Some properties of the Boolean Quadric Polytope

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Abstract

The topic of our research is unconstrained 0-1 quadratic programming. We find some properties of the Boolean Quadric Polytope, which is obtained from the standard linearization of the given 0-1 quadratic function. By using a result obtained by Deza and Laurent on a class of hypermetric inequalities for the Cut Polytope, we find a necessary and sufficient condition for a class of inequalities to be facet defining for the Boolean Quadric Polytope. Furthermore we find a property characterizing the non integral vertices of a class of relaxations of the Boolean Quadric Polytope.

1 Introduction

We present some properties of the Boolean Quadric Polytope which is obtained from the standard linearization of a 0-1 quadratic programming problem

$$\begin{array}{l} \min f(x) \\ x \in \{0,1\}^n \end{array}$$

where

$$f(x) = q_0 + \sum_{1 \le j \le n} q_j x_j + \sum_{1 \le i < j \le n} q_{ij} x_j x_j.$$

The standard linearization of the given quadratic problem is obtained by introducing new variables y_{ij}, $1 \le i < j \le n$, and imposing the following linear constraints (see, e.g., [7]):

$$\left.\begin{array}{l} -y_{ij} + x_i + xj \le 1 \\ y_{ij} - x_i \le 1 \\ y_{ij} - x_j \le 1 \\ y_{ij} \ge 0 \end{array}\right. \qquad 1 \le i < j \le n \tag{1}$$

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which, for all binary vectors x, enforce the identities $y_{ij} = x_i x_j$, $1 \le i < j \le n$. The quadratic problem can then be formulated as a linear integer problem

$$\begin{array}{l} \min\ q_0 + \sum_{1\le j\le n} q_j x_j + \sum_{1\le i<j\le n} q_{ij} y_{ij} \\ s.t.\ (1) \\ x \in \{0,1\}^n. \end{array}$$

The *Boolean Quadric Polytope* is the convex hull of the feasible solutions to this problem. In the following we will denote by BQ_V the Boolean Quadric Polytope defined on the variables x_i, $i \in V = \{1, \dots, n\}$ and y_{ij}, $i, j \in V, i < j$.

The Boolean Quadric Polytope has been first studied by Padberg in 1989 [9], and by Boros, Crama and Hammer [1], [2]. In his paper, Padberg, found many properties of the Boolean Quadric Polytope. Among these we recall the following ones: inequalities (1) are facets defining for the Boolean Quadric Polytope; if $n = 2$ the above inequalities are enough to define the Boolean Quadric Polytope so that it coincides with its continuous relaxation; for $n = 3$, the continuous relaxation has four fractional vertices, where all the x variables are equal to 1/2 and these vertices are cut off by four facets defining inequalities which, together with (1), define the Boolean Quadric Polytope; for arbitrary values of n the components of the vertices of the continuous relaxation of the Boolean Quadric Polytope are equal to 0, 1/2 or 1. Finally he found three classes of facet defining inequalities called cut inequalities, clique inequalities and generalized cut inequalities.

It has been proved by several authors that there exists a linear bijective transformation mapping the Cut Polytope onto the Boolean Quadric Polytope so that any result for the former one can be translated into a result for the latter one and vice versa (see e.g. [4]).

In this paper we present a new result on the fractional vertices of a class of relaxations of the Boolean Quadric Polytope (Section 3) and, using a result by Deza and Laurent [5] for the Cut Polytope, we find a necessary and sufficient condition for a class of inequalities to be facet defining for the Boolean Quadric Polytope (Section 4). In Section 2 we describe some previous results necessary in our exposition and we introduce some further definitions and notations.

2 Previous results, notations and definitions

Given a quadratic function

$$h(x) = a_0 + \sum_{j\in V} a_j x_j + \sum_{i,j\in V: i<j} a_{ij} x_i x_j$$

where $V = \{1, \dots, n\}$, we denote by

$$L_h(x, y) = a_0 + \sum_{j\in V} a_j x_j + \sum_{i,j\in V: i<j} a_{ij} y_{ij}$$

the linear function obtained by substituting in h the quadratic terms $x_i x_j$ with the variables y_{ij}.

Let $C_V \subset R^{n(n+1)/2}$ be the cone of nonnegative 0-1 quadratic functions of the variables x_i, $i \in V$, and for a given $k \in \{2, \dots, n\}$, let $C_V^k \subseteq C_V$ be the cone of nonnegative 0-1 quadratic functions of k variables.

If constrains (1) hold, then $L_h(x, y) \geq 0$ is valid for BQ_V if and only if $h(x) \geq 0$ for all binary vectors x ([1]). Based on this fact Boros, Crama and Hammer ([1]) defined a hierarchy of relaxations of BQ_V:

$$Q_V^k = \{(x, y) \in R^{n(n+1)/2} : L_h(x, y) \geq 0, h \in C_V^k\}, k = 2, \dots, n$$

such that $Q_V^n = BQ_V$.

For any given set $K \subseteq V$, we define also the cone $C_V[K] \subset R^{n(n+1)/2}$ of nonnegative 0-1 quadratic functions of the variables in K and the relaxation of BQ_V

$$Q_V[K] = \{(x, y) \in R^{n(n+1)/2} : L_g(x, y) \geq 0, g \in C_V[K]\}.$$

Given a vector $a \in R^{n(n+1)/2}$, $a = (a_1, \dots, a_n, a_{12}, \dots, a_{n-1,n})$, and $K \subset V$ such that $|K| = k$, we define the *canonical restriction* of a to K as the vector $a^K \in R^{k(k+1)/2}$ obtained by discarding all components a_j such that $j \in V - K$ and all components a_{ij} such that $i \in V - K$ or $j \in V - K$. Based on the definition of canonical restriction of a vector a, we can define:

- the canonical restriction $b^K(x, y)^K \geq 0$ of an inequality $b(x, y) \geq 0$;
- the canonical restriction P^K of a polytope P.

Remark 2.1 *The canonical restrictions to K of the elements in $C_V[K]$ are all and only the elements of C_K, i.e., they define the Boolean Quadric Polytope BQ_K on the set K.*

3 A property of the vertices of some relaxations of the Boolean Quadric Polytope

In this section we prove a property of the non integral vertices of the relaxations $Q_V[K]$ and Q_V^k.

Theorem 3.1 *Let K be a subset of V having cardinality at least two and let $(\bar{x}, \bar{y})$ be a vertex of $Q_V[K]$ such that $\bar{x}_j$ is not integral for some $j \in K$. Then $\exists s \in N - K$ such that $\bar{x}_s$ is not integral.*

Proof. By Remark 2.1, the canonical restriction to K of $Q_V[K]$ is BQ_K and then all its vertices are integral. It follows that the canonical restriction $(\bar{x}, \bar{y})^K$ of $(\bar{x}, \bar{y})$ is not a vertex of BQ_K since it is not integral. By the Caratheodory's theorem there exist the vertices $(x^1, y^1), \dots, (x^p, y^p)$ of BQ_K such that $(\bar{x}, \bar{y})^K = \sum_{h=1,\dots,p} \alpha_h (x^h, y^h)$, $\alpha_h \geq 0, h = 1, \dots, p$ and $\sum_{h=1,\dots,p} \alpha_h = 1$.

Suppose that $\bar{x}_s$ is integral for each $s \in V - K$. For $h = 1, \dots, p$, define the vectors $(\bar{x}^h, \bar{y}^h) \in R^{n(n+1)/2}$ as follows:

$$\bar{x}_j^h = \begin{cases} x_j^h & j \in K \\ \bar{x}_j & j \in V - K \end{cases}$$

$$\bar{y}^h_{ij} = \begin{cases} y^h_{ij} & i \in K, j \in K \\ \bar{y}_{ij} & i \in V-K, j \in V-K \\ \min\{x^h_i, \bar{x}_j\} & i \in K, j \in V-K \\ \min\{\bar{x}_i, x^h_j\} & i \in V-K, j \in K \end{cases}$$

These vectors are integral and belong to BQ_V and to $Q_V[K]$ (since $BQ_V \subseteq Q_V[K]$). Moreover:

$$(\bar{x}, \bar{y}) = \sum_{h=1,\dots,p} \alpha_h (\bar{x}^h, \bar{y}^h).$$

But this contradicts the hypothesis that $(\bar{x}, \bar{y})$ is a vertex of $Q_V[K]$. Hence there exists $s \in N-K$ such that x_s is not integral. □

Corollary 3.2 *For any non integral vertex $(\bar{x}, \bar{y})$ of Q^k_V, at least $k+1$ among the components $\bar{x}_1, \dots, \bar{x}_n$ are non integral.*

4 A Class of facets of the Boolean Quadric Polytope

Let $P_C(K_N)$ be the cut polytope defined on a complete graph having N vertices. It has been proved by several authors that there exists a linear bijective transformation mapping the cut polytope $P_C(K_N)$ onto the Boolean Quadric Polytope BQ_V, $|V| = N-1$, so that any result for $P_C(K_N)$ can be translated into a result for BQ_V and vice versa (see e.g. [4]). In particular the following proposition holds.

Proposition 4.1 *The inequality*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_{ij} z_{ij} \leq d$$

is valid (facet defining) for $P_C(K_N)$ if and only if the inequality

$$\sum_{i=1}^{N-1} a_i x_i + \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} a_{ij} y_{ij} \leq d$$

is valid (facet defining) for BQ_V, $|V| = N-1$, where:

$$c_{iN} = a_i + \tfrac{1}{2} \textstyle\sum_{j=1,\dots,N-1: i \neq j} a_{ij} \quad 1 \leq i \leq N-1 \tag{2}$$

and

$$c_{ij} = -\tfrac{1}{2} a_{ij} \quad 1 \leq i < j < N. \tag{3}$$

The facets of the cut polytope have been extensively studied. Deza and Laurent, 1992 [5], present some classes of facets defining inequalities: Hypermetric, Cycle, Parachute, Grishukhin, Barahona-Grotschel-Mahajoub, Kelly, Poljak-Turzik.

Given N integers $b_1, \ldots, b_N$ such that $\sum_{i=1}^{N} b_i = 1$, the inequality

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} b_i b_j x_{ij} \leq 0 \tag{4}$$

is an *hypermetric inequality*. All hypermetric inequalities are valid for the cut polytope. If all negative b_i are equal to -1, the hypermetric inequality is called *linear*; if at most one negative coefficient is different from -1 the hypermetric inequality is called *quasilinear*. Deza and Laurent, 1992, proved the following theorem.

Theorem 4.2 *[5] We are give an hypermetric inequality such that:*

$$b_1 \geq b_2 \geq \ldots \geq b_f > 0 > b_{f+1} \geq \ldots \geq b_N.$$

1) If the inequality is linear, then it is facet defining if and only if, either $b = (1, 1, -1)$*, or* $b = (1, 1, 1, -1, -1)$ *or* $3 \leq f \leq N - 3$*.*
2) If the inequality is quasi-linear, then it is facet defining if and only if, either $b = (1, 1, -1)$*, or* $b = (1, \ldots, 1, -1, -N + 4)$ *or* $3 \leq f \leq N - 3$ *and* $b_1 + b_2 \leq n - f - 1 + sign(b_1 - b_f)$*.*

Given the set $V = \{1, \ldots, n\}$ and an integral vector $\rho_1, \ldots, \rho_n$, consider the pseudo-Boolean quadratic function

$$g(x) = (\sum_{j=1}^{n} \rho_j x_j)(\sum_{j=1}^{n} \rho_j x_j - 1). \tag{5}$$

Since $g(x) \geq 0$ for all integral vectors x, the corresponding linear inequality $L_g(x, y) \geq 0$ is valid for the Boolean Quadric Polytope. It can be shown that these inequalities correspond precisely to hypermetric inequalities. In [3], a more general class of facet defining inequalities for the Boolean Quadric Polytope has been introduced and sufficient conditions for these inequalities to be facet defining have been presented. The corresponding inequalities for the cut polytope include hypermetric and cycle inequalities. In [6], a class of valid inequalities for the cut cone including hypermetric and cycle inequalities has been introduced.

In the following we suppose:

$$\rho_1 \geq \rho_2 \geq \ldots \geq \rho_p > 0 > \rho_{p+1} \geq \ldots \geq \rho_n \tag{6}$$

and

$$\rho_i = -1, i = p + 1, \ldots, n. \tag{7}$$

Boros, Crama and Hammer, 1990 [1], proved that for $n \geq 3$, if $\rho_1 = 1$ and $2 \leq p \leq n - 1$ then $L_g(x, y) \geq 0$ is facet defining.

Theorem 4.3 *If the 0-1 quadratic function $g(x)$ satisfies (5), (6) and (7), the inequality $L_g(x,y) \geq 0$ valid (facet defining) for BQ_V can be transformed into a linear or quasilinear hypermetric inequality valid (facet defining) for $P_C(K_{n+1})$.*

Vice versa a linear or quasi linear hypermetric inequality valid (facet defining) for $P_C(K_{n+1})$ can be transformed into the inequality $L_g(x,y) \geq 0$ valid (facet defining) for BQ_V, for some 0-1 quadratic function $g(x)$ satisfying (5), (6) and (7).

Proof. Using the bijection defined in (2) and (3) we obtain a linear or quasi linear hypermetric inequality such that

$$\begin{array}{ll} b_i = \rho_i & i = 1, \dots, n \\ b_{n+1} = -\sum_{i=1}^{p} \rho_i + n - p + 1. & \end{array} \tag{8}$$

Notice that the number f of positive elements in vector b, depends on the value of b_{n+1}. In particular if $b_{n+1} > 0$ (i.e. $\sum_{i=1}^{p} \rho_i \leq n - p$) then $f = p + 1$; if $b_{n+1} = 0$ (i.e. if $\sum_{i=1}^{p} \rho_i = n - p + 1$), $f = p$ and the obtained hypermetric inequality is defined on n variables; if $b_{n+1} < 0$ (i.e. if $\sum_{i=1}^{p} \rho_i \geq n - p + 2$), $f = p$.

In a specular way, given a linear or quasilinear hypermetric inequality such that:

$$b_1 \geq b_2 \geq \ldots \geq b_f \geq 0 > b_{f+1} \geq \ldots \geq b_N$$

by using the bijection defined in (2) and (3), we obtain the inequality $L_g(x,y) \geq 0$ such that relations (8) hold. □

The following theorem is a consequence of Theorems (4.2) and (4.3).

Theorem 4.4 *The inequality $L_g(x,y) \geq 0$ is facet defining for BQ_V if and only if one of the following conditions holds:*

1. *either $n = 2$, $\rho_1 = 1$ and $p = 1, 2$;*
2. *or $n = 3, 4$, $\rho_1 = 1$ and $2 \leq p \leq n - 1$;*
3. *or $n \geq 5$, $\rho_1 = 1$ and $p = n - 1$;*
4. *or $n \geq 5$, $2 + max\{0, sign(\sum_{i=1,\ldots,p} \rho_i - n + p\} \leq p \leq n - 2$ and $\rho_1 + \rho_2 \leq n - p + sign(\rho_1 - \rho_p)$.*

Proof. Transform $L_g(x,y) \geq 0$ into an hypermetric inequality for $P_C(K_{n+1})$ as in the proof of Theorem 4.3 so that $L_g(x,y) \geq 0$ is facet defining for BQ_V if and only if the obtained linear or quasilinear hypermetric inequality is facet defining for $P_C(K_{n+1})$.

In order to apply Theorem 4.2 to the obtained hypermetric inequality, we distinguish different cases.

a) $\sum_{i=1}^{p} \rho_i \leq n - p$

In this case $b_{n+1} \geq 1$, the hypermetric inequality is linear and $f = p + 1$, hence it is facet defining if and only if:

- $n = 2$, $\rho_1 = 1$ and $p = 1$;
- $n = 4$, $\rho_1 = 1$ and $p = 2$;
- $n \geq 5$, $2 \leq p \leq n - 3$.

b) $\sum_{i=1}^{p} \rho_i = n - p + 1$
In this case $b_{n+1} = 0$, the hypermetric inequality is linear and $f = p$ and it is facet defining if and only if:

- $n = 3$, $\rho_1 = 1$ and $p = 2$;
- $n = 5$, $\rho_1 = 1$ and $p = 3$;
- $n \geq 6$, $3 \leq p \leq n - 3$.

c) $\sum_{i=1}^{p} \rho_i = n - p + 2$
In this case $b_{n+1} = -1$, the hypermetric inequality is linear and $f = p$ and it is facet defining if and only if:

- $n = 2$, $\rho_1 = 1$ and $p = 2$;
- $n = 4$, $\rho_1 = 1$ and $p = 3$;
- $n \geq 5$, $3 \leq p \leq n - 2$.

d) $\sum_{i=1}^{p} \rho_i \geq n - p + 3$
In this case $b_{n+1} \leq -2$, the hypermetric inequality is quasi-linear, $f = p$ and it is facet defining if and only if:

- $n \geq 5$, $\rho_1 = 1$ and $p = n - 1$;
- $n \geq 5$, $3 \leq p \leq n - 2$ and (*) $\rho_1 + \rho_2 \leq n - p + \mathrm{sign}(\rho_1 - \rho_p)$.

Notice that condition (*) always holds for linear inequalities such that $3 \leq f \leq n - 3$.

The above conditions can be summarized as follows:

1. either $n = 2$, $\rho_1 = 1$ and $p = 1, 2$;
2. or $n = 3, 4$, $\rho_1 = 1$ and $1 \leq p \leq n - 1$;
3. or $n \geq 5$, $\rho_1 = 1$ and $p = n - 1$;
4. or $n \geq 5$, $2 + \max\{0, \mathrm{sign}(\sum_{i=1,\dots,p} \rho_i - n + p\} \leq p \leq n - 2$ and $\rho_1 + \rho_2 \leq n - p + \mathrm{sign}(\rho_1 - \rho_p)$.

By Theorem 4.3 $L_g(x, y) \geq 0$ is facet defining if and only if one among cases 1., ..., 4. holds. □

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