# Explicit solutions of fractional diffusion equations via Generalized Gamma Convolution

Mirko D'Ovidio*

April 16, 2010

**Abstract** In this paper we deal with Mellin convolution of generalized Gamma densities which brings to integrals of modified Bessel functions of the second kind. Such convolutions allow us to write explicitly the solutions of the time-fractional diffusion equations involving the adjoint operators of a square Bessel process and a Bessel process.

**Keywords**: Mellin convolution formula, generalized Gamma r.v.'s, Stable subordinators, Fox functions, Bessel processes, Modified Bessel functions.

## 1 Introduction

We study the role of the Mellin convolution formula in finding solutions of fractional diffusion equations. In particular, our result allows us to write explicitly the distribution of both stable subordinator and its inverse process. By so doing we find out the explicit solution to space-fractional or time-fractional equation governing respectively stable or inverse process. This result turns out to be useful for representing the solutions to the following fractional diffusion equations

$$D_t^\nu \tilde{u}_\nu^{1,\mu} = \left( x \frac{\partial^2}{\partial x^2} - (\mu - 2) \frac{\partial}{\partial x} \right) \tilde{u}_\nu^{1,\mu}, \quad x \geq 0,\ t > 0,\ \mu > 0 \tag{1.1}$$

and

$$D_t^\nu \tilde{u}_\nu^{2,\mu} = \frac{1}{2^2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{(2\mu - 1)}{x} \right) \tilde{u}_\nu^{2,\mu}, \quad x \geq 0,\ t > 0,\ \mu > 0. \tag{1.2}$$

We present, for $\nu \in 1/(2n+1)$, $n \in \mathbb{N}$, the explicit solutions of (1.1) and (1.2) in terms of integrals of modified Bessel function of the second kind ($K_\nu$) whereas, for $\nu \in (0,1)$, we obtain the solutions of (1.1) and (1.2) in terms of Fox's functions.

## 2 Preliminaries

Let us introduce the Fox's H-functions as the class of functions uniquely identified by their Mellin transforms. A function $f$ for which the following Mellin transform exists

$$\mathcal{M}[f(\cdot)](\eta) = \int_0^\infty x^\eta f(x) \frac{dx}{x}, \quad \Re\{\eta\} > 0 \tag{2.1}$$

can be written in terms of H-functions by observing that

$$\int_0^\infty x^\eta H_{p,q}^{m,n} \left[ x \left| \begin{array}{l} (a_i, \alpha_i)_{i=1,..,p} \\ (b_j, \beta_j)_{j=1,..,q} \end{array} \right. \right] \frac{dx}{x} = \mathcal{M}_{p,q}^{m,n}(\eta), \quad \Re\{\eta\} > 0 \tag{2.2}$$

where

$$\mathcal{M}_{p,q}^{m,n}(\eta) = \frac{\prod_{j=1}^m \Gamma(b_j + \eta\beta_j) \prod_{i=1}^n \Gamma(1 - a_i - \eta\alpha_i)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \eta\beta_j) \prod_{i=n+1}^p \Gamma(a_i + \eta\alpha_i)}. \tag{2.3}$$

*Dipartimento di Statistica, Probabilità e Statistiche Applicate. Sapienza Università di Roma, P.le Aldo Moro, 5 - 00185 Rome (Italy). e-mail: mirko.dovidio@uniroma1.it

The inverse Mellin transform is defined as

$$f(x) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \mathcal{M}[f(\cdot)](\eta) x^{-\eta} d\eta \tag{2.4}$$

at all points $x$ where $f$ is continuous and for some real $\theta$. For an extensive discussion on this function see Fox [5]; Mathai and Saxena [10]. For the Mellin convolution

$$f_1 \star f_2(x) = \int_0^\infty f_1(x/s) f_2(s) \frac{ds}{s}, \quad x > 0 \tag{2.5}$$

we have (the Mellin convolution formula)

$$\mathcal{M}\left[f_1 \star f_2(\cdot)\right](\eta) = \mathcal{M}\left[f_1(\cdot)\right](\eta) \times \mathcal{M}\left[f_2(\cdot)\right](\eta). \tag{2.6}$$

Also, we recall the following connections between Mellin transform and both integer and fractional order derivatives. In particular, we consider a rapidly decreasing function $f : [0, \infty) \mapsto [0, \infty)$, if there exists $a \in \mathbb{R}$ such that

$$\lim_{x \to 0^+} x^{a-k-1} \frac{d^k}{dx^k} f(x) = 0, \quad k = 0, 1, \dots, n-1, \quad n \in \mathbb{N}, \ x \in \mathbb{R}_+ \tag{2.7}$$

then we have

$$\mathcal{M}\left[\frac{d^n}{dx^n} f(\cdot)\right](\eta) = (-1)^n \frac{\Gamma(\eta)}{\Gamma(\eta - n)} \mathcal{M}\left[f(\cdot)\right](\eta - n) \tag{2.8}$$

and, for $0 < \alpha < 1$

$$\mathcal{M}\left[\frac{d^\alpha}{dx^\alpha} f(\cdot)\right](\eta) = \frac{\Gamma(\eta)}{\Gamma(\eta - \alpha)} \mathcal{M}\left[f(\cdot)\right](\eta - \alpha) \tag{2.9}$$

(see [8] for details). The fractional derivative appearing in (2.9) must be understood as follows

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} \frac{d^n f}{ds^n}(s)\, ds, \quad n-1 < \alpha < n \tag{2.10}$$

that is the Caputo sense. Consider the Riemann-Liouville fractional derivative

$$D_x^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\alpha-1} f(s)\, ds, \quad n-1 < \alpha < n. \tag{2.11}$$

Gorenflo and Mainardi (see e.g. [6]) have shown that

$$D_x^\alpha f = \frac{d^\alpha}{dx^\alpha} f - \sum_{k=0}^{n-1} \frac{d^k}{dx^k} f \Bigg|_{x=0^+} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}, \quad n-1 < \alpha < n. \tag{2.12}$$

# 3 Mellin convolution of generalized Gamma densities

In this section we introduce and study the Mellin convolution of generalized gamma densities. In the literature it is well-known that generalized Gamma r.v.'s possesses density laws given by

$$Q_\mu^\gamma(z) = \gamma \frac{z^{\gamma\mu - 1} e^{-z^\gamma}}{\Gamma(\mu)}, \quad z > 0, \gamma > 0, \mu > 0. \tag{3.1}$$

Our discussion here concerns the function

$$g_\mu^\gamma(x,t) = \frac{1}{t} Q_\mu^{|\gamma|}\left(\frac{x}{t}\right), \quad x > 0, \ t > 0, \gamma \neq 0, \ \mu > 0$$

for which we define the convolution

$$g_{\mu_1}^{\gamma_1} \star g_{\mu_2}^{\gamma_2}(x,t) = \int_0^\infty g_{\mu_1}^{\gamma_1}(x,s) g_{\mu_2}^{\gamma_2}(s,t) ds = \frac{1}{t} \int_0^\infty Q_{\mu_1}^{|\gamma_1|}(x/s) Q_{\mu_2}^{|\gamma_2|}(s/t) \frac{ds}{s}. \tag{3.2}$$

Formula (3.2) is a Mellin convolution in the sense that

$$\mathcal{M}\left[g_{\mu_1}^{\gamma_1} \star g_{\mu_2}^{\gamma_2}(\cdot,t)\right](\eta) = \mathcal{M}\left[g_{\mu_1}^{\gamma_1}(\cdot,t^{1/2})\right](\eta) \times \mathcal{M}\left[g_{\mu_2}^{\gamma_2}(\cdot,t^{1/2})\right](\eta) \tag{3.3}$$

as a straightforward calculation shows. Throughout the paper we also deal with the integral

$$f_1 \circ f_2(x,t) = \int_0^\infty f_1(x,s) f_2(s,t) ds \tag{3.4}$$

(for some well-defined $f_1$, $f_2$) which is not, in general, a Mellin convolution.

We now introduce the generalized Gamma process. Roughly speaking, the function

$$g_\mu^\gamma(x,t) = |\gamma| \frac{x^{\gamma\mu-1}\, e^{-\frac{x^\gamma}{t^\gamma}}}{t^{\gamma\mu}\Gamma(\mu)}, \qquad x \geq 0,\ t > 0,\ \gamma \neq 0,\ \mu > 0 \tag{3.5}$$

can be viewed as the distribution of a generalized Gamma process $\{G_t^{\gamma,\mu},\ t > 0\}$ in the sense that $\forall t$ the distribution of the r.v. $G_t^{\gamma,\mu}$ is a generalized Gamma distribution. Thus, we make some abuse of language by considering a process without its covariance structure. In the literature there are several non-equivalent definitions of the distribution on $\mathbb{R}_+^n$ of Gamma distributions. See e.g. Kotz et al. [9] for a comprehensive discussion. Distribution (3.5) satisfies the p.d.e.

$$\frac{\partial}{\partial t} g_\mu^\gamma = |\gamma|\, t^{\gamma-1}\, \mathcal{G}_{\gamma,\mu} g_\mu^\gamma, \qquad x \geq 0,\ t > 0 \tag{3.6}$$

where

$$\mathcal{G}_{\gamma,\mu} f = \frac{1}{\gamma^2}\left\{ \frac{\partial}{\partial x} x^{2-\gamma} \frac{\partial}{\partial x} f - (\gamma\mu - 1) \frac{\partial}{\partial x} x^{1-\gamma} f \right\}, \quad x \geq 0,\ t > 0 \tag{3.7}$$

and $\gamma \neq 0$ (see D'Ovidio [4]). For $\gamma = 1$ and $\gamma = 2$ in (3.5) we obtain respectively the density law of a $2\mu$-dimensional squared Bessel process $\{BESSQ_{t/2}^{(2\mu)},\ t > 0\}$ and a $2\mu$-dimensional Bessel process $\{BES_{t/2}^{(2\mu)},\ t > 0\}$ both starting from zero. Some interesting distributions can be realized through Mellin convolution of distribution $g_\mu^\gamma$. Indeed, after some algebra we have

$$g_{\mu_1}^\gamma \star g_{\mu_2}^{-\gamma}(x,t) = \frac{\gamma}{B(\mu_1,\mu_2)} \frac{x^{\gamma\mu_1-1} t^{\gamma\mu_2}}{(t^\gamma + x^\gamma)^{\mu_1+\mu_2}}, \quad x > 0,\ t > 0,\ \gamma > 0 \tag{3.8}$$

and

$$g_{\mu_1}^{-\gamma} \star g_{\mu_2}^{\gamma}(x,t) = \frac{\gamma}{B(\mu_1,\mu_2)} \frac{x^{\gamma\mu_2-1} t^{\gamma\mu_1}}{(t^\gamma + x^\gamma)^{\mu_1+\mu_2}}, \quad x > 0,\ t > 0,\ \gamma > 0 \tag{3.9}$$

where $B(\cdot,\cdot)$ is the Beta function (see e.g. Gradshteyn and Ryzhik [7, formula 8.384]). Moreover, in the light of the Mellin convolution formula (2.6), the following holds

$$\mathcal{M}\left[g_{\mu_1}^\gamma \star g_{\mu_2}^{-\gamma}(\cdot,t)\right](\eta) = \mathcal{M}\left[g_{\mu_1}^{-\gamma} \star g_{\mu_2}^{\gamma}(\cdot,t)\right](\eta). \tag{3.10}$$

A further density arising from convolution can be presented. In particular, for $\gamma \neq 0$, we have

$$g_{\mu_1}^\gamma \star g_{\mu_2}^\gamma(x,t) = \frac{2|\gamma|\ (x^\gamma/t^\gamma)^{\frac{\mu_1+\mu_2}{2}}}{x\,\Gamma(\mu_1)\Gamma(\mu_2)} K_{\mu_2-\mu_1}\left(2\sqrt{\frac{x^\gamma}{t^\gamma}}\right), \quad x > 0,\ t > 0 \tag{3.11}$$

which turns out to be very useful later on. The function $K_\nu$ appearing in (3.11) is the modified Bessel function of imaginary argument (see e.g [7, formula 8.432]). For the sake of completeness we write down the following Mellin transforms

$$\mathcal{M}\left[g_\mu^\gamma(\cdot,t)\right](\eta) = \frac{\Gamma\left(\frac{\eta-1}{\gamma} + \mu\right)}{\Gamma(\mu)} t^{\eta-1}, \quad t > 0,\ \Re\{\eta\} > 1 - \gamma\mu,\ \gamma \neq 0, \tag{3.12}$$

and

$$\mathcal{M}\left[g_\mu^\gamma(x,\cdot)\right](\eta) = \frac{\Gamma\left(\mu - \frac{\eta}{\gamma}\right)}{\Gamma(\mu)} x^{\eta-1}, \quad x>0,\ \Re\{\eta\} > \gamma\mu,\ \gamma \neq 0. \tag{3.13}$$

Formula (3.13) suggests that

$$\mathcal{M}\left[g_{\mu_1}^{\gamma_1} \star g_{\mu_2}^{\gamma_2}(x,\cdot)\right](\eta) = \mathcal{M}\left[g_{\mu_1}^{\gamma_1}(x^{1/2},\cdot)\right](\eta) \times \mathcal{M}\left[g_{\mu_2}^{\gamma_2}(x^{1/2},\cdot)\right](\eta). \tag{3.14}$$

For the one-dimensional GGP depicted above we are able to define the inverse generalized Gamma process $\{E_t^{\gamma,\mu},\ t>0\}$ (IGGP in short) by means of the following relation

$$Pr\{E_t^{\gamma,\mu} < x\} = Pr\{G_x^{\gamma,\mu} > t\}.$$

The density law $e_\mu^\gamma = e_\mu^\gamma(x,t)$ of IGGP can be carried out by observing that

$$e_\mu^\gamma(x,t) = Pr\{E_t^{\gamma,\mu} \in dx\}/dx = \int_t^\infty \frac{d}{dx} g_\mu^\gamma(s,x)\, ds, \quad x>0,\ t>0 \tag{3.15}$$

and

$$\begin{aligned}
\mathcal{M}\left[e_\mu^\gamma(\cdot,t)\right](\eta) &= \int_t^\infty \mathcal{M}\left[\frac{d}{dx} g_\mu^\gamma(s,\cdot)\right](\eta)\, ds, \quad \Re\{\eta\} < 1 \\
&= -(\eta-1)\int_t^\infty \mathcal{M}\left[g_\mu^\gamma(s,\cdot)\right](\eta-1)\, ds = [\text{by (3.13)}] \\
&= -(\eta-1)\int_t^\infty \frac{\Gamma\left(\mu - \frac{\eta-1}{\gamma}\right)}{\Gamma(\mu)} s^{\eta-2}\, ds = \frac{\Gamma\left(\mu - \frac{\eta-1}{\gamma}\right)}{\Gamma(\mu)} t^{\eta-1}
\end{aligned}$$

where we have used formula (2.8). From (2.3) and the fact that

$$H_{p,q}^{m,n}\left[x \left| \begin{array}{l} (a_i,\alpha_i)_{i=1,..,p} \\ (b_j,\beta_j)_{j=1,..,q} \end{array}\right.\right] = c\, H_{p,q}^{m,n}\left[x^c \left| \begin{array}{l} (a_i,c\alpha_i)_{i=1,..,p} \\ (b_j,c\beta_j)_{j=1,..,q} \end{array}\right.\right] \tag{3.16}$$

for all $c>0$, we have that

$$e_\mu^\gamma(x,t) = \frac{\gamma}{x} H_{1,1}^{1,0}\left[\frac{t^\gamma}{x^\gamma} \left| \begin{array}{l} (\mu,0) \\ (\mu,1) \end{array}\right.\right], \quad x>0,\ t>0,\ \gamma>0. \tag{3.17}$$

By observing that $\mathcal{M}\left[e_\mu^\gamma(\cdot,t)\right](1) = 1$, we immediately verify that (3.17) integrates to unity. The density law $g_\mu^\gamma$ can be expressed in terms of H functions as well. We have

$$g_\mu^\gamma(x,t) = \frac{\gamma}{x} H_{1,1}^{1,0}\left[\frac{x^\gamma}{t^\gamma} \left| \begin{array}{l} (\mu,0) \\ (\mu,1) \end{array}\right.\right], \quad x>0,\ t>0,\ \gamma>0. \tag{3.18}$$

In view of (3.17) and (3.18) we can argue that

$$E_t^{\gamma,\mu} \stackrel{law}{=} G_t^{-\gamma,\mu} \stackrel{law}{=} 1/G_t^{\gamma,\mu}, \quad t>0,\ \gamma>0,\ \mu>0$$

and $e_\mu^\gamma(x,t) = g_\mu^{-\gamma}(x,t)$, $\gamma>0$, $x>0$, $t>0$.

**Remark 1.** We notice that the inverse process $\{E_t^{1,1/2},\ t>0\}$ can be written as

$$E_t^{1,1/2} = \inf\{s;\ B(s) = \sqrt{2t}\}$$

where $B$ is a standard Brownian motion. Thus, $E^{1,1/2}$ can be interpreted as the first-passage time of a standard Brownian motion through the level $\sqrt{2t}$.

In what follows we will consider the Mellin convolution $e_{\bar{\mu}}^{\star n}(x,t) = e_{\mu_1} \star \ldots \star e_{\mu_n}(x,t)$ (see formulae (2.6) and (3.2)) where $\bar{\mu} = (\mu_1, \ldots, \mu_n)$, $\mu_i > 0$, $i = 1, 2, \ldots, n$ and for the sake of simplicity, $e_\mu(x,t) = e_\mu^1(x,t)$. For the density law $e_{\bar{\mu}}^{\star n}(x,t)$, $x > 0$, $t > 0$ we have

$$\mathcal{M}\left[e_{\bar{\mu}}^{\star n}(\cdot,t)\right](\eta) = \prod_{i=1}^{n} \mathcal{M}\left[e_{\mu_i}(\cdot, t^{1/n})\right](\eta) = \prod_{i=1}^{n} \frac{\Gamma(\mu_i + 1 - \eta)}{\Gamma(\mu_i)} t^{\eta - 1} \tag{3.19}$$

with $\Re\{\eta\} < 1$. Furthermore, for the Mellin convolution $g_{\bar{\mu}}^{\gamma,\star n}(x,t) = g_{\mu_1}^{\gamma} \star, \ldots, \star g_{\mu_n}^{\gamma}(x,t)$ we have

$$\mathcal{M}\left[g_{\bar{\mu}}^{\gamma,\star n}(\cdot,t)\right](\eta) = \prod_{i=1}^{n} \mathcal{M}\left[g_{\mu_i}^{\gamma}(\cdot, t^{1/n})\right](\eta) = \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\eta-1}{\gamma} + \mu_i\right)}{\Gamma(\mu_i)} t^{\eta-1} \tag{3.20}$$

with $\Re\{\eta\} > 1 - \min_i\{\mu_i\}$.

## 4 Stable subordinators

The $\nu$-stable subordinators $\{\tilde{\tau}_t^{(\nu)},\ t > 0\}$, $\nu \in (0,1)$, are defined as non-decreasing, (totally) positively skewed, Lévy processes with Laplace transform

$$E\exp\{-\lambda \tilde{\tau}_t^{(\nu)}\} = \exp\{-t\lambda^\nu\}, \quad t > 0, \quad \lambda > 0 \tag{4.1}$$

and characteristic function

$$E\exp\{i\xi\tilde{\tau}_t^{(\nu)}\} = \exp\{-t\Psi_\nu(\xi)\}, \quad \xi \in \mathbb{R} \tag{4.2}$$

where

$$\Psi_\nu(\xi) = \int_0^\infty (1 - e^{-i\xi u}) \frac{\nu}{\Gamma(1-\nu)} \frac{du}{u^{\nu+1}} \tag{4.3}$$

(see Bertoin [3]; Zolotarev [15]). After some algebra we get

$$\Psi_\nu(\xi) = \sigma|\xi|^\nu \left(1 - i\,\mathrm{sgn}(\xi)\tan\left(\frac{\pi\nu}{2}\right)\right) = |\xi|^\nu \exp\left\{-i\frac{\pi\nu}{2}\frac{\xi}{|\xi|}\right\}.$$

For the density law of the $\nu$-stable subordinator $\{\tilde{\tau}_t^{(\nu)}, t > 0\}$, say $h_\nu = h_\nu(x,t)$, $x > 0$, $t > 0$ we have the $t$-Mellin transforms

$$\mathcal{M}\left[\hat{h}_\nu(\xi,\cdot)\right](\eta) = |\xi|^{-\eta\nu} \exp\left\{i\frac{\pi\eta\nu}{2}\frac{\xi}{|\xi|}\right\}\Gamma(\eta) \tag{4.4}$$

and

$$\mathcal{M}\left[\tilde{h}_\nu(\lambda,\cdot)\right](\eta) = \lambda^{-\eta\nu}\Gamma(\eta) \tag{4.5}$$

where $\hat{h}_\nu(\xi,t) = \mathcal{F}[h_\nu(\cdot,t)](\xi)$ is the Fourier transform appearing in (4.2) and $\tilde{h}_\nu(\lambda,t) = \mathcal{L}[h_\nu(\cdot,t)](\lambda)$ is the Laplace transform (4.1). By inverting (4.4) we obtain the Mellin transform with respect to $t$ of the density $h_\nu$ which reads

$$\begin{aligned}
\mathcal{M}[h_\nu(x,\cdot)](\eta) =& \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} \mathcal{M}\left[\hat{h}(\xi,\cdot)\right](\eta) d\xi \\
=& \frac{\Gamma(\eta)\Gamma(1-\eta\nu)}{2\pi}\left\{\frac{e^{i\frac{\pi\eta\nu}{2}}}{(ix)^{1-\eta\nu}} + \frac{e^{-i\frac{\pi\eta\nu}{2}}}{(-ix)^{1-\eta\nu}}\right\} \\
=& \frac{\Gamma(\eta)\Gamma(1-\eta\nu)}{2\pi\, x^{1-\eta\nu}}\left\{\exp\left\{-i\frac{\pi}{2} + i\pi\eta\nu\right\} + \exp\left\{i\frac{\pi}{2} - i\pi\eta\nu\right\}\right\} \\
=& \frac{\Gamma(\eta)\Gamma(1-\eta\nu)}{\pi\, x^{1-\eta\nu}} \sin\pi\eta\nu = \frac{\Gamma(\eta)}{\Gamma(\eta\nu)} x^{\eta\nu-1}, \quad x > 0,\ \nu \in (0,1)
\end{aligned} \tag{4.6}$$

where $\Re\{\eta\nu\} \in (0,1)$. Formula (4.6) can be also obtained by inverting (4.5). We are also able to evaluate the Mellin transform with respect to $x$ of the density law $h_\nu$. From (4.4) and the fact that

$$\int_0^\infty x^{\eta-1} e^{-i\xi x} dx = \frac{\Gamma(\eta)}{(i\xi)^\eta}, \quad \text{where} \quad (\pm i\xi)^\nu = |\xi|^\nu \exp\left\{\pm i \frac{\nu\pi}{2} \frac{\xi}{|\xi|}\right\}, \quad \nu \in (0,1) \tag{4.7}$$

we obtain

$$\begin{aligned}
\mathcal{M}\left[h_\nu(\cdot,t)\right](\eta) =& \frac{\Gamma(\eta)}{2\pi} \int_{\mathbb{R}} |\xi|^{-\eta} \exp\left\{-i\frac{\pi\eta}{2}\frac{\xi}{|\xi|} - t\Psi_\nu(\xi)\right\} d\xi \\
=& \frac{\Gamma(\eta)}{2\pi}\left\{e^{-i\frac{\pi\eta}{2}} \int_0^\infty \xi^{-\eta} e^{-t\Phi_\nu(\xi)} d\xi + e^{i\frac{\pi\eta}{2}} \int_0^\infty \xi^{-\eta} e^{-t\Phi_\nu(-\xi)} d\xi\right\} \\
=& \frac{\Gamma(\eta)}{2\pi\nu} \Gamma\left(\frac{1-\eta}{\nu}\right) t^{\frac{\eta-1}{\nu}} \left\{e^{i\pi(1-\eta)} + e^{-i\pi(1-\eta)}\right\} \\
=& \Gamma\left(\frac{1-\eta}{\nu}\right) \frac{t^{\frac{\eta-1}{\nu}}}{\nu\,\Gamma(1-\eta)}, \quad \Re\{\eta\} \in (0,1),\ t>0.
\end{aligned} \tag{4.8}$$

We investigate the relationship between stable subordinators and their inverse processes. For a $\nu$-stable subordinator $\{\tilde{\tau}_t^{(\nu)},\ t>0\}$ and an inverse process $\{L_t^{(\nu)},\ t>0\}$ (ISP in short) such that

$$Pr\{L_t^{(\nu)} < x\} = Pr\{\tilde{\tau}_x^{(\nu)} > t\}$$

we have the following relationship between density laws

$$l_\nu(x,t) = Pr\{L_t^{(\nu)} \in dx\}/dx = \int_t^\infty \frac{d}{dx} h_\nu(s,x) ds, \qquad x>0,\ t>0. \tag{4.9}$$

The density law (4.9) can be written in terms of Fox functions by observing that

$$\begin{aligned}
\mathcal{M}\left[l_\nu(\cdot,t)\right](\eta) =& \int_t^\infty \mathcal{M}\left[\frac{d}{dx} h_\nu(s,\cdot)\right](\eta)\, ds = [\text{by (2.8)}] \\
=& -(\eta-1) \int_t^\infty \mathcal{M}\left[h_\nu(s,\cdot)\right](\eta-1)\, ds = [\text{by (4.6)}] \\
=& -\int_t^\infty \frac{\Gamma(\eta)}{\Gamma(\eta\nu-\nu)} s^{\eta\nu-\nu-1} ds \\
=& \frac{\Gamma(\eta)}{\Gamma(\eta\nu-\nu+1)} t^{\nu(\eta-1)}, \quad \Re\{\eta\} < 1/\nu,\ t>0.
\end{aligned} \tag{4.10}$$

Thus, by direct inspection of (2.3), we recognize that

$$l_\nu(x,t) = \frac{1}{t^\nu} H_{1,1}^{1,0}\left[\frac{x}{t^\nu}\left|\begin{array}{c} (1-\nu,\nu) \\ (0,1) \end{array}\right.\right], \qquad x>0,\ t>0,\ \nu(0,1). \tag{4.11}$$

Density (4.11) integrates to unity, indeed $\mathcal{M}\left[l_\nu(\cdot,t)\right](1) = 1$. The $t$-Laplace transform

$$\mathcal{L}[l_\nu(x,\cdot)](\lambda) = \lambda^{\nu-1} \exp\left\{-x\lambda^\nu\right\}, \qquad \lambda>0,\ \nu\in(0,1) \tag{4.12}$$

comes directly from the fact that

$$\int_0^\infty e^{-\lambda t} \mathcal{M}\left[l_\nu(\cdot,t)\right](\eta)\, dt = \frac{\Gamma(\eta)}{\lambda^{\eta\nu-\nu+1}} = \int_0^\infty x^{\eta-1} \mathcal{L}\left[l_\nu(x,\cdot)\right](\lambda)\, dx.$$

From (4.12) we recognize that $\mathcal{L}\left[l_\nu(\cdot,t)\right](\lambda) = E_\nu(-\lambda t^\nu)$ where $E_\beta$ is the well-known Mittag-Leffler function which can be also written as follows

$$E_\nu(-\lambda t^\nu) = \frac{1}{\pi} \int_0^\infty \exp\left\{-\lambda^{1/\nu} t x\right\} \frac{x^{\nu-1}\sin\pi\nu}{1+2x^\nu\cos\pi\nu + x^{2\nu}} dx, \quad t>0,\ \lambda>0 \tag{4.13}$$

Distribution $l_\nu$ satisfies the fractional Cauchy problem $\frac{\partial^\nu}{\partial t^\nu} l_\nu = \frac{\partial}{\partial x} l_\nu$, $x > 0$, $t > 0$ subject to $l_\nu(x,0) = \delta(x)$ and $l_\nu(0,t) = 0$ where the fractional derivative must be understood as in (2.10). The governing equation of $l_\nu$ can be also presented by considering the Riemann-Liouville derivative (2.11) and the relation (2.12) (see e.g. Meerschaert and Scheffler [11]; Baeumer et al. [1]). It is well-known that the ratio involving two independent stable subordinator $\{ {}_1\tilde{\tau}_t^{(\nu)}, t > 0\}$ and $\{ {}_2\tilde{\tau}_t^{(\nu)}, t > 0\}$ has distribution, $\forall t$, given by

$$Pr\{ {}_1\tilde{\tau}_t^{(\nu)} / {}_2\tilde{\tau}_t^{(\nu)} \in dw\}/dw = \frac{1}{\pi} \frac{w^{\nu-1} \sin \pi\nu}{1 + 2w^\nu \cos \pi\nu + w^{2\nu}}, \quad w > 0, t > 0. \tag{4.14}$$

Here we study the ratio of two independent inverse stable processes $\{ {}_1L_t^{(\nu)}, t > 0\}$ and $\{ {}_2L_t^{(\nu)}, t > 0\}$ by evaluating its Mellin transform as follows

$$E\left\{ {}_1L_t^{(\nu)} / {}_2L_t^{(\nu)} \right\}^{\eta-1} = \mathcal{M}\left[l_\nu(\cdot,t)\right](\eta) \times \mathcal{M}\left[l_\nu(\cdot,t)\right](2-\eta) = \frac{1}{\nu} \frac{\sin \nu\pi - \eta\nu\pi}{\sin \eta\pi} \tag{4.15}$$

with $\Re\{\eta\} \in (0,1)$. By inverting (4.15) we obtain

$$\frac{1}{\nu\pi} \frac{\sin \nu\pi}{1 + 2x \cos \nu\pi + x^2} = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \frac{\sin \nu\pi - \eta\nu\pi}{\sin \eta\pi} x^{-\eta} d\eta \tag{4.16}$$

for some real $\theta \in (0,1)$. From (4.14) and (4.16) we can argue that

$$\left( {}_1\tilde{\tau}_t^{(\nu)} / {}_2\tilde{\tau}_t^{(\nu)} \right)^\nu \stackrel{law}{=} {}_1L_t^{(\nu)} / {}_2L_t^{(\nu)}, \quad \forall t > 0. \tag{4.17}$$

We notice that the equivalence in law (4.17) is independent of $t$ as the formulae (4.14) and (4.16) entail. The distribution $h_\nu \circ l_\nu(x,t)$ of the process $\{\tilde{\tau}^{(\nu)}_{L_t^{(\nu)}}, t > 0\}$ has Mellin transform (by making use of the formulae (4.8) and (4.10)) given by

$$\mathcal{M}\left[h_\nu \circ l_\nu(\cdot,t)\right](\eta) = \mathcal{M}\left[h_\nu(\cdot,1)\right](\eta) \times \mathcal{M}\left[l_\nu(\cdot,t)\right]\left(\frac{\eta-1}{\nu} + 1\right) = \frac{1}{\nu} \frac{\sin \pi\eta}{\sin \pi \frac{1-\eta}{\nu}} t^{\eta-1}, \quad t > 0 \tag{4.18}$$

with $\Re\{\eta\} \in (0,1)$. Thus, we can infer that

$$\tilde{\tau}^{(\nu)}_{L_t^{(\nu)}} \stackrel{law}{=} t \times {}_1\tilde{\tau}_t^{(\nu)} / {}_2\tilde{\tau}_t^{(\nu)} \quad t > 0 \tag{4.19}$$

and $h_\nu \circ l_\nu(x,t) = t^{-1} r(x/t)$ where $r(w)$ is that in (4.14). For the process $\{L^{(\nu)}_{\tilde{\tau}_t^{(\nu)}}, t > 0\}$ with distribution $l_\nu \circ h_\nu(x,t)$ we obtain (from (4.10) and (4.8))

$$\mathcal{M}\left[l_\nu \circ h_\nu(\cdot,t)\right](\eta) = \mathcal{M}\left[l_\nu(\cdot,1)\right](\eta) \times \mathcal{M}\left[h_\nu(\cdot,t)\right](\eta\nu - \nu + 1) = \frac{1}{\nu} \frac{\sin \pi\nu - \pi\eta\nu}{\sin \pi\eta} t^{\eta-1} \tag{4.20}$$

with $\Re\{\eta\} \in (0,1)$ and thus

$$L^{(\nu)}_{\tilde{\tau}_t^{(\nu)}} \stackrel{law}{=} t \times {}_1L_t^{(\nu)} / {}_2L_t^{(\nu)}, \quad t > 0. \tag{4.21}$$

We have that $l_\nu \circ h_\nu(x,t) = t^{-1} k(x/t)$ where $k(x)$ is that in (4.16).

# 5 Main results

In this section we consider compositions of processes whose governing equations are fractional diffusion equations. These processes represent anomalous diffusions. When we consider compositions involving Markov processes and stable subordinators we still have Markov processes. Here we study Markov processes with random time which is the inverse of a stable subordinator. Such a process is not belonging to the family of stable subordinators (see (4.13)) and the resultant composition is not, in general, a Markov process. This somehow explains the role of the fractional derivative appearing in the governing equations. Such governing equations and the compositions governed by them, have been studied by various authors, see e.g. Orsingher and Beghin [14], Baeumer et al. [1, 2]. Hereafter we exploit the Mellin convolution of generalized Gamma densities in order to write explicitly the solutions of fractional diffusion equations. We firstly present a result which relates the density law $h_\nu$ with the convolution $e^{\star n}_{\bar{\mu}}$ introduced in Section 3. To do this we also introduce the time-stretching function $\varphi_m(s) = (s/m)^m$, $m \geq 1$, $s \in (0, \infty)$. We state the following Lemma.

**Lemma 1.** *The Mellin convolution $e^{\star n}_{\bar{\mu}}(x, \varphi_{n+1}(t))$ where $\mu_i = i\,\nu$, for $i = 1, 2, \ldots, n$ is the density law of a $\nu$-stable subordinator $\{\tilde{\tau}^{(\nu)}_t,\ t > 0\}$ with $\nu = 1/(n+1)$, $n \in \mathbb{N}$.*

*Proof.* From (3.19) we have that

$$\mathcal{M}\left[e^{\star n}_{\bar{\mu}}(\cdot, \varphi_{n+1}(t))\right](\eta) = \frac{\prod_{i=1}^{n} \Gamma\left(1 - \eta + \mu_i\right)}{\prod_{i=1}^{n} \Gamma\left(\mu_i\right)} \left(\varphi_{n+1}(t)\right)^{\eta - 1}. \tag{5.1}$$

From Gradshteyn and Ryzhik [7, formula 8.335.3] we deduce that

$$\prod_{k=1}^{n} \Gamma\left(\frac{k}{n+1}\right) = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{n+1}}, \quad n \in \mathbb{N} \tag{5.2}$$

and formula (5.1) reduces to

$$\mathcal{M}\left[e^{\star n}_{\bar{\mu}}(\cdot, \varphi_{n+1}(t))\right](\eta) = \frac{\prod_{i=1}^{n} \Gamma\left(1 - \eta + \mu_i\right)}{(2\pi)^{n/2}\sqrt{\nu}} \left(\varphi_{n+1}(t)\right)^{\eta - 1}. \tag{5.3}$$

Furthermore, by making use of the (product theorem) relation

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx - 1/2} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \tag{5.4}$$

(see Gradshteyn and Ryzhik [7, formula 3.335]) formula (5.3) becomes

$$\mathcal{M}\left[e^{\star n}_{\bar{\mu}}(\cdot, \varphi_n(t))\right](\eta) = \frac{\Gamma\left(\frac{1-\eta}{\nu}\right)(2\pi)^{n/2}(n+1)^{\eta/\nu - n}}{\Gamma\left(1 - \eta\right)(2\pi)^{n/2}} \left(\varphi_{n+1}(t)\right)^{\eta - 1} = \frac{\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma\left(1 - \eta\right)} t^{\frac{\eta - 1}{\nu}}$$

(with $\Re\{\eta\} \in (0, 1)$) which coincides with (4.8). □

In the light of the last result we are able to write explicitly the density law of stable subordinators. For $\nu = 1/2$, Lemma 1 says that

$$e^{\star 1}_{\bar{\mu}}(x, \varphi_2(t)) = e_{1/2}(x, (t/2)^2) = \frac{x^{-1/2-1} e^{-\frac{t^2}{4x}}}{t^{-1}\sqrt{4}\,\Gamma\left(\frac{1}{2}\right)}, \quad x > 0,\ t > 0 \tag{5.5}$$

which is the well-known density law of a 1/2-stable subordinator or the first-passage time of a standard Brownian motion trough the level $t/\sqrt{2}$. For $\nu = 1/3$, from (3.11), we obtain

$$e^{\star 2}_{\bar{\mu}}(x, \varphi_3(t)) = e_{1/3} \star e_{2/3}(x, (t/3)^3) = \frac{1}{3\pi} \frac{t^{3/2}}{x^{3/2}} K_{\frac{1}{3}}\left(\frac{2}{3^{3/2}} \frac{t^{3/2}}{\sqrt{x}}\right), \quad x > 0,\ t > 0. \tag{5.6}$$

For $\nu = 1/4$, by (3.11) (and the commutativity under $\star$), we have

$$e^{\star 3}_{\bar{\mu}}(x, \varphi_4(t)) = e_{2/4} \star (e_{1/4} \star e_{3/4})(x, (t/4)^4)$$

where $K_{1/2}(z) = \sqrt{\pi/2z}\exp\{-z\}$ (see [7, formula 8.469]). We notice that

$$\mathcal{M}\left[e^{\star 3}_{\bar{\mu}}(\cdot, \varphi_4(t))\right](\eta) = \mathcal{M}\left[h_{1/2} \circ h_{1/2}(\cdot, t)\right](\eta)$$

which is in line with the well-known fact that

$$E \exp\left\{-\lambda\, {}_1\tilde{\tau}^{(\nu_1)}_{{}_2\tilde{\tau}^{(\nu_2)}_t}\right\} = E \exp\left\{-\lambda^{\nu_1}\, {}_2\tilde{\tau}^{(\nu_2)}_t\right\} = \exp\{-t\lambda^{\nu_1 \nu_2}\}, \tag{5.7}$$

$0 < \nu_i < 1$, $i = 1, 2$. For $\nu = 1/5$, by exploiting twice (3.11) (and the commutativity under $\star$), we can write down

$$\begin{aligned} e_{\bar{\mu}}^{\star 4}(x, (t/5)^5) =& (e_{1/5} \star e_{2/5}) \star (e_{3/5} \star e_{4/5})(x, (t/5)^5) \\ =& \frac{t^{7/2}}{5^3 \pi^2 \, x^{3/10+1}} \int_0^\infty s^{-2/5-1} K_{\frac{1}{5}} \left( 2\sqrt{\frac{s}{x}} \right) K_{\frac{1}{5}} \left( \frac{2}{5^{5/2}} \frac{t^{5/2}}{\sqrt{s}} \right) ds \end{aligned} \tag{5.8}$$

or equivalently

$$\begin{aligned} e_{\bar{\mu}}^{\star 4}(x, (t/5)^5) =& (e_{1/5} \star e_{3/5}) \star (e_{2/5} \star e_{4/5})(x, (t/5)^5) \\ =& \frac{t^3}{5^{5/2} \pi^2 \, x^{2/5+1}} \int_0^\infty s^{-1/5-1} K_{\frac{2}{5}} \left( 2\sqrt{\frac{s}{x}} \right) K_{\frac{2}{5}} \left( \frac{2}{5^{5/2}} \frac{t^{5/2}}{\sqrt{s}} \right) ds. \end{aligned} \tag{5.9}$$

For $\nu = 1/(2n+1)$, $n \in \mathbb{N}$, by using repeatedly (3.11) we arrive at

$$h_\nu(x,t) = \frac{x^{\nu/2} \, t^{1/\nu - 3/2}}{\nu^{2-1/\nu} \pi^{1/2\nu - 1/2}} \mathcal{K}_\nu^{\circ\, n} \left( x, (\nu t)^{1/\nu} \right), \quad x > 0, \, t > 0 \tag{5.10}$$

where

$$\mathcal{K}_\nu^{\circ\, n}(x,t) = \int_0^\infty \dots \int_0^\infty \mathcal{K}_\nu(x, s_1) \dots \mathcal{K}_\nu(s_{n-1}, t) \, ds_1 \dots ds_{n-1} \tag{5.11}$$

is the integral (3.4) (as the symbol $\circ\, n$ denote) where $n$ functions are involved and $\mathcal{K}_\nu(x,t) = x^{-2\nu-1} K_\nu \left( 2\sqrt{t/x} \right)$, $x > 0$, $t > 0$. We state a similar result for the density law $l_\nu$ and the convolution $g_{\bar{\mu}}^{\gamma, \star n}$ (see Section 3). Let us consider the time-stretching function $\psi_m(s) = m\, s^{1/m}$, $s \in (0, \infty)$, $m \in \mathbb{N}$, ($\psi = \varphi^{-1}$ where $\varphi$ has been introduced in the previous Lemma).

**Lemma 2.** *The Mellin convolution* $g_{\bar{\mu}}^{1/\nu, \star(1/\nu - 1)}(x, \psi_{1/\nu}(t))$ *where* $\mu_i = i\,\nu$, $i = 1, 2, \dots, (1/\nu - 1)$ *and* $1/\nu \in \mathbb{N}$, *is the density law of a* $\nu$*-inverse process* $\{L_t^{(\nu)}, \, t > 0\}$.

*Proof.* The proof can be carried out as the proof of Lemma 1. □

We obtain that $l_{1/2}(x,t) = g_{1/2}^2(x, 2t^{1/2}) = e^{-\frac{x^2}{4t}} / \sqrt{\pi t}$, $x > 0$, $t > 0$. Moreover, by making use of (3.11) and (5.2), we have that

$$l_{1/3}(x,t) = g_{2/3}^3 \star g_{1/3}^3 (x, 3t^{1/3}) = \frac{1}{\pi} \sqrt{\frac{x}{t}} K_{\frac{1}{3}} \left( \frac{2}{3^{3/2}} \frac{x^{3/2}}{\sqrt{t}} \right), \quad x > 0, \, t > 0 \tag{5.12}$$

and $l_{1/4}(x,t) = g_{3/4}^4 \star g_{2/4}^4 \star g_{1/4}^4 (x, 4t^{1/4})$ follows (thank to the commutativity under $\star$) from

$$g_{3/4}^4 \star g_{2/4}^4 \star g_{1/4}^4 (x,t) = g_{1/2}^4 \star (g_{3/4}^4 \star g_{1/4}^4)(x,t) = \frac{2^{3/2}}{\pi} \frac{x}{t} \int_0^\infty \exp\left\{ -(sx)^4 - \frac{2}{(st)^2} \right\} ds$$

where $g_{3/4}^4 \star g_{1/4}^4(x,t)$ is given by (3.11) and $K_{1/2}(z) = \sqrt{\pi/2z} \exp\{-z\}$ (see [7, formula 8.469]). In a more general setting, by making use of (3.11) we can write down

$$g_{\bar{\mu}}^{1/\nu, \star(1/\nu - 1)}(x,t) = \frac{1}{\nu^{1/2\nu}} \left( \frac{x}{\pi^2 t^3} \right)^{\frac{1-\nu}{4\nu}} \mathcal{Q}_{\frac{1-\nu}{2}}^{\circ n}(x,t), \quad \nu = 1/(2n+1), \, n \in \mathbb{N} \tag{5.13}$$

where the symbol "$\circ\, n$ " stands for the integral (3.4) where $n$ functions $\mathcal{Q}_{\frac{1-\nu}{2}}$ are involved and

$$\mathcal{Q}_{\frac{1-\nu}{2}}(x,t) = K_{\frac{1-\nu}{2}} \left( 2\sqrt{(x/t)^{1/\nu}} \right), \quad x > 0, \, t > 0. \tag{5.14}$$

Now, we present the main result of this paper concerning the explicit solutions of fractional diffusion equations. The solution to $\frac{\partial^\alpha}{\partial t^\alpha} u = \frac{\partial^2}{\partial x^2} u$, for some $0 < \alpha < 1$ where the fractional derivative is that in (2.10), has been studied in Meerschaert and Scheffler [12], Orsingher and Beghin [13],Orsingher and Beghin [14]. This is the simplest case of fractional diffusion. Here we study a generalized problem which brings to fractional diffusion equations involving the adjoint operators of both Bessel and squared Bessel processes. Consider the distribution $\tilde{u}_\nu^{\gamma,\mu} = \tilde{g}_\mu^\gamma \circ l_\nu$ where $\tilde{g}_\mu^\gamma(x,t) = g_\mu^\gamma(x,t^{1/\gamma})$ and the Mellin transform of $\tilde{u}_\nu^{\gamma,\mu}$ which reads

$$\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta) = \frac{\Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)\Gamma\left(\frac{\eta-1}{\gamma}+1\right)}{\Gamma(\mu)\Gamma\left(\frac{\eta-1}{\gamma}\nu+1\right)} t^{\frac{\eta-1}{\gamma}\nu}, \quad 1-\gamma\mu < \Re\{\eta\} < 1+\gamma/\nu-\gamma. \tag{5.15}$$

We can state the following result.

**Theorem 1.** *Let the previous setting prevail. The solution to*

$$D_t^\alpha \tilde{u}_\nu^{\gamma,\mu} = \mathcal{G}_{\gamma,\mu}\tilde{u}_\nu^{\gamma,\mu}, \quad x\geq 0,\ t>0 \tag{5.16}$$

*can be represented, for* $\nu = 1/(2n+1)$, $n\in\mathbb{N}$, *in terms of generalized Gamma convolution as*

$$\tilde{u}_\nu^{\gamma,\mu}(x,t) = \gamma\frac{x^{2\mu-1}\nu^{\frac{1-3\nu}{4\nu}}}{(\pi^2 t^{3\nu})^{\frac{1-\nu}{4\nu}}}\int_0^\infty s^{\frac{1}{4\nu}-\frac{1}{4}-\mu}e^{-x^\gamma/s}\, v_\nu(s,t)\, ds, \quad x\geq 0,\ t>0 \tag{5.17}$$

*where* $\mathcal{G}_{\gamma,\mu}$ *is the operator appearing in* (3.7) *and*

$$v_\nu(s,t) = \int_0^\infty \dots \int_0^\infty \mathcal{Q}_{\frac{1-\nu}{2}}(s,s_1)\dots\mathcal{Q}_{\frac{1-\nu}{2}}(s_{n-1},t^\nu/\nu)ds_1\dots ds_{n-1}.$$

*Moreover, for* $\nu\in(0,1)$, *we have*

$$\tilde{u}_\nu^{\gamma,\mu}(x,t) = \frac{\gamma}{xt^{\nu/\gamma}}H_{2,2}^{2,0}\left[\frac{x^\gamma}{t^\nu}\left|\begin{matrix}(1,\nu); & (\mu,0)\\ (1,1); & (\mu,1)\end{matrix}\right.\right] \tag{5.18}$$

*in terms of H Fox functions.*

*Proof.* By exploiting the property (2.8) of the Mellin transform and the fact that

$$\int_0^\infty x^{\eta-1}x^\theta f(x)dx = \mathcal{M}\left[f(\cdot)\right](\eta+\theta),$$

for the operator (3.7) we have that

$$\begin{aligned}
&\mathcal{M}\left[\mathcal{G}_{\gamma,\mu}\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta)\\
&=-\frac{1}{\gamma^2}(\eta-1)\mathcal{M}\left[\frac{\partial}{\partial x}\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta-\gamma+1)+\frac{1}{\gamma^2}(\gamma\mu-1)(\eta-1)\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta-\gamma)\\
&=\frac{1}{\gamma^2}(\eta-1)(\eta-\gamma)\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta-\gamma)+\frac{1}{\gamma^2}(\gamma\mu-1)(\eta-1)\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta-\gamma)\\
&=\frac{1}{\gamma^2}(\eta-1)(\eta-1+\gamma\mu-\gamma)\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta-\gamma)
\end{aligned}$$

where $\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta)$ is that in (5.15). We obtain

$$\begin{aligned}
&\mathcal{M}\left[\mathcal{G}_{\gamma,\mu}\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta)\\
&=\frac{1}{\gamma^2}(\eta-1)(\eta-1+\gamma\mu-\gamma)\frac{\Gamma\left(\frac{\eta-\gamma-1}{\gamma}+\mu\right)\Gamma\left(\frac{\eta-\gamma-1}{\gamma}+1\right)}{\Gamma(\mu)\Gamma\left(\frac{\eta-\gamma-1}{\gamma}\nu+1\right)}t^{\frac{\eta-\gamma-1}{\gamma}\nu}\\
&=\frac{1}{\gamma}(\eta-1)\frac{\Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)\Gamma\left(\frac{\eta-1}{\gamma}\right)}{\Gamma(\mu)\Gamma\left(\frac{\eta-1}{\gamma}\nu-\nu+1\right)}t^{\frac{\eta-1}{\gamma}\nu-\nu}\\
&=\frac{\Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)\Gamma\left(\frac{\eta-1}{\gamma}+1\right)}{\Gamma(\mu)\Gamma\left(\frac{\eta-1}{\gamma}\nu-\nu+1\right)}t^{\frac{\eta-1}{\gamma}\nu-\nu}=D_t^\nu\mathcal{M}\left[\tilde{u}_\nu^{\gamma,\mu}(\cdot,t)\right](\eta)
\end{aligned}$$

and $\tilde{u}^{\gamma,\mu}_\nu(x,t)$ solves (5.16) for $\nu \in (0,1)$. In view of Lemma 2 we can write

$$\tilde{u}^{\gamma,\mu}_\nu(\cdot,t) = \int_0^\infty \tilde{g}^\gamma_\mu(x,s)\, g^{1/\nu,\star(1/\nu-1)}_{\bar{\mu}}(s,\psi_{1/\nu}(t))\, ds \tag{5.19}$$

and by means of (5.13) result (5.17) appears. Formula (5.18) follows directly from (2.3) by considering formula (3.16) and the fact that

$$H^{m,n}_{p,q}\left[x\middle|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right] = \frac{1}{x^c}\, H^{m,n}_{p,q}\left[x\middle|\begin{array}{l}(a_i+c\alpha_i,\alpha_i)_{i=1,..,p}\\(b_j+c\beta_j,\beta_j)_{j=1,..,q}\end{array}\right] \tag{5.20}$$

for all $c \in \mathbb{R}$. □

We specialize the previous result by keeping in mind formula (2.12).

**Corollary 1.** *Let us write* $\tilde{u}^\mu_\nu(x,t) = \tilde{u}^{1,\mu}_\nu(x,t)$. *The distribution* $\tilde{u}^\mu_\nu(x,t)$, $x>0$, $t>0$ $\mu>0$, $\nu\in(0,1)$, *solves the following fractional equation*

$$\frac{\partial^\nu}{\partial t^\nu} u^\mu_\nu = \left( x\frac{\partial^2}{\partial x^2} - (\mu-2)\frac{\partial}{\partial x}\right) u^\mu_\nu. \tag{5.21}$$

In particular, for $\nu = 1/2$, we have

$$\tilde{u}^\mu_{1/2}(x,t) = \frac{x^{\mu-1}}{\sqrt{\pi t}\,\Gamma(\mu)} \int_0^\infty s^{-\mu} \exp\left\{ -\frac{x}{s} - \frac{s^2}{4t}\right\} ds, \quad x>0,\, t>0,\ \mu>0 \tag{5.22}$$

which can be seen as the distribution of the process $\{G^{1,\mu}_{|B(2t)|},\ t>0\}$ where $B$ is a standard Brownian motion run at twice its usual speed and $G^{\gamma,\mu}$ is a GGP. We notice that the governing equation of the process $\{B_{G^{1,\mu}(t)},\ t>0\}$ is given by (see D'Ovidio [4])

$$\frac{\partial}{\partial t} q = -\frac{1}{2^2}\frac{\partial}{\partial x}\left( x\frac{\partial^2}{\partial x^2} - (2\mu-2)\frac{\partial}{\partial x}\right) q, \quad x>0,\ t>0. \tag{5.23}$$

**Corollary 2.** *The distribution* $\tilde{u}^{2,\mu}_\nu = \tilde{u}^{2,\mu}_\nu(x,t)$, $x>0$, $t>0$, $\mu>0$, $\nu\in(0,1)$ *solves the following fractional equation*

$$\frac{\partial^\nu}{\partial t^\nu}\tilde{u}^{2,\mu}_\nu = \frac{1}{2^2}\left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{(2\mu-1)}{x}\right)\tilde{u}^{2,\mu}_\nu. \tag{5.24}$$

In particular, for $\nu = 1/3$, we have

$$\tilde{u}^{2,\mu}_{1/3}(x,t) = \frac{2\,x^{2\mu-1}}{\pi\Gamma(\mu)\sqrt{t}} \int_0^\infty \frac{e^{-\frac{x^2}{s}}}{s^{\mu-1/2}} K_{\frac{1}{3}}\left(\frac{2}{3^{3/2}}\frac{x^{3/2}}{\sqrt{t}}\right) ds, \quad x>0,\ t>0,\ \mu>0. \tag{5.25}$$

and for $\mu = 1/2$ we obtain

$$\tilde{u}^{2,1/2}_{1/3}(x,t) = \frac{2}{\pi^{3/2}\sqrt{t}} \int_0^\infty e^{-\frac{x^2}{s}} K_{\frac{1}{3}}\left(\frac{2}{3^{3/2}}\frac{x^{3/2}}{\sqrt{t}}\right) ds, \quad x>0,\ t>0. \tag{5.26}$$

# References


[1] B. Baeumer, M. Meerschaert, and E. Nane. Space-time duality for fractional diffusion. *Journal of Applied Probability*, 46:1100 – 1115, 2009.

[2] B. Baeumer, M. Meerschaert, and E. Nane. Fractional cauchy problems on bounded domains. *Ann. of Probab.*, 37(3):979 – 1007, 2009.

[3] J. Bertoin. *Lévy Processes.* Cambridge University Press, 1996.

[4] M. D'Ovidio. Generalized gamma process: composition and governing equations. *(Submitted)*, 2010.

[5] C. Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 68: 395 – 429, 1962.

[6] R. Gorenflo and F. Mainardi. Fractional calculus: integral and differential equations of frational order, in A. Carpinteri and F. Mainardi (Editors). *Fractals and Fractional Calculus in Continuum Mechanics*, 1997. Wien and New York, Springher Verlag.

[7] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series and products.* Academic Press, 2007. Seventh edition.

[8] A. Kilbas, H. Srivastava, and J. Trujillo. *Theory and applications of fractional differential equations (North-Holland Mathematics Studies)*, volume 204. Elsevier, Amsterdam, 2006.

[9] S. Kotz, N. Balakrishnan, and N. L. Johnson. *Continuous multivariate distributions*, volume 1. New York, Wiley, 2nd edition, 2000.

[10] A. Mathai and R. Saxena. *Generalized Hypergeometric functions with applications in statistics and physical sciences.* Lecture Notes in Mathematics, n. 348, 1973.

[11] M. Meerschaert and H. P. Scheffler. Triangular array limits for continuous time random walks. *Stoch. Proc. Appl.*, 2008.

[12] M. Meerschaert and H. P. Scheffler. Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.*, 41:623 – 638, 2004.

[13] E. Orsingher and L. Beghin. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Related Fields*, 128(1):141 – 160, 2004.

[14] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.*, 37:206 – 249, 2009.

[15] V. M. Zolotarev. *One-dimensional stable distributions, volume 65 of Translations of Mathematical Monographs.* American Mathematical Society, 1986. ISBN 0-8218-4519-5. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.