(t,x) = \int_{-b_n}^{b_n} I(t,x,\lambda)\,dy(\lambda). \tag{3.8}$$

It is self-evident that

$$\frac{\partial^s U_{b_n}(t,x)}{\partial x^s} = \int_{-b_n}^{b_n} (i\lambda)^s I(t,x,\lambda)\,dy(\lambda),\ s = 0,1,2,\ldots,2N{+}1, \tag{3.9}$$

and

$$\frac{\partial U_{b_n}(t,x)}{\partial t} = \int_{-b_n}^{b_n}\left(i\sum_{k=1}^{N} a_k\lambda^{2k+1}(-1)^k\right) I(t,x,\lambda)\,dy(\lambda), \tag{3.10}$$

for $s = 0, 1, 2, \ldots, 2N+1$. It follows from (3.9) and (3.10) that

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} U_{b_n}(t,x)}{\partial x^{2k+1}} = \frac{\partial U_{b_n}(t,x)}{\partial t}, \quad t > 0, \quad x \in R^1, \tag{3.11}$$

since

$$\frac{\partial^{2k+1} U_{b_n}(t,x)}{\partial x^{2k+1}} \text{ converges to } \frac{\partial^{2k+1} U(t,x)}{\partial x^{2k+1}}$$

and

$$\frac{\partial U_{b_n}(t,x)}{\partial t} \text{ converges to } \frac{\partial U(t,x)}{\partial t}$$

uniformly for $|x| \le A$, $0 \le t \le T$ with probability one. Therefore $U(t,x)$ satisfies equation (3.1) and $U(0,x) = \int_R e^{i\lambda x} dy(x) = \eta(x)$.□

Remark 3.1. The integrals $\int_R \lambda^s I(t,x,\lambda)\, dy(\lambda)$ exist if the twofold integrals $\int_R \int_R \lambda^s \mu^s I(t,x,\lambda) I(t,x,\mu)\, d\Gamma_y(\lambda,\mu)$ exist or otherwise if $\int_R \int_R |\lambda|^s |\mu|^s d\Gamma_y(\lambda,\mu) < \infty$.

On the other side all the integrals $\int_R \lambda^s I(t,x,\lambda)\, dy(\lambda)$, $s = 0, 1, 2, \ldots,$ $2N+1$, exist if

$$\int_R \int_R |\lambda|^{2N+1} |\mu|^{2N+1} I(t,x,\lambda) I(t,x,\mu)\, d\Gamma_y(\lambda,\mu) < \infty.$$

Remark 3.2. Under the conditions of Theorem 3.1 we can write down the expression for the covariance function of the random field $U(t,x)$ given by formula (3.6):

$$cov\,(U(t,x), U(s,y)) = \int_R \int_R I(t,x,\lambda)\, \overline{I(s,y,\mu)} d\Gamma_y(\lambda,\mu).$$

In particular, in the case where the process $\eta(x)$ representing the initial condition is centered and stationary with a spectral function $F(\lambda)$, we have that

$$\begin{aligned} cov\,(U(t,x), U(s,y)) &= \int_R e^{i\left(\lambda(x-y)+(t-s)\sum_{k=0}^{N} a_k \lambda^{2k+1}(-1)^k\right)} dF(\lambda) \\ &= \int_R I(t-s, x-y, \lambda)\, dF(\lambda) \end{aligned}$$

and thus the solution $U(t,x)$ is stationary in space and time.

4. φ-subGaussian random variables and processes

Definition 4.1. ([14]) Let $\varphi = \{\varphi(x), x \in R\}$ be a continuous even convex function. The function φ is an Orlicz $N-$function if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and the following conditions hold:

$$\lim_{x \to 0} \frac{\varphi(x)}{x} = 0, \quad \lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty.$$

Definition 4.2. ([14]) Let $\varphi = \{\varphi(x), x \in R\}$ be an $N-$function. The function φ^* defined by

$$\varphi^*(x) = \sup_{y \in R} (xy - \varphi(y))$$

is called the Young-Fenchel transform of φ.

Remark 4.1. ([14]) The Young-Fenchel transform of an $N-$function is again an $N-$function and the following inequality holds (Young-Fenchel inequality)

$$xy \leq \varphi(x) + \varphi^*(y) \quad \text{as } x > 0,\ y > 0. \tag{4.1}$$

Condition Q. Let φ be an $N-$function which satisfies

$$\liminf_{x \to 0} \frac{\varphi(x)}{x^2} = C > 0.$$

It may happen that $C = \infty$.

Example 4.1. The functions

$$\varphi(x) = c|x|^{\alpha},\ c > 0,\ 1 < \alpha \leq 2;$$

$$\varphi(x) = \begin{cases} \alpha^{-1}|x|^2, & |x| \leq 1 \\ \alpha^{-1}|x|^{\alpha}, & |x| > 1 \end{cases},\ \alpha > 2$$

are $N-$functions which satisfy the Condition Q.

Definition 4.3. ([8]) Let φ be an $N-$function satisfying Condition Q and $\{\Omega, B, P\}$ be a standard probability space. The random variable ξ belongs to the space $Sub_{\varphi}(\Omega)$, if $E\xi = 0$, $E\exp\{\lambda\xi\}$ exists for all $\lambda \in R$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in R$

$$E\exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda a)\}. \tag{4.2}$$

The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm ([13])

$$\tau_\varphi(\xi) = \sup_{\lambda \neq 0} \frac{\varphi^{(-1)}\left(\ln E \exp\{\lambda\xi\}\right)}{|\lambda|},$$

where $\varphi^{(-1)}$ denotes the inverse function of φ.

Examples of $\varphi-$subGaussian random variables can be found in the paper [8] and in the book [5]. In particular, all bounded centered random variables belong to all $Sub_\varphi(\Omega)$. Some random variables having a centered Weibull distribution belong to a certain space $Sub_\varphi(\Omega)$. The Normal centered random variable $\xi = N(0, \sigma^2)$ belongs to the space $Sub_\varphi(\Omega)$, with $\varphi(x) = \frac{x^2}{2}$, $\tau^2(\xi) = \sigma^2$.

Definition 4.4. ([14]) A family Δ of random variables $\xi \in Sub_\varphi(\Omega)$ is called strictly $\varphi-$subGaussian if there exists a constant C_Δ such that $\forall$ finite set I of random variables $\xi_i \in \Delta$ the following inequality holds

$$\tau_\varphi\left(\sum_{i \in I} \lambda_i \xi_i\right) \leq C_\Delta \left| E\left(\sum_{i \in I} \lambda_i \xi_i\right)^2 \right|^{1/2}. \tag{4.3}$$

The constant C_Δ is called the determining constant of the family Δ.

Lemma 4.1. ([14]) *The linear closure of a strictly $\varphi-$subGaussian family Δ in the space $L_2(\Omega)$ is the strictly $\varphi-$subGaussian family with the same determining constant.*

Definition 4.5. The random process $\xi = \{\xi(t), t \in T\}$ is called $\varphi-$subGaussian if all random variables $\xi(t), t \in T$, are $\varphi-$subGaussian and $\sup_{t \in T} \tau_\varphi(\xi(t)) < \infty$.

The random process $\xi = \{\xi(t), t \in T\}$ is called strictly $\varphi-$subGaussian if the family of random variables $\xi(t), t \in T$, is strictly $\varphi-$subGaussian.

Examples ([14]).

1. Let $\{\xi_k, k = 1, ..., \infty\}$ be a family of strictly $\varphi-$subGaussian random variables with determining constant C. Let $\xi(t) = \sum_{k=1}^{\infty} \xi_k f_k(t)$ be a mean square convergent series for all $t \in T$. Then $\xi(t)$ is a strictly $\varphi-$subGaussian random process with the same determining constant.

2. Let $\xi(t) = \sum_{k=1}^{\infty} \eta_k f_k(t), t \in T$, where η_k are independent random variables, $\eta_k \in Sub_\varphi(\Omega)$, $\varphi(x)$ is an $N-$function such that the function $\psi(x) = \varphi(\sqrt{x})$ is convex. If $\tau_\varphi(\eta_k) \leq R\left(E\eta_k^2\right)^{1/2}$, and the series

$\sum_{k=1}^{\infty} \eta_k f_k(t)$ converges in mean square for all $t \in T$, then $\xi(t)$ is the strictly $\varphi-$subGaussian random process with the determining constant R.

3. A Gaussian centered random process $\xi(t)$ is strictly $\varphi-$subGaussian with $\varphi(x) = \frac{x^2}{2}$ and the determining constant is equal to 1.

Definition 4.6. A harmonized random process

$$\eta(x) = \int_R e^{iux} dy(u)$$

is a strictly $\varphi-$subGaussian harmonized random process if the process y is strictly $\varphi-$subGaussian.

Remark 4.1. It follows from Lemma 4.1 that in this case the process η and all the processes

$$\eta_{a,b}(x) = \int_a^b e^{iux} dy(u)$$

are strictly $\varphi-$subGaussian.

5. The conditions of convergence in probability in $C(T)$ of a sequence of φ-subGaussian random processes

Let (T, d) be a compact metric space and $C(T)$ is the Banach space of continuous functions with uniform norm. Let $X_k = \{X_k(t), t \in T\}$ be a sequence of $\varphi-$subGaussian random processes such that $X_k \in C(T)$. The general conditions of convergence in probability of X_k in the space $C(T)$ are presented in the book [5]. In the paper [15] these conditions are presented for the case where T is a finite-dimensional space.

Theorem 5.1. ([15]) *Let R^k be a k-dimensional space, $d(t,s) = \max_{1 \le i \le k} |t_i - s_i|$, $T = \{0 \le t_i \le T_i,\ i = 1, 2, ..., k\}$, $T_i > 0$; $X_n = \{X_n(t), t \in T\}$ be a sequence of $\varphi-$subGaussian random processes such that $X_n \in C(T)$. Let us assume also that there exists a continuous increasing function $\sigma = \{\sigma(h), h > 0\}$, $\sigma(h) \to 0$ as $h \to 0$, such that*

$$\sup_{d(t,s) \le h} \tau_\varphi (X_n(t) - X_n(s)) \le \sigma(h) \tag{5.1}$$

and

$$\int_{0+}\Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right)d\varepsilon<\infty, \tag{5.2}$$

where $\Psi(u)=\frac{u}{\varphi^{(-1)}(u)}$, $\sigma^{(-1)}(u)$ *is the inverse function of* $\sigma(u)$, $\varphi^{(-1)}(u)$ *is the inverse function of* $\varphi(u)$, *for* $u>0$, *and* $\int_{0+}f(\varepsilon)\,d\varepsilon$ *denotes* $\int_0^\delta f(\varepsilon)\,d\varepsilon$ *for sufficiently small* $\delta>0$. *If the sequence of processes* $X_n(t)$, $n\geq 1$, *converges in probability to* $X(t)$ *for all* $t\in T$, *then* $X_n(t)$ *converges in probability to* $X(t)$ *in the space* $C(T)$.

6. The main result

Lemma 6.1.([14]) *Let* $\theta(u)$, $u\geq u_0\geq 0$, *be a continuous, increasing function such that* $\theta(u)>0$ *and the function* $\frac{u}{\theta(u)}$ *is non-decreasing for* $u>u_0$, *where* $u_0\geq 0$ *is a constant. Then for all* $u,v\neq 0$

$$\left|\sin\frac{u}{v}\right|\leq\frac{\theta(|u|+u_0)}{\theta(|v|+u_0)}. \tag{6.1}$$

Example 6.1. The functions $\theta(u)=u^\alpha$, $u\geq 0$, $0<\alpha\leq 1$ $(u_0=0)$ and $\theta(u)=(\ln u)^\alpha$, $\alpha>0$, $u>u_0\geq e^\alpha$, satisfy the conditions of Lemma 6.1.

Assumption Ψ. Let φ be an N-function satisfying the condition Q; $\Psi(u)=\frac{u}{\varphi^{(-1)}(u)}$, where $\varphi^{(-1)}(u)$ is the inverse function of $\varphi(u)$. Let the function $\theta(u)$, $u>u_0$, satisfy the condition of Lemma 6.1. We say that the function $\theta(u)$, $u\geq u_0\geq 0$, satisfies the assumption Ψ if the following integral converges

$$\int_{0+}\Psi\left(\ln\left(\theta^{(-1)}\left(\varepsilon^{-1}\right)\right)\right)d\varepsilon<\infty, \tag{6.2}$$

where $\int_{0+}f(\varepsilon)\,d\varepsilon$ denotes the integral $\int_0^\delta f(\varepsilon)\,d\varepsilon$ for sufficiently small $\delta>0$.

Example 6.2. Let, for sufficiently large x, $\varphi(x)=\frac{|x|^p}{p}$, $p>1$ and $\theta(x)=(\ln x)^\alpha$, $x>e^\alpha$. Then $\theta(x)$ satisfies the assumption Ψ if $\alpha>1-\frac{1}{p}$. Indeed

$$\begin{aligned}\int_{0+}\Psi\left(\ln\left(\theta^{(-1)}\left(\varepsilon^{-1}\right)\right)\right)d\varepsilon &= \int_{0+}\Psi\left(\varepsilon^{-1/\alpha}\right)d\varepsilon\\ &= \frac{1}{p^{1/p}}\int_{0+}\varepsilon^{-\frac{1}{\alpha}\left(1-\frac{1}{p}\right)}d\varepsilon<\infty.\end{aligned}$$

The function $\theta(x)=x^\alpha$, $\alpha>0$, $x>0$ also satisfies the assumption Ψ.

Theorem 6.1. *Let us consider the linear equation (3.1)*

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t,x)}{\partial x^{2k+1}} = \frac{\partial u(t,x)}{\partial t}, \quad t>0, \quad x \in R^1$$

subject to the random initial condition

$$u(0,x) = \eta(x),\ x \in R^1.$$

Let $\eta(x)$ be the harmonized process defined above, which is a strictly φ-subGaussian random process. Let $\theta(x)$, $x > u_0$ be a function satisfying the assumption Ψ. Let us assume that the following integral converges

(6.3)

$$\int_R \int_R |\lambda|^{2N+1} |\mu|^{2N+1} \theta\left(u_0 + |\lambda|^{2N+1}\right) \theta\left(u_0 + |\mu|^{2N+1}\right) d\Gamma_y(\lambda,\mu) < \infty.$$

Then

$$U(t,x) = \int_R I(t,x,\lambda)\, dy(\lambda),$$

where

$$I(t,x,\lambda) = \exp\left\{ i\left(\lambda x + t\sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \right\}$$

is the classical solution of the problem (3.1)-(3.2).

Proof. It follows from Theorem 3.1 that it is sufficient to prove that there exists a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that the sequences $U_{n,s}(t,x) = \int_{-a_n}^{a_n} \lambda^s I(t,x,\lambda)\, dy(\lambda)$, $s = 0,1,2,...,2N+1$ converge uniformly in probability for $|x| \le A$, $0 \le t \le T$, where $A > 0$ and $T > 0$ are some constants. Since the random processes $U_{n,s}(t,x)$ are strictly subGaussian, then

(6.4)

$$\begin{aligned}
& \tau_\varphi^2 \left(U_{n,s}(t,x) - U_{n,s}(t_1,x_1)\right) \\
\le\ & C_\xi E\left|U_{n,s}(t,x) - U_{n,s}(t_1,x_1)\right|^2 \\
=\ & C_\xi \int_{-a_n}^{a_n}\int_{-a_n}^{a_n} \lambda^s \mu^s \left(I(t,x,\lambda) - I(t_1,x_1,\lambda)\right) \\
& \times \left(I(t,x,\mu) - I(t_1,x_1,\mu)\right) d\Gamma_y(\lambda,\mu) \\
\le\ & C_\xi \int_R\int_R |\lambda|^s |\mu|^s \left|I(t,x,\lambda) - I(t_1,x_1,\lambda)\right| \\
& \times \left|I(t,x,\mu) - I(t_1,x_1,\mu)\right| d\left|\Gamma_y(\lambda,\mu)\right|,
\end{aligned}$$

where C_ξ is the determining constant of the family $\{\xi(t), t \in T\}$. It is evident that

$$
\begin{aligned}
&(6.5)\\
&|I(t,x,\lambda) - I(t_1,x_1,\lambda)|\\
= \ & [(\cos(\lambda x + t\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k) - \cos(\lambda x_1 + t_1 \sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k))^2\\
& + (\sin(\lambda x + t\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k) - \sin(\lambda x_1 + t_1 \sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k))^2]^{1/2}\\
= \ & 2\left|\sin\frac{1}{2}(\lambda(x-x_1) + (t-t_1)\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k)\right|\\
\leq \ & 2\left(\left|\sin\frac{x-x_1}{2}\lambda\right| + \left|\sin\frac{t-t_1}{2}\left(\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k\right)\right|\right).
\end{aligned}
$$

It follows from (6.1) that

$$
\begin{aligned}
(6.6) \quad & |I(t,x,\lambda) - I(t_1,x_1,\lambda)|\\
\leq \ & 2\left(\theta\left(u_0 + \frac{|\lambda|}{2}\right)\left(\theta\left(u_0 + \frac{1}{|x-x_1|}\right)\right)^{-1}\right.\\
& \left. + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k\right|\right)\left(\theta\left(u_0 + \frac{1}{|t-t_1|}\right)\right)^{-1}\right),
\end{aligned}
$$

where the function $\theta(u)$ satisfies the assumption Ψ. Now it follows from (6.4.) and (6.6) that

$$
(6.7) \qquad \sup_{\substack{|t-t_1|\leq h\\ |x-x_1|\leq h}} \tau_\varphi(U_{n,s}(t,x) - U_{n,s}(t_1,x_1)) \leq \frac{C_s}{\theta\left(u_0 + \frac{1}{h}\right)},
$$

where

$$
\begin{aligned}
& C_s\\
= \ & 2C_\xi \int_R \int_R |\lambda|^s |\mu|^s \left|\theta\left(u_0 + \frac{|\lambda|}{2}\right) + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k\right|\right)\right|\\
& \times \left|\theta\left(u_0 + \frac{|\mu|}{2}\right) + \theta\left(u_0 + \frac{1}{2}\left|\sum_{k=1}^{N} a_k \mu^{2k+1}(-1)^k\right|\right)\right| d\Gamma_y(\lambda,\mu).
\end{aligned}
$$

It is evident that the last integrals converge since the integral (6.3) converges.

Now the theorem follows from Theorem 5.1 since

$$\sigma(h) = \frac{C_s}{\theta\left(u_0 + \frac{1}{h}\right)} \quad \text{and} \quad \sigma^{(-1)}(\varepsilon) = \frac{1}{\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right) - u_0}, \quad 0 < \varepsilon < \frac{\theta(u_0)}{C_s}$$

that is

$$\begin{aligned} \int_{0+} \Psi\left(\ln\left(\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right) - u_0\right)\right) d\varepsilon &< \int_{0+} \Psi\left(\ln\left(\theta^{(-1)}\left(\frac{C_s}{\varepsilon}\right)\right)\right) d\varepsilon \\ &= C_s \int_{0+} \Psi\left(\ln\left(\theta^{(-1)}\left(\frac{1}{\varepsilon}\right)\right)\right) d\varepsilon \\ &< \infty. \end{aligned}$$

Corollary 6.1. Let $\varphi(x) = \frac{|x|^p}{p}$, $p > 1$ for sufficiently large x. Then the statement of Theorem 6.1 holds if the following integral converges

$$\int_R \int_R |\lambda\mu|^{2N+1} \left(\ln(1+\lambda)\ln(1+\mu)\right)^{\alpha} d\Gamma_y(\lambda,\mu), \tag{6.8}$$

where α is a constant such that $\alpha > 1 - \frac{1}{p}$.

Proof. It follows from Example 6.2 that in this case the function $\theta(x) = (\ln x)^{\alpha}$, where $\alpha > 1 - \frac{1}{p}$, satisfies the assumption Ψ. Therefore the assertion of Theorem 6.1 holds if the following integral converges

$$\int_R \int_R |\lambda\mu|^{2N+1} \left(\ln\left(e^{\alpha} + |\lambda|^{2N+1}\right) \ln\left(e^{\alpha} + |\mu|^{2N+1}\right)\right)^{\alpha} d\Gamma_y(\lambda,\mu) < \infty.$$

But this integral converges if the integral (6.8) converges.

Remark 6.1. If in Theorem 6.1 the process $\eta(x)$ is a strictly φ-subGaussian stationary random process that is

$$\eta(x) = \int_R e^{iux} d\xi(u),$$

where $\xi(u)$ is a centered process with uncorrelated increments ($E\eta(x+\tau)\overline{\eta(x)} = \int_R e^{i\tau\lambda} dF(\lambda)$), then the assumptions (6.3) and (6.8) are of the form

$$\int_R |\lambda|^{4N+2} \theta^2\left(u_0 + |\lambda|^{2N+1}\right) dF(\lambda) < \infty,$$

$$\int_R |\lambda|^{4N+2} \left(\ln(1+\lambda)\right)^{2\alpha} dF(\lambda) < \infty,$$

if $\alpha > 1 - \frac{1}{p}$.

Corollary 6.2. Let us assume that $\eta(x)$ (representing the initial condition) is a Gaussian process. Then $\eta(x)$ is a strictly φ-subGaussian random process, where $\varphi(x) = \frac{x^2}{2}$ and $\theta(u) = (\ln u)^{\alpha}$, where $\alpha > \frac{1}{2}$.

7. Uniform convergence of random series with independent terms

Lemma 7.1. *Let $\{\xi_k, k = 1, 2, ...\}$ be a sequence of centered independent random variables such that $E|\xi_k|^2 = 1$. Let T be a bounded interval on R and let $f_k(t)$, $k \geq 1$ be a sequence of continuous functions on T such that*

$$\sum_{k=1}^{\infty} f_k^2(t) < \infty, \qquad t \in T. \tag{7.1}$$

Assume that one can find a continuous function $\sigma(h)$, $h > 0$, such that $\sigma(h)$ is increasing, $\sigma(0) = 0$, and for all sufficiently small $\varepsilon > 0$

$$\int_0^{\varepsilon} \left|\ln \sigma^{(-1)}(v)\right|^{1/2} dv < \infty \tag{7.2}$$

($\sigma^{(-1)}(v)$ denotes the inverse function for $\sigma(v)$) and the following inequalities hold

$$\sup_{\substack{t,s\in T\\ |t-s|\leq h}} |f_k(t) - f_k(s)| \leq b_k \sigma(h), \tag{7.3}$$

$$\sum_{k=1}^{\infty} b_k^2 < \infty. \tag{7.4}$$

Then the series $\sum_{k=1}^{\infty} \xi_k f_k(t)$ converges uniformly for $t \in T$ with probability one.

Proof. This theorem is a modification of Theorem 3.5.5 of the book by Buldygin and Kozachenko [5]. Consider the random pseudometric, on the space T, $\Psi(t,s) = \left(\sum_{k=1}^{\infty} \xi_k^2 |f_k(t) - f_k(s)|^2\right)^{1/2}$. Let $H_\Psi(\varepsilon) = \ln(N_\Psi(\varepsilon))$, where $N_\Psi(\varepsilon)$ is the smallest number of elements of an ε-covering of the space $(T, \Psi(t,s))$. In the proof of Theorem 3.5.5 in [5] Buldygin and Kozachenko proved the following assertion:

The series $\sum_{k=1}^{\infty} \xi_k f_k(t)$ converges uniformly for $t \in T$ with probability one if with probability one

$$\int_0^{\varepsilon} |H_\Psi(v)|^{1/2} dv < \infty \tag{7.5}$$

for any sufficiently small $\varepsilon > 0$.

We now prove that (7.5) holds. It follows from the assumption (7.3) that

$$\sup_{\substack{t,s\in T\\ |t-s|\leq h}} \Psi(t,s) \leq \left(\sum_{k=1}^{\infty} \xi_k^2 b_k^2\right)^{1/2} \sigma(h) = \eta^{1/2}\sigma(h).$$

The series $\sum_{k=1}^{\infty} \xi_k^2 b_k^2 = \eta$ converges with probability one since $\sum_{k=1}^{\infty} E\xi_k^2 b_k^2 = \sum_{k=1}^{\infty} b_k^2 < \infty$. By consulting the book [5] we can see that

$$N_\Psi(u) \leq \frac{|T|}{2\sigma^{(-1)}(\frac{u}{\eta})} + 1,$$

where $|T|$ is the length of T. Therefore for sufficiently small $\varepsilon > 0$

$$\begin{aligned} (7.6)\quad \int_0^{\varepsilon} |\ln(N_\Psi(u))|^{1/2} du &\leq \int_0^{\varepsilon} \left|\ln\left(\frac{|T|}{2\sigma^{(-1)}(\frac{u}{\eta})} + 1\right)\right|^{1/2} du \\ &= \int_0^{\varepsilon/\eta} \left|\ln\left(\frac{|T|}{2\sigma^{(-1)}(v)} + 1\right)\right|^{1/2} \eta dv \\ &\leq \sqrt{2}\int_0^{\varepsilon} \left|\ln\sigma^{(-1)}(v)\right|^{1/2} dv \end{aligned}$$

because

$$\begin{aligned} \ln\left(\frac{T}{2\sigma^{(-1)}(v)} + 1\right) &\leq \ln\left(\frac{T}{\sigma^{(-1)}(v)}\right) \leq \ln T + \left|\ln\sigma^{(-1)}(v)\right| \\ &\leq 2\left|\ln\sigma^{(-1)}(v)\right|, \end{aligned}$$

for sufficiently small v. Therefore the integral (7.6) converges with probability one if $\int_0^{\varepsilon} \left|\ln\sigma^{(-1)}(v)\right|^{1/2} dv < \infty$. □

8. Stochastic integrals with respect to processes with independent increments

Let $\xi(\lambda)$, $\lambda \in R$, be a random process with independent increments such that $E\xi(\lambda) = 0$, $E|\xi(\lambda)|^2 < \infty$. Let $F(\lambda)$ be the spectral function of this process, that is $E|\xi(\lambda_2) - \xi(\lambda_1)|^2 = F(\lambda_2) - F(\lambda_1)$ if $\lambda_2 > \lambda_1$, $F(-\infty) = 0$, $F(+\infty) = 1$.

Let $f(\lambda)$, $\lambda \in R$, be a function which possesses continuous derivative $f'(\lambda)$. We suppose that $\int_{-\infty}^{\infty} f(\lambda)\, d\xi(\lambda)$ exists, that is $\int_{-\infty}^{\infty} |f(\lambda)|^2 dF(\lambda) < \infty$.

There exists a lot of stochastically equivalent modifications of the process $\xi(\lambda)$. J.L.Doob [6] proved that there exists a modification of the process $\xi(\lambda)$ such that, with probability one, the sample paths of $\xi(\lambda)$ are measurable, bounded on any interval $[a, b]$, right continuous and have only a countable set of discontinuities. It is also assumed that the process $\xi(\lambda)$ possesses limits for $\lambda \to \pm\infty$.

In the sequel we shall consider such a version of $\xi(\lambda)$, for which the Riemann integral $\int_a^b f'(\lambda)\xi(\lambda)\,d\lambda$ exists and coincides with the Lebesgue integral. Define the following integral by means of the equality

$$\int_a^b f(\lambda)\,d\xi(\lambda) = f(b)\,\xi(b) - f(a)\,\xi(a) - \int_a^b \xi(\lambda)\,f'(\lambda)\,d\lambda.$$

Such integrals, in some particular cases, were introduced by Hunt [10] and in a more general case were considered in the paper [12]. It is easy to show that these integrals coincide with integrals of the form $\int_a^b f(\lambda)\,d\xi(\lambda)$ in the mean square sense, with probability one (see also the paper [12]).

Define $\int_{-\infty}^{\infty} f(\lambda)\,d\xi(\lambda)$ as the limit with probability one of the integrals $\int_a^b f(\lambda)\,d\xi(\lambda)$ as $a \to -\infty$, $b \to \infty$ (if this limit exists).

Theorem 8.1. *Let $g(t,\lambda)$ be a continuous function for $t \in T$, $\lambda \in R$, and let us assume also that $g'_\lambda(t,\lambda)$ exists and is continuous. Let $\xi(\lambda)$, $\lambda \in R$, be a centered random process with independent increments, $F(\lambda)$ be the spectral function of $\xi(\lambda)$. Let the following assumptions hold*

$$\int_{-\infty}^{\infty} A^2(|\lambda|)\,dF(\lambda) < \infty, \tag{8.1}$$

where

$$A(\lambda) = \max_{\substack{|u|\le\lambda \\ t\in T}} |g(t,u)|,$$

$$\sup_{|t-s|\le h} |g(t,\lambda) - g(s,\lambda)| \le Z(|\lambda|)\,\sigma(h), \tag{8.2}$$

where $Z(|\lambda|)$ is a monotonously increasing function such that $\int_{-\infty}^{\infty} Z^2(|\lambda|)\,dF(\lambda) < \infty$, and $\sigma(h)$, $h > 0$, is a continuous function such that $\sigma(0) = 0$ and the assumption (7.2) holds for this function.

Then the integral $\int_{-\infty}^{\infty} g(t,\lambda)\,d\xi(\lambda)$ converges uniformly for $t \in T$ with probability one.

Proof. To prove this theorem we use Lemma 7.1 and the method worked out by Hunt [10]. Let us introduce the random process $y_n(u) =$

$\xi\left(\frac{k}{n}\right)$ as $\frac{k}{n} \le u < \frac{k+1}{n}$, and consider the difference of the integrals

$$\begin{aligned} I_m &= \left|\int_m^{m+1} g(t,\lambda)\, d\xi(\lambda) - \int_m^{m+1} g(t,\lambda)\, dy_n(\lambda)\right| \\ &= |g(t,m+1)\,\xi(m+1) - g(t,m+1)\, y_n(m+1) - g(t,m)\,\xi(m) + \\ &\quad + g(t,m)\, y_n(m) - \int_m^{m+1} g'_\lambda(t,\lambda)\,(\xi(\lambda) - y_n(\lambda))\, d\lambda| \\ &\le A(m+1)\,|\xi(m+1) - y_n(m+1)| + A(m)\,|\xi(m) - y_n(m)| + \\ &\quad + B(m) \int_m^{m+1} |\xi(\lambda) - y_n(\lambda)|\, d\lambda, \end{aligned}$$

where

$$B(m) = \max_{\substack{t\in T \\ m\le\lambda\le m+1}} |g'_\lambda(t,\lambda)|.$$

The properties of the process $\xi(\lambda)$ guarantee that $I_m \to 0$ as $m \to \infty$ uniformly for $t \in T$ with probability one.

For any $\varepsilon > 0$, there exists a number $n_{\varepsilon,m}$ such that, with probability larger than $1 - \frac{\varepsilon}{2^{|m|+2}}$, the following inequality holds

$$\left|\int_m^{m+1} g(t,\lambda)\, d\xi(\lambda) - \int_m^{m+1} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda)\right| < \frac{\varepsilon}{2^{|m|+2}}. \tag{8.3}$$

Consider now the random process $y_\varepsilon(\lambda) = y_{n_{\varepsilon,m}}(\lambda)$ as $m \le \lambda \le m+1$. For $A_1 < A_2$ the following inequality holds

$$\begin{aligned} \left|\int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda)\right| &\le \left|\int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda) - \right. \\ &\left. - \int_{A_1}^{A_2} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda)\right| + \left|\int_{A_1}^{A_2} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda)\right|. \end{aligned} \tag{8.4}$$

It follows from (8.3) that

$$\left|\int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda) - \int_{A_1}^{A_2} g(t,\lambda)\, dy_{n_{\varepsilon,m}}(\lambda)\right| \le \varepsilon$$

with probability larger than $1-\varepsilon$. Therefore there exists a sequence ε_k, $\varepsilon_k \to 0$ as $k \to \infty$, such that with probability one uniformly for all A_1, A_2, $t \in T$

$$\int_{A_1}^{A_2} g(t,\lambda)\, dy_{n,\varepsilon_k}(\lambda) \to \int_{A_1}^{A_2} g(t,\lambda)\, d\xi(\lambda)$$

as $\varepsilon_k \to 0$. Therefore the assertion of the theorem holds true if the integral $\int_{-\infty}^{\infty} g(t,\lambda)\, dy_{n,\varepsilon}(\lambda)$ converges uniformly as $t \in T$ for any

$\varepsilon > 0$ with probability one (see inequality (8.4)). Note that $I(t) = \int_{-\infty}^{\infty} g(t,\lambda)\, dy_{n,\varepsilon}(\lambda)$ is the random series of the form

$I(t) = \sum_{s=-\infty}^{\infty} g(t,\lambda_s)\left(\xi(\lambda_{s+1}) - \xi(\lambda_s)\right)$ where $\lambda_{s+1} > \lambda_s$.

Denote $\delta_s^2 = F(\lambda_{s+1}) - F(\lambda_s)$. Then

$$\begin{aligned} I(t) &= \sum_{s=-\infty}^{\infty} g(t,\lambda_s)\,\delta_s 1(\delta_s \neq 0)\,\frac{\xi(\lambda_{s+1}) - \xi(\lambda_s)}{\delta_s} \\ &= \sum_{s=-\infty}^{\infty} g(t,\lambda_s)\,\delta_s 1(\delta_s \neq 0)\,\eta_s, \end{aligned}$$

where η_s are independent random variables such that $E|\eta_s|^2 = 1$. We check that the assumptions of Lemma 7.1 hold true for the series $I(t)$. It follows from the assumption (8.1) that

$$\sum_{s=-\infty}^{\infty} g^2(t,\lambda_s)\,\delta_s^2 1^2(\delta_s \neq 0)$$

$$\leq \sum_{s=-\infty}^{\infty} A^2(|\lambda_s|)\left(F(\lambda_{s+1}) - F(\lambda_s)\right) \leq \int_{-\infty}^{\infty} A^2(\lambda)\, dF(\lambda) < \infty.$$

We now check that the assumptions (7.3) and (7.4) hold. Indeed, it follows from assumption (8.2) that

$$\sup_{\substack{t,u\in T \\ |t-u|\leq h}} |g(t,\lambda_s) - g(u,\lambda_s)|\,\delta_s \leq Z(|\lambda_s|)\,\delta_s \sigma(|h|)$$

and

$$\begin{aligned} \sum_{s=-\infty}^{\infty} Z^2(|\lambda_s|)\,\delta_s^2 &= \sum_{s=-\infty}^{\infty} Z^2(|\lambda_s|)\left(F(\lambda_{s+1}) - F(\lambda_s)\right) \\ &\leq \int_{-\infty}^{\infty} Z^2(|\lambda|)\, dF(\lambda) < \infty. \end{aligned}$$

□

Theorem 8.2. *Consider the linear equation (3.1)*

$$\sum_{k=1}^{N} a_k \frac{\partial^{2k+1} u(t,x)}{\partial x^{2k+1}} = \frac{\partial u(t,x)}{\partial t}, \quad t > 0, \quad x \in R^1$$

subject to the random initial condition

$$u(0,x) = \eta(x),\ x \in R^1,$$

where

$$\eta(x) = \int_{-\infty}^{\infty} e^{i\lambda x} d\xi(\lambda),$$

$\xi(\lambda)$ is a random process with independent increments and the spectral function $F(\lambda)$. Let $\theta(x)$, $x > x_0$, be a function satisfying the conditions of Lemma 6.1 and such that for sufficiently small $\varepsilon > 0$

$$\int_0^{\varepsilon} \ln \theta^{(-1)}(u^{-1}) du < \infty.$$

Let the following integral converge

$$\int_R |\lambda|^{4N+2} \theta^2 \left(u_0 + |\lambda|^{2N+1}\right) dF(\lambda) < \infty.$$

Then

$$U(t,x) = \int_R I(t,x,\lambda)\, dy(\lambda),$$

where

$$I(t,x,\lambda) = \exp\left\{ i \left(\lambda x + t \sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \right\}$$

is the classical solution of problem (3.1)-(3.2).

Proof. The proof coincides with that of Theorem 6.1 with Theorem 5.1 replaced by Theorem 8.1.

Example. The function $\theta(x) = (\ln x)^{\alpha}$, $\alpha > \frac{1}{2}$, $x > e^{\alpha}$ satisfies the conditions of Theorem 8.2 (see Examples 6.1 and 6.2). Therefore, the statement of Theorem 8.2 holds true if the following integral converges

$$\int_R |\lambda|^{4N+2} \left(\ln\left(e^2 + |\lambda|^{2N+1}\right)\right)^{2\alpha} dF(\lambda) < \infty,$$

as $\alpha > \frac{1}{2}$. This integral converges when

$$\int_R |\lambda|^{4N+2} (\ln(1+|\lambda|))^{2\alpha} dF(\lambda) < \infty$$

for $\alpha > \frac{1}{2}$.

9. Note on generalized solutions

Generalized solutions for equation (3.1), with the random initial data (3.2), where $\eta(x)$ a harmonized process

$$\eta(x) = \int_R e^{iux} dy\,(u)\,,$$

are given by processes of the form

$$U\,(t,x) = \int_R I\,(t,x,\lambda)\, dy\,(\lambda)\,. \tag{9.1}$$

where

$$I\,(t,x,\lambda) = \exp\left\{ i\left(\lambda x + t\sum_{k=1}^{N} a_k \lambda^{2k+1} (-1)^k \right) \right\},$$

provided that the integral (9.1) converges uniformly in probability for $|x| \le A$, $0 < t \le T$ for all A, T. The condition under which the integral (9.1) converges is given below.

Condition **G**. There exists a sequence $a_n > 0$, $a_n \to \infty$ as $n \to \infty$, such that for all $A > 0$ and $T > 0$ the sequence of the integrals $\int_{-a_n}^{a_n} I\,(t,x,\lambda)\, dy\,(\lambda)$ converges in probability to $U\,(t,x)$
$= \int_R I\,(t,x,\lambda)\, dy\,(\lambda)$ uniformly for $|x| \le A$, $0 \le t \le T$.

Condition **G** implies that there exist a subsequence $a_{n_k} > 0$ of the sequence a_n such that $\int_{-a_{n_k}}^{a_{n_k}} I\,(t,x,\lambda)\, dy\,(\lambda)$ converges almost surely to $\int_R I\,(t,x,\lambda)\, dy\,(\lambda)$ uniformly for $|x| \le A$, $0 \le t \le T$.

Analyzing the proofs of the results of sections 6 and 8 we arrive at the following statements.

Let $\eta(x)$ be a harmonized process which is strictly φ-subGaussian and the function $\theta\,(x)\,,\, x > u_0$, be a function satisfying the assumption Ψ. Then condition **G** holds if the following integral converges

$$\int_R \int_R \theta\left(u_0 + |\lambda|^{2N+1}\right) \theta\left(u_0 + |\mu|^{2N+1}\right) d\Gamma_y\,(\lambda,\mu) < \infty. \tag{9.2}$$

When $\eta(x)$ is a strictly φ-subGaussian stationary process

$$\eta(x) = \int_R e^{iux} d\xi\,(u)\,,$$

where $\xi\,(u)$ is a centered process with uncorrelated increments ($E\eta(x+\tau)\overline{\eta(x)} = \int_R e^{i\tau\lambda} dF\,(\lambda)$), the condition (9.2) becomes

$$\int_R \theta^2\left(u_0 + |\lambda|^{2N+1}\right) dF\,(\lambda) < \infty. \tag{9.3}$$

Finally, let

$$\eta(x) = \int_{-\infty}^{\infty} e^{i\lambda x} d\xi(\lambda),$$

where $\xi(\lambda)$ is the random process with independent increments with spectral function $F(\lambda)$. Let $\theta(x)$, $x > x_0$, be a function satisfying the conditions of Lemma 6.1 such that for sufficiently small $\varepsilon > 0$

$$\int_0^{\varepsilon} \ln \theta^{(-1)}(u^{-1}) du < \infty.$$

Then condition **G** holds if the following integral converges

$$\int_R \theta^2 \left(u_0 + |\lambda|^{2N+1} \right) dF(\lambda) < \infty.$$

Example. For the function $\theta(u) = (\ln u)^{\alpha}$, $\alpha > \frac{1}{2}$, the last integral converges when

$$\int_R (\ln(1 + |\lambda|))^{2\alpha} dF(\lambda) < \infty$$

for $\alpha > \frac{1}{2}$.

10. Concluding remarks

A possible direction of future research is the analysis of the probabilistic properties of the solutions, such as the probabilities of exceeding of a given level, the distribution of the supremum and other functionals, the asymptotic behavior of solutions (eventually, rescaled). An important practical problem is the development of methods of computer simulation of the solutions.

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