Boolean harmonies, and a tractable class of set covering problems

Isabella Lari, Bruno Simeone

Dipartimento di Scienze Statistiche, Sapienza, Università di Roma

Abstract

In this paper we deal with a class of set covering problems arising in polytonal harmony. Actually, in a recent paper Simeone, Nouno, Mezzadri and Lari introduced the notion of Boolean signatures of a tonal scale, depending on a Boolean operator. They proved that any minimum cardinality signature may by obtained through the solution of an appropriate set covering problem and that all the signatures are the prime implicants of a suitable monotone Boolean function represented by a CNF. Here we focus on the solution of the set covering problems. It turns out that all of them are easily solvable by a combination of standard preprocessing rules and small-size linear programs, without any need of enumeration at all.

1. Introduction

The notion of signature of a musical scale was introduced and put into musical practice by the French composer and flutist Malik Mezzadri. As the name suggests, the signature of a scale S is a set of notes of S that collectively give the “gist” of the scale: a formal definition will be given below.

Recently, Simeone, Nouno, Mezzadri and Lari [6] generalized this concept, introducing the notion of Boolean signature, which depends not only on a scale, but also on a Boolean operator; and proposed effective computational methods to find a smallest Boolean signature and to generate a full catalogue of all Boolean signatures.

The former problem can be formulated as a set covering problem, while the latter one can be solved by a simple, but performant, dualization algorithm for monotone Boolean functions.

In the present note, we shall focus on the solution of the above set covering problems. As it turns out, no enumeration at all is required. All one needs are simple and standard preprocessing rules, sometimes followed by the solution of small linear programs.

In Section 2 we recall from [6] the notions of signatures and Boolean signatures. Section 3 deals with preprocessing and its outcomes. Section 4 discusses the use of linear programming to solve the simplified set covering problems.

As argued in [6] the richness and variety of Boolean signatures has a great potential for the development of new paradigms in polytonal harmony. The possibility of getting those signatures cheaply enhances this potential by making their full catalogue easily accessible to composers and interpreters.

2. From signatures to Boolean signatures

In the present section we briefly present some relevant background on signatures and Boolean signatures as discussed in [6]. Given an ordered set of notes **N**, a scale S is any nonempty subset of **N**. The most important scales in the western tradition are the major, the minor (melodic) and the (minor) harmonic scales, all having |S| = 7 on the same base set of 12 notes **N** ={C, C#, **…,** B♭, B}.

All major, minor, harmonic scales can be obtained by a transposition of the basic scales

C major: C, D, E, F, G, A, B;
C minor: C, D, E♭, F, G, A, B;
C harmonic: C, D, E♭, F, G, A♭, B,

respectively.

Every subset S of **N** can be represented by its *characteristic vector* v(S):

$$v_j(S) = \begin{cases} 1, & if \;\; j \in S \\ 0, & if \;\; j \notin S \end{cases}, \quad j \in \mathbf{N}.$$

From now on, for the sake of brevity but with a slight abuse of language and notation, we shall refer to subsets of **N** and to their characteristic vectors interchangeably.

In the following table the characteristic vectors of the three basic scales are shown.

	C	**C#**	**D**	**E♭**	**E**	**F**	**F#**	**G**	**A♭**	**A**	**B♭**	**B**
C major	1	0	1	0	1	1	0	1	0	1	0	1
C minor	1	0	1	1	0	1	0	1	0	1	0	1
C harmonic	1	0	1	1	0	1	0	1	1	0	0	1

Table 1

The characteristic vectors of the other scales are obtained by a cyclic permutation of the "basic" characteristic vectors.

Let A = [a_{ij}] be the 36x12 *scale-note* binary matrix whose rows are the characteristic vectors of the 36 major, minor and harmonic scales and let S_i be the scale in the i-th row.

A *typical* set T for a given reference scale S_r is any subset of S_r that is not contained in any scale $S_i \neq S_r$. A minimal typical set is called a (*tonal*) *signature*. Let P = [p_{ij}] be the 35×12 (standard) *product matrix* defined as follows:

$$p_{ij} = \begin{cases} 1, & \text{if}\, j \in S_r,\, j \notin S_i \\ 0, & \text{else} \end{cases} \qquad i = 1,\ldots,36, i \neq r;\, j \in N$$

A typical set for S_r is any solution x to the following *Satisfiability* (SAT) problem

$$\bigwedge_{i \neq r} \bigvee_{j \in N} p_{ij} x_j = 1. \qquad (1)$$

(For the explanation of the Boolean notions dealt with in this paper, the reader may consult [4]). A solution that does not satisfy formula (1) whenever the value of any of its components equal to 1 is decreased to 0, is said to be *minimal*. The tonal signatures are precisely the minimal solutions to (1). From a well-known theorem on Boolean functions, it follows that there is a one-to-one correspondence between the minimal solutions to (1) and the prime implicants of the *conjunctive normal form* (CNF) on the l.h.s. of (1). The prime implicants are actually the terms of the (unique) irredundant *disjunctive normal form* (DNF) of the l.h.s. of (1). In order to get such DNF, a variety of dualization algorithms is available in the literature (see [4]).

A tonal signature having minimum cardinality can be found through the solution of the following set covering problem

$$\min \sum_{j \in N} x_j \qquad (2.1)$$

$$SC(r) \qquad \sum_{j \in N} p_{ij}\, x_j \;\geq\; 1, \quad i \neq r \qquad (2.2)$$

$$x_j = 0 \; or \; 1, \quad j \in N \qquad (2.3)$$

The feasible solutions to *SC*(r) are precisely the solutions to the SAT problem (1).

In [6] the definition of typical set and tonal signature was generalized on the basis of more general binary product matrices. The product matrix P defined above for tonal signatures can be equivalently defined as follows:

$$p_{ij} = a_{rj} * a_{ij} \qquad i \neq r;\, j \in \mathbf{N} \qquad (3)$$

where * is the Boolean operator defined by the truth table

x	y	x * y
0	0	0
0	1	0
1	0	1
1	1	0

That is, $x * y = x\,\overline{y}$ = x AND (NOT y).

By replacing in (3) the above operator by any Boolean operator *, one obtains 16 binary product matrices P(r,*). A *boolean signature* for a given reference scale S_r and a given Boolean operator is any minimal subset of S_r whose characteristic vector satisfies the SAT problem (1) where, this time P = P(r,*). In order to generate all these Boolean signatures one may apply the dualization procedure mentioned above for the tonal signatures. Furthermore, a Boolean signature having minimum cardinality is an optimal solution to the set covering problem (2), where again P = P(r,*). This more general set covering problem will be denoted *SC*(r,*).

The 0-constant and the 1-constant operators give rise to product matrices consisting of all 0's and all 1's, respectively. In the former case, the associated set covering problem has no solution; in the latter case, any single note corresponds to an optimal solution. Hence, these two Boolean operators are trivial and they may be ignored. When * is the NAND operator and the reference scale is either major or minor, the set covering problem has no feasible solution. Thus also the NAND operator may be discarded unless the reference scale is harmonic.

The following invariance results hold.

Theorem 1: For any given Boolean operator *, all the product matrices relative to a major (minor, harmonic) scale can be obtained from those relative to the C major (minor, harmonic, resp.) scale through cyclic permutations of the rows and of the columns.

More precisely, if $P^{(t,k)}$ is the product matrix relative to the scale $S_r \equiv S_{t+k}$ (t =0, 12, 24 ;

k= 2, …,12) and to the Boolean operator *, the following identity holds:

$$p^{(t,k)}_{s+[h+k-1],[j+k-1]} = p^{(t,1)}_{s+h,\,j} \quad , \quad s = 0, 12, 24\,;\; h\,, j = 1, \dots, 12; \qquad (s,h) \neq (t,k) \qquad (4)$$

Proof. See [6].

Corollary 1 (*Invariance*) : The set covering problem and the signature function are the same for all major (minor, harmonic) scales, up to cyclic variable renaming.

The above Invariance results relative to the major, minor and harmonic scales imply that a full catalogue of all Boolean signatures for all tonal scales and all Boolean operators can be obtained through the generation of all the prime implicants of 40 (=13×2+14: recall that F8 makes sense only for harmonic scales) Boolean functions with 12 variables and 35 clauses. Similarly, for all tonal scales and all Boolean operators, Boolean signatures having minimum cardinality can be obtained via the solution of 40 set covering problems with 12 variables and 35 constraints each. It turns out that, even though arbitrary set covering problems are NP-complete [3], all such 40 set covering problems are easily solvable by a combination of standard preprocessing rules and small-size linear programs, without any need of enumeration at all.

3. Preprocessing

The next two sections deal with the solution of the set covering problems (2). Actually, they were inspired by our curious finding that, when we solved these problems by CPLEX/MIP, no branching was ever needed! This was somehow surprising, since set covering problems are known to be NP-complete; and even some small-size instances, such as the terrible ST_{15} arising from Steiner triple systems, require massive branching [5].

The first idea that came to our mind, when we tried to discover the causes of the above surprising phenomenon, was that perhaps preprocessing might play a significant role. As a matter of fact, in standard commercial codes such as CPLEX, the solution of set covering problems is greatly enhanced through the use of simple *preprocessing rules* which usually allow for a considerable reduction in problem size.

We shall employ here four simple such rules, applied to the 35×12 binary coefficient matrix of the set covering problem, that is, the product matrix $P = [p_{ij}]$, and then iteratively on the current matrix:

R1) if a row has only a 1 in column j, then variable x_j is forced to 1 and column j may be deleted;

R2) if row h contains row i, that is, $p_{ij} \leq p_{hj}$ for $j = 1, \ldots, 12$ (equivalently, every 1 in row i is present, in the same column, also in row h), then row h may be deleted;

R3) if the matrix has a column of all 0's, then such column may be deleted;

R4) if column j contains column k, that is, $p_{ij} \geq p_{ik}$ for $i = 1, \ldots, 35$, then column k may be deleted.

Since the above four rules are essentially logical rules, they can be directly translated into simplification rules R1'), R2'). R3') and R4') for the CNF's of the signature functions. In this way one gets:

R1') if a clause consists of a single variable, then the variable is forced to 1 and all the clauses containing it may be deleted;

R2') if clause *i* absorbs clause *h*, then clause *h* may be deleted from the CNF;

R3') is void, since the presence of a column of 0's in the matrix is tantamount to the absence of the corresponding variable from the CNF;

R4') if variable x_j appears in every clause where variable x_k appears , then the variable x_k may be set to 0. Moreover, if x_j appears in no extra clause, then the values of x_j and x_k may be interchanged in any solution.

One should notice that rule R4') does apply since we are interested only in *minimal* false points of the CNF.

Starting from the initial product matrix, rules R1), R2), R3) and R4) are applied as far as possible, until a final *irreducible* matrix is obtained. Similarly, starting from the initial CNF, rules R1'), R2') and R4') are applied as far as possible, until a final *irreducible* CNF is obtained.

In this way, we obtain for the major, minor, and harmonic scales the following table, where:

- Each of the 13 relevant Boolean operators is labelled $f_1\, f_2\, f_3\, f_4$, where f_1 , f_2 , f_3 , f_4 are the values of p*q for (p,q) = (0,0), (0,1), (1,0) and (1,1), respectively; and the 13 operators are sorted in reverse lexicographic order: F1 = 0001, F2 =0010, F3 = 0011, ... Thus, e.g., F6 = 0110 is the XOR operator, and the NAND operator F8 = 1000 is missing, since the corresponding set covering problem is infeasible - the corresponding CNF is identically null.
- m = number of constraints in the final irreducible set covering problem = number of clauses in the final irreducible CNF.
- n = number of variables in the final irreducible set covering problem = number of variables in the final irreducible CNF.

Scale		major		minor		harmonic	
Operator	Label	m	n	m	n	m	n
F1	0001	6	5	5	5	9	7
F2	0010	–	–	–	–	–	–
F3	0011	–	–	–	–	–	–
F4	0100	–	–	–	–	–	–
F5	0101	35	12	35	12	35	12
F6	0110	–	–	–	–	–	–
F7	0111	–	–	–	–	–	–
F9	1001	6	5	5	5	13	9
F10	1010	35	12	35	12	35	12
F11	1011	–	–	–	–	–	–
F12	1100	–	–	–	–	–	–
F13	1101	–	–	–	–	–	–
F14	1110	–	–	–	–	–	–

Table 2

In all those cases where both m and n are labelled "–", preprocessing alone solves the set covering problem. At the opposite extreme, for F5 and F10 the initial product matrices are irreducible. In the case of the C major scale, the irreducible matrix both for F1 and F9 is the following one (for the time being, please ignore the meaning of the boldface entries, which will be explained in the next section).

C	**D**	**E♭**	**G**	**B**
0	**1**	1	**1**	0
1	**1**	0	0	0
0	0	1	0	1
0	0	0	1	1
1	0	0	**1**	0
1	0	1	0	0

Table 3 - Irreducible matrix for C major and F1 and F9

Also in the case of the C minor scale the irreducible matrices for F1 and F9 coincide: here both are equal to the incidence matrix of a C_5 .

Instead, in the case of the C harmonic scale the irreducible matrices for F1 and F9 are different: they are given by the matrices

C	D	E♭	F	G	A♭	B
0	0	1	0	0	1	1
0	0	1	0	1	0	1
1	1	0	0	1	0	0
0	1	0	0	**0**	**1**	**1**
1	0	1	0	0	1	0
0	1	0	1	1	0	0
1	0	0	1	**1**	**1**	**0**
1	0	1	1	0	0	0
0	1	0	0	**1**	**0**	**1**

Table 4 - Irreducible matrix for C harmonic and F1

and

C	D	E♭	F	G	A♭	A	B♭	B
0	1	0	0	1	0	0	1	1
0	0	1	0	0	1	0	1	1
0	1	0	0	0	1	0	1	1
0	0	1	0	0	1	1	0	1
1	0	1	0	0	1	1	0	0
0	0	**1**	**0**	**1**	0	0	1	1
1	1	0	0	1	0	0	0	0
0	1	0	0	0	1	1	0	1
1	0	1	0	0	1	0	1	0
0	1	**0**	**1**	**1**	0	0	0	0
1	0	0	1	1	1	1	0	0
1	0	**1**	**1**	**0**	0	0	0	0
0	1	0	0	1	0	1	0	1

Table 5 - Irreducible matrix for C harmonic and F9

respectively.

Remark 1. Notice that in the F5 case the initial product matrix is irreducible and, since in this case $p*q = q$, it coincides with the scale-note matrix with the reference scale (here, C major) omitted. Similarly, in the F10 case the initial product matrix is irreducible and, since in this case $p*q = \overline{q}$, its entries are actually the complements of the entries of the F5 matrix.

4. Easy solution of the set covering problems

In the previous section, we have seen that, for all reference scales and all Boolean operators but F1, F5, F9, and F10, the set covering problem can be fully and easily solved by simple standard preprocessing rules. Here we focus our attention on the remaining four operators, for which the above signalled no-branching phenomenon was left unanswered. We thought that an explanation might be given in terms of the special structure of their coefficient matrices. Could they possibly be totally unimodular or at least totally balanced? (recall that a binary matrix is totally unimodular if the determinant of each square submatrix is equal to 1, 0, or -1; and it is totally balanced if no square submatrix is the incidence matrix of a cycle C_k of length $k \geq 3$).

It is well known [2] that totally unimodular matrices are also totally balanced, and that any set covering problem whose coefficient matrix is totally balanced can be efficiently solved by linear programming (LP). In this case LP, although looking in principle also at fractional solutions, always yields a binary optimal solution.

Unfortunately**,** we found the following negative result.

Fact: None of the nonempty irreducible matrices (i.e., those relative to F1, F5, F9, and F10) is totally balanced.

Actually, in view of Thm. 1, we can restrict ourselves to the three C scales. We have pointed out above that in the case of the C minor scale, both the irreducible matrices for F1 and F9 are equal to the incidence matrix of a C_5 . It can be directly checked that all the other nonempty irreducible matrices contain as a 3×3 submatrix the incidence matrix of a C_3 . One such submatrix is shown in boldface in the previous section for each of the irreducible matrices relative to the C major and the C minor scales and the operators F1 and F9. For all the C scales, the C_3 submatrix for F5 shows up in the lower right corner; and the one for F10 is induced by the rows 29, 31, 35; and the columns 9, 10, 12.

The above negative result weakens our hopes to solve the above 4 set covering problems directly as linear programs. In fact, for example, CPLEX 10.0 produces for the C major scale and the F1, F5, F9, F10 operators the following fractional optimal solutions:

F1:	2/3	0	1/3	0	1/3	1/3	0	0	0	0	0	2/3
F5:	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7
F9:	2/3	0	0	0	1/3	1/3	0	0	1/3	0	0	2/3
F10:	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5	1/5

On the other hand, it is well known [1] that, in principle, any set covering problem can be solved as a linear program, provided that suitable additional linear inequalities (cutting planes) are added to it. There is one problem, though: the number of these extra inequalities, if the set covering problem is arbitrary, might be huge.

However, it turns out that, quite luckily, for the above 10 set covering problems one cutting plane is enough! Moreover, the cutting plane has the same expression "sum of the variables ≥ 3" for all the 10 problems but one, which has an easy direct solution anyhow.

In the first place, we settle the simple, but exceptional, case of the C harmonic scale and the F5 operator. We need some preliminary definition. Given a real vector [matrix] with n components [columns], consecutively numbered 1, … , n, the *circular distance* between components [columns] *i* and *j* is defined to be $| i - j | \bmod \lfloor n/2 \rfloor$. In particular, *i* and *j* are said to be circularly consecutive if their circular distance is 1.

Theorem 2. The unique minimum cardinality signature for the C harmonic scale and the F5 operator is given by {A, B♭}.

Proof. The only scales whose characteristic vectors feature two (circularly) consecutive 0's are the harmonic ones. For example, in the C harmonic scale the two consecutive 0's show up in columns A, B♭. No other scale, harmonic or not, features two 0's in columns A, B♭. It follows that all the rows of the irreducible product matrix P – which is equal to the scale-note matrix with the C harmonic row dropped - are covered by the two columns A, B♭. On the other hand, no single column can cover all the rows of P, since the column sums of P are either 20 or 21. ☐

Now we prove the main result of this section.

Theorem 3: For all the four Boolean operators F1, F5, F9, F10 and for all scales, (except for the single combination of the F5 operator and the harmonic scales) the (binary) feasible solutions to the corresponding set covering problem satisfy the inequality

$$\text{sum of the variables} \geq 3. \qquad (5)$$

Proof. In view of Thm. 1, we can restrict ourselves to the three C scales (in one case, it will be more convenient to refer to A scales). In every case, the proof is by contradiction. In the following x_i , i = 1, ..., 12, is the variable of the set covering problem corresponding to the i-th note of the set **N**.

C major; F1, F9

In this case, the reduced matrix is the one shown in Table 5. Suppose that there is a binary feasible solution **x** such that

$$x_1 + x_5 + x_6 + x_{10} + x_{12} \leq 2. \qquad (6)$$

Adding up the all the set covering constraints, and recalling that the coefficient matrix is given by Table 5, one gets

$$3x_1 + 2x_5 + 3x_6 + 3x_{10} + 2x_{12} \geq 6$$

or equivalently,

$$x_1 + 2/3\, x_5 + x_6 + x_{10} + 2/3\, x_{12} \geq 2,$$

which together with (6) implies

$$1/3\, x_5 + 1/3\, x_{12} \leq 0.$$

Hence, $x_5 = x_{12} = 0$. But then the set covering constraints imply $x_1 = x_6 = x_{10} = 1$, contradicting (6).

C major, minor; F5

After Remark 1, in the F5 case the initial product matrix P is irreducible and it is equal to the scale-note matrix with the reference scale (either C major or C minor) dropped. We preliminarily notice the following

Fact: For each k = 2, …, 6, the 3×12 matrix whose rows are the characteristic vectors of the three C scales

	C	C#	D	E♭	E	F	F#	G	A♭	A	B♭	B
C	1	0	1	0	1	1	0	1	0	1	0	1
C m	1	0	1	1	0	1	0	1	0	1	0	1
C h	1	0	1	1	0	1	0	1	1	0	0	1

has a 2×2 submatrix whose columns are at circular distance k and whose entries are all 0's.

The following table provides the rows and the columns of one such submatrix for each k = 2, … , 6.

k	Rows		Columns	
2	C	C m	F#	A♭
3	C m	C h	C#	E
4	C	C m	F#	B♭
5	C	C m	C#	F#
6	C m	C h	E	B♭

Next, we notice that no single column can cover all the rows of the product matrix P, since the column-sums of P are again either 20 or 21. Furthermore, no pair i, j of columns can cover all the rows of P, since there exists anyhow some row intersecting i and j in a pair of 0's. This is certainly true if i and j are circularly consecutive, since then one such row is provided by a suitable harmonic scale; but it remains true when i and j are at any other circular distance k = 2, …, 6. Indeed, in view of the above Fact, there is in P some row h with two 0's at distance k. But then there is some row, possibly obtained from h by a suitable cyclic permutation – which leaves circular distances unchanged – having two 0's in columns i and j. Therefore, at least 3 columns are needed to cover all the rows of P, so the inequality (5) must hold also in the present case.

A major, minor, harmonic; F10

Consider at first the A major scale. Suppose that there is a binary feasible solution **x** such that the inequality

$$x_1 + \ldots + x_{12} \leq 2 \tag{7}$$

holds. Adding up the three constraints corresponding to three C scales, we obtain

$$3x_2 + x_4 + 2x_5 + 3x_7 + 2x_9 + x_{10} + 3x_{11} \geq 3,$$

that is,

$$2x_1 + 2\backslash 3\ x_4 + 4\backslash 3\ x_5 + 2x_7 + 4\backslash 3\ x_9 + 2\backslash 3\ x_{10} + 2x_{11} \geq 2,$$

which together with (7) implies the inequality

$$x_1 + 1\backslash 3\ x_5 + x_7 + 1\backslash 3\ x_9 + x_{11} \geq x_2 + x_3 + 1\backslash 3\ x_4 + x_6 + x_8 + 1\backslash 3\ x_{10} + x_{12}\ , \tag{8}$$

The same reasoning, but with reference to the three C# scales, yields the inequality

$$x_2 + 1\backslash 3\ x_6 + x_8 + 1\backslash 3\ x_{10} + x_{12} \geq x_1 + x_3 + x_4 + 1\backslash 3\ x_5 + x_7 + x_9 + 1\backslash 3\ x_{11}. \tag{9}$$

Adding up (8) and (9) and then simplifying, one obtains the inequality

$$2\backslash 3\ x_{11} \geq 2x_3 + 4\backslash 3 x_4 + 2\backslash 3\ x_6 + 2\backslash 3\ x_9\ ,$$

which implies

$$x_3 = x_4 = 0, \quad x_{11} \geq x_6 + x_9 , \tag{10}$$

By the cyclicity of the coefficient matrix, we obtain the following system of relations

$$\begin{aligned}
&x_3 = x_4 = 0, \quad x_{11} \geq x_6 + x_9 , \\
&x_4 = x_5 = 0, \quad x_{12} \geq x_7 + x_{10} , \\
&x_5 = x_6 = 0, \quad x_1 \geq x_8 + x_{11} , \\
&x_6 = x_7 = 0, \quad x_2 \geq x_9 + x_{12} , \qquad (11)\\
&x_7 = x_8 = 0, \quad x_3 \geq x_{10} + x_1 , \\
&x_8 = x_9 = 0, \quad x_4 \geq x_{11} + x_2 ,
\end{aligned}$$

whose only solution is the zero vector (0, … ,0). Clearly, such vector cannot satisfy the set covering constraints, so we get a contradiction to the feasibility of **x**.

The above proof for the F10 operator can be repeated almost *verbatim* for the minor and the harmonic scales, since the fact that the A minor or the A harmonic row, rather than the A major one, is missing from the coefficient matrix has no influence on the rest of the proof.

C minor; F1, F9

In this case, as remarked above the reduced matrix is the incidence matrix of a C_5. Then the desired cutting plane is but the *Chvátal cut* [1], [2] obtained by the addition of all set covering constraints,

$$2x_1 + 2x_3 + 2x_4 + 2x_8 + 2x_{12} \geq 5,$$

followed by division by 2,

$$x_1 + x_3 + x_4 + x_8 + x_{12} \geq 5/2,$$

and rounding up of the r.h.s.

C harmonic; F1

Suppose that there exists a binary feasible solution **x** such that

$$x_1 + x_3 + x_4 + x_6 + x_8 + x_9 + x_{12} \leq 2. \tag{12}$$

Adding up all the set covering constraints corresponding to the irreducible matrix (see Table 4), we obtain

$$4x_1 + 4x_3 + 4x_4 + 3x_6 + 5x_8 + 4x_9 + 4x_{12} \geq 9,$$

that is,

$$8/9x_1 + 8/9x_3 + 8/9x_4 + 2/3x_6 + 10/9x_8 + 8/9x_9 + 8/9x_{12} \geq 2,$$

which together with (12) implies the inequality

$$1/9x_8 \geq 1/9x_1 + 1/9x_3 + 1/9x_4 + 1/3x_6 + 1/9x_9 + 1/9x_{12}. \tag{13}$$

If x_8 would be equal to 0, then by (13) all the other variables would be equal to 0, and the set covering constraints would be violated; hence

$$x_8 = 1; \tag{14}$$

furthermore, since the left hand side of (13) must be equal to 1/9, one has

$$x_6 = 0. \tag{15}$$

Hence (13) becomes

$$1 \geq x_1 + x_3 + x_4 + x_9 + x_{12}. \tag{16}$$

By adding up the constraints corresponding to the 4th and the 8th row of the irreducible matrix (see Table 4) and in view of the relation (15), one obtains:

$$x_1 + x_3 + x_4 + x_9 + x_{12} \geq 2,$$

contradicting (16).

C harmonic; F9

As in the previous cases, suppose that there is a binary feasible solution **x** such that

$$x_1 + x_3 + x_4 + x_6 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \leq 2. \tag{17}$$

Adding up all the constraints corresponding to the irreducible matrix (see Table 5), we obtain

$$5x_1 + 6x_3 + 6x_4 + 3x_6 + 6x_8 + 7x_9 + 5x_{10} + 5x_{11} + 7x_{12} \geq 13,$$

that is,

$$10/13x_1 + 12/13x_3 + 12/13x_4 + 6/13x_6 + 12/13x_8 + 14/13x_9 + 10/13x_{10} + 10/13x_{11} + 14/13x_{12} \geq 2,$$

which together with (17) implies the inequality

$$1/13x_9 + 1/13x_{12} \geq 3/13x_1 + 1/13x_3 + 1/13x_4 + 7/13x_6 + 1/13x_8 + 3/13x_{10} + 3/13x_{11}. \tag{18}$$

Since the left hand side is less than or equal to 2/13, one gets

$$x_1 = x_6 = x_{10} = x_{11} = 0. \tag{19}$$

Hence (18) becomes

$$x_9 + x_{12} \geq x_3 + x_4 + x_8. \tag{20}$$

By the constraints corresponding to the 10th and 12th row of the irreducible matrix (see Table 5) and by the relations (19), one obtains:

$$x_3 + x_8 \geq 1 \text{ and } x_4 \geq 1$$

which, together with (20), imply

$$x_9 + x_{12} \geq x_3 + x_4 + x_8 \geq 2.$$

Hence the following relation must hold

$$x_9 + x_{12} + x_4 \geq 3$$

contradicting (17). □

Given a binary linear program, its *continuous relaxation* is the linear program obtained when one replaces the restriction that the variables be binary by the requirement that they take their values in the real interval [0,1].

Consider the continuous relaxation of the binary linear program obtained by the addition of the constraint (5) to the set covering problem corresponding to any scale and any of the operators F1, F5, F9, and F10 (but the single combination of the F5 operator and the harmonic scales, for which, as seen above, an easy solution is available anyhow). Then the following stunning outcome can be observed: it turns out that the optimal solution output by CPLEX 10.0 is binary for any such linear program, even though there might exist also optimal fractional solutions (e. g., for the operator F10 and the C h scale, (1/2, 1/2, 1/2, 0, 0, 0, 0, 1/2, 1/2, 0, 0, 1/2) is an optimal solution). The above integrality result holds if one starts from the initial product matrix (together with inequality (5)) and it still holds also when such initial matrix is replaced by the reduced matrix after the preprocessing stage, with the single exception of the operator F9 for the C harmonic scale: in this case CPLEX 10.0 yields an half-integer optimal solution. It should be stressed that these results are obtained without any resort to the rounding heuristic [7] of CPLEX 10.0 .

Summing up, the set covering problem is easily solvable as a small linear program for the operators F1, F5, F9, and F10 and for all the 36 tonal scales. For all the remaining operators and all the scales, preprocessing alone by four simple (and standard) rules is enough.

Acknowledgements

We thank Gilbert Nouno and Malik Mezzadri for stimulating discussions.

References

[1] V. Chvátal, Edmonds Polytopes and a Hierarchy of Combinatorial Problems, *Discrete Mathematics* 4 (1973) 305-337.

[2] G. Cornuéjols, Combinatorial Optimization: Packing and Covering, SIAM, 2001.

[3] M. R. Garey and D. S. Johnson, *Computers and Intractability: A guide to the theory of NP-completeness*, W. H. Freeman and co., New York, 1979.

[4] P.L. Hammer and Y.Crama, *Boolean Functions*, (Cambridge University Press, 2010), forthcoming.

[5] C. Mannino and A. Sassano, Solving hard set covering problems, *Operations Research Letters*, (18) 1995, 1-5.

[6] B. Simeone, G. Nouno, M. Mezzadri, I. Lari: “A Boolean theory of signatures for tonal scales”, Technical Report n. 13, 2010, Department of Statistical Sciences, La Sapienza University, Rome.

[7] ILOG CPLEX 10.0 User's Manual, ILOG S.A. and ILOG, Inc, 2006.