Heuristic approaches for the Minimum Labelling Hamiltonian Cycle Problem

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Abstract

Given a graph G with a label (color) assigned to each edge (not necessarily properly) we look for an hamiltonian cycle of G with the minimum number of different colors. The problem has several applications in telecommunication networks, electric networks, multimodal transportation networks, among others, where one aims to ensure connectivity or other properties by means of limited number of different connections. We analyze the complexity of the problem on special graph classes and propose, for the general case, heuristic resolution algorithms. Performances of the algorithms are experimentally evaluated on a set of instances and compared with the exact solution value provided by a solver.

Key words: Labelled graph algorithms, Hamiltonian cycles, Tabu search

1 Introduction

The minimum hamiltonian problem may be called one of the most important combinatorial optimization problem, with application in many real world situations such as routing and ordering problems. In this paper we study a variant of the hamiltonian cycle problem, namely, the *Minimum Labelling Hamiltonian Cycle problem* (MLHC). Given a graph G with a label (color) assigned to each edge (not necessarily properly) we look for an hamiltonian cycle of G with

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the minimum number of different colors. The problem belongs to a recently studied class of problems defined on colored graphs having several applications in telecommunication networks, electric networks, multimodal transportation networks, among others, where one aims to ensure connectivity or other properties by means of a limited number of different connections. In particular, problems belonging to this class already addressed in literature include the *Minimum Labelling Spanning Tree Problem* (MLST) [2]-[5]-[7], the *Minimum Labelling Steiner Problem* [4]-[6], the *Minimum Labelling Generalized Forest* [3] and the *Minimum Labelling Path Problem* [9].

The sequel of the paper is organized as follows. Section 2 resumes the needed notations and analyzes the complexity of the problem on general graph and on special graph classes. In section 3 we propose two heuristic approaches. In particular we present a fast heuristic algorithm, *ColorHAM*, and a tabu search approach that massively uses *ColorHAM* either to find the initial solution and to define neighborhoods. In the last part of the section some experimental results are presented. Conclusions and further research are discussed in section 4.

2 Problem description

In this section we formally describe the problem and study its complexity both on general graphs and on special graph classes.

Notation
Given an undirected graph $G = (V, E)$ with V being the set of nodes and E denoting the set of edges, let c_e be the color (label) associated with edge $e \in E$ and $L = \{c_1, c_2, \dots, c_l\}$ be the set of all the colors. We denote by $C(S) = \bigcup_{e \in S} c_e$ the set of colors assigned with edges in $S \subseteq E$. Any hamiltonian cycle H of G has associated the set of its colors $C(H)$. We look for an hamiltonian cycle of G such that the set of its colors $C(H)$ has minimum size.

Looking for an hamiltonian cycle on general graph is an $NP-complete$ problem [8], therefore looking for an hamiltonian cycle with the minimum number of colors on general graphs is a difficult problem too. It is, then, of interest the analysis of the complexity of the problem on graphs where the existence of an hamiltonian cycle is known in advance (i.e., hamiltonian graphs). Hence, in the sequel of this section, for sake of completeness, we give a formal proof of the complexity of the problem on the class of complete graphs (Theorem 1).

Theorem 1 *The minimum labelling hamiltonian cycle problem on complete graphs is NP-complete*
Proof. The problem belongs to the NP class. Indeed, given an hamiltonian

cycle of G, it is possible to verify in polynomial time whether it contains less than k colors.
We show the problem is NP-complete on general graphs by reduction from the hamiltonian cycle problem. That is, let $G = (V, E)$ be a general instance of the hamiltonian problem, and consider the associated complete graph $G' = (V, E')$ where with each edge $e = (u, v) \in E'$ is associated the color $c(e) = 0$ if $e \in E$ and $c(e) = uv$ otherwise.
Since G' is complete any permutation of its vertices is an hamiltonian cycle. It is simple to show that G admits an hamiltonian cycle H iff the corresponding set $C(H)$ on G' has size equal to one, that is iff G' admits a monochromatic hamiltonian cycle.

3 The Proposed Heuristic Approaches

3.1 ColorHAM heuristic

The ColorHam heuristic is based on a former heuristic for Hamiltonian Cycles, namely the HAM heuristic proposed by Bollobas et al. [1]. The main idea of such an algorithm consists in extending a partial actual path $P = \{u_1, u_2, \ldots, u_k\}$ from one of its extreme node (**extension from the extreme** operation) until either an hamiltonian cycle is found or not. If it is not possible to extend P from its extreme nodes and there exists the arc (u_1, u_k) connecting them, then it is possible to select one of the internal node of the path, say x, as a new extreme, by deleting from P one of the arcs connecting x with one of its neighbors in P and inserting (u_1, u_k) (**cycle extension** operation). If it is not possible to apply the cycle extension operation, for each neighbor $x \in P$ of u_1 and u_k the algorithm performs a **rotational transformation**, an operation that builds a new path containing the same set of vertices but with different extreme nodes. More in detail, let (u_1, u_j), $1 < j < k$, be an arc between the extreme node u_1 of P and one of its internal node, i.e. (u_1, u_j) is the *rotating arc*. The new path P' is built inserting the rotating arc and changing the order of the nodes between u_1 and u_j, that is, the new path is $P' = \{u_{j-1}, u_{j-2}, \ldots, u_2, u_1, u_j, u_{j+1}, \ldots, u_k\}$
The resulting new paths are stored in a list, and once all the neighbors of the first path have been explored without resulting in an hamiltonian cycle then the research starts again from the paths in the list.

During the extension operations, in case of multiple choices the HAM heuristic selects one arc at random among all the possible ones. Our *ColorHAM* heuristic adopts two greedy selection criterions: the *used color criterion* and the *max coverage criterion*. Let P be a path in the graph, $C(P)$ be the set of colors associated with the arcs in P, and, $u \in P$ be a node of the path from

which one of the three extension operations has to be performed:
Used Color Criterion: chooses the node $i \notin P$ such that the label l of the arc (u, i) to be inserted into P already belongs to $C(P)$.
Max Coverage Criterion: chooses the node $i \notin P$ such that the label l of the arc (u, i) that has to be inserted into P covers the greatest number of nodes not already in the path.

Let us now describe the *ColorHAM* heuristic by looking at how these two criteria work in each of the above mentioned extension operations.
Extension from the extreme nodes
Let us suppose that al least one among the extreme node u_1 and u_k of the actual partial path has a neighbor $i \notin P$. *ColorHAM* chooses node satisfying the *used color criterion*, if such a node does not exist then the selected node would be the one satisfying the *max coverage criterion*. Ties are broken randomly.
Cycle Extension
If it is not possible to extend the actual partial path P from its extreme nodes then the cycle extension operation is carried out. Differently from the HAM heuristic that chooses randomly the internal node u, *ColorHAM* chooses the node satisfying the *used color criterion*, if such a node does not exist then the selected node would be the one satisfying the *max coverage criterion*. Ties are broken randomly.
Extended Rotational Transformation
When it is not possible to apply none of the two above mentioned extension operations, then HAM heuristic starts analyzing all the paths that one can create by performing a rotational transformation on P. In this case HAM and *ColorHam* operate the same. We however considered SemiHam, a modification proposed by [10] to limit the number of paths stored in the list.
We want just to point out that the rotational transformation operation as well as the cycle extension operation exchange internal arcs with external ones. Such an exchange may modify the value of the objective function. However, from an experimental point of view such modifications, when they occur, are negligible.

3.2 Tabu Search

In this section we describe a tabu search approach we developed to solve the problem. We decided to build up the neighbors of a given feasible solution H by applying our *ColorHAM* heuristic on a perturbation of the graph G. More in detail, let $H = \{u_1, u_2, \ldots, u_n, u_1\}$ be any current feasible solution (i.e. an hamiltonian cycle of G), and let $e_i = (u_i, u_{i+1})$, $i = 1, \ldots, n$, be the collection of its arcs. A neighbor $H_k \in N(H)$ of H is an hamiltonian cycle produced by our *ColorHAM* heuristic on the graph G_k obtained from G after deleting

the arc $e_k = (u_k, u_{k+1})$. Each neighborhood of a given solution contains, then, at most n elements and can be explored by, for example, a steepest descent technique.
The tabu list memorizes *moves* corresponding to feasible solutions in order to avoid cycling on local optima. Let us suppose we move from the hamiltonian cycle $H = \{u_1, u_2, \ldots, u_n, u_1\}$ to one of its neighbor $H' = \{u'_1, u'_2, \ldots, u'_n, u'_1\}$. The corresponding move should ideally store n possible pairs (u_i, u'_i) corresponding to all the exchanges performed to move from H to H'. Since memorizing all the n couples could be heavy from a memory point of view, we fixed a parameter h defining the number of pairs exchange to memorize. In our implementation we set the length of the tabu list equal to 10 and $h = 4$. The algorithm terminates when either a given neighborhood $N(H) = \emptyset$ or after a given limit on the total number of iterations without any improvement.

3.3 Experimental Results

We evaluated experimentally our algorithms on a set of instances, comparing the results with the exact solution value provided by the CPLEX solver that solves a single-commodity flow formulation of the problem. We fixed a threshold on the execution time of the solver equal to 3 hours and a limit of 100 iterations since the last improvement in the solution provided by the tabu search technique.
The tabu search approach shows generally good performances, but should be improved when the number of colors is low compared to the dimension of the graph and a large amount of local minima can occur.
Test instances are available at:
www.dmi.unisa.it/people/cerulli/www/GroupPages/SoftwareGroup.htm.
Our test instances are divided into two groups of scenarios based on different parameter settings: n: number of nodes of the graph, l: total number of colors assigned to the graph, m: total number of edges of the graph computed by $m = \frac{d(n-1)n}{2}$ where d is a measure of density of the graph. Parameter settings for scenarios in Group 1 are: $l = n = 20, 30, 40, 50$ and $d = 0.2, 0.5, 0.8, 1$, for a total of 16 different scenarios. Instances in Group 2 are characterized by $n = 50, 100$, $l = 0.25 * n, 0.5 * n, n, 1.25 * n$ and $d = 0.2, 0.5, 0.8, 1$, for a total of 32 different scenarios. The following tables summarize some of the results we obtained. For each scenario we generated ten different instances; each value given in these tables is the mean of the values obtained on the instances of that scenario. Time values are approximated in seconds; we used a * symbol to mark CPLEX time values where the solver did not find the exact solution for every instance of the scenario because the time limit was exceeded.

n = 20, l = 20					
d	ColorHam Sol	Tabu Search Sol	Tabu Search Time	Cplex Sol	Cplex Time
0.2	10	10	1	10	0
0.5	9.5	7.2	0.9	5.5	5
0.8	6.5	4.9	0.1	3.6	24.4
1	5.3	4	0	3	32.1

n = 40, l = 40					
d	ColorHam Sol	Tabu Search Sol	Tabu Search Time	Cplex Sol	Cplex Time
0.2	21.4	17.4	23.6	13.6	32.2
0.5	14.3	10.6	14.3	6.4	806.3
0.8	10.4	7.4	10.8	4.1	2677.1
1	9	6.1	9.1	3.3	4610.9

n = 50, l = 50					
d	ColorHam Sol	Tabu Search Sol	Tabu Search Time	Cplex Sol	Cplex Time
0.2	29	22	69.5	18	33.5
0.5	21	15.8	80.5	7.8	2050.1
0.8	11.3	7.8	26.1	4.5*	7249.5
1	9.6	6.9	24.3	4.3*	10554.3

n = 50, l = 62					
d	ColorHam Sol	Tabu Search Sol	Tabu Search Time	Cplex Sol	Cplex Time
0.2	30	24.5	154.5	20.5	82.5
0.5	21.1	17.2	125.4	9.2	3395.2
0.8	12.4	8.9	31.9	6.3*	10800
1	10.3	7.3	24.5	5.4*	10800

4 Conclusions

In this paper we addressed a new problem namely the *Minimum Labelling Hamiltonian Cycle Problem*, that is a variant of the well-known Hamiltonian Cycle problem and has several applications in telecommunication networks, electric networks, multimodal transportation networks, among others, where one aims to ensure connectivity or other properties by means of limited number of different connections.
Starting from the analysis of the best heuristics existing in the literature we define a local search heuristic (*ColorHAM*) and a tabu search paradigm to solve the problem. Performances of the algorithms are experimentally evaluated on a set of instances and compared with the exact solution value provided by a solver.

Further research is focused in developing an exact approach to solve the problem and improving the performance of our tabu search algorithm.

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