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Fractional Non-Linear, Linear and Sublinear Death Processes

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Abstract This paper is devoted to the study of a fractional version of non-linear $\mathcal{M}^{\nu}(t)$, $t > 0$, linear $M^{\nu}(t)$, $t > 0$ and sublinear $\mathfrak{M}^{\nu}(t)$, $t > 0$ death processes. Fractionality is introduced by replacing the usual integer-order derivative in the difference-differential equations governing the state probabilities, with the fractional derivative understood in the sense of Dzhrbashyan–Caputo. We derive explicitly the state probabilities of the three death processes and examine the related probability generating functions and mean values. A useful subordination relation is also proved, allowing us to express the death processes as compositions of their classical counterparts with the random time process $T_{2\nu}(t)$, $t > 0$. This random time has one-dimensional distribution which is the solution to a Cauchy problem of the fractional diffusion equation with a reflecting boundary in the origin.

Keywords Fractional Diffusion · Dzhrbashyan–Caputo Fractional Derivative · Mittag–Leffler Functions · Linear Death Process · Non-Linear Death Process · Sublinear Death Process · Subordinated Processes

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1 Introduction

We assume that we have a population of n_0 individuals or objects. The components of this population might be the set of healthy people during an epidemic or the set of items being sold in a store, or even, say, melting ice pack blocks. However even a coalescence of particles can be treated in this same manner, leading to a large ensemble of physical analogues suited to the method. The main interest is to model the fading process of these objects and, in particular, to analyse how the size of the population decreases.

The classical death process is a model describing this type of phenomena and, its linear version is analysed in [1], page 90. The most interesting feature of the extinguishing population is the probability distribution

$$p_k(t) = \Pr\{M(t) = k \mid M(0) = n_0\}, \qquad t > 0,\ 0 \le k \le n_0, \tag{1.1}$$

where $M(t)$, $t > 0$ is the point process representing the size of the population at time t. If the death rates are proportional to the population size, the process is called *linear* and the probabilities (1.1) are solutions to the initial-value problem

$$\begin{cases} \frac{d}{dt}p_k(t) = \mu(k+1)p_{k+1}(t) - \mu k p_k(t), & 0 \le k \le n_0, \\ p_k(0) = \begin{cases} 1, & k = n_0, \\ 0 & 0 \le k < n_0, \end{cases} \end{cases} \tag{1.2}$$

with $p_{n_0+1}(t) = 0$.

The distribution satisfying (1.2) is

$$p_k(t) = \binom{n_0}{k} e^{-\mu k t}\left(1 - e^{-\mu t}\right)^{n_0-k}, \qquad 0 \le k \le n_0. \tag{1.3}$$

The equations (1.2) are based on the fact that the death rate of each component of the population is proportional to the number of existing individuals.

In the non-linear case, where the death rates are μ_k, $0 \le k \le n_0$, equations (1.2) must be replaced by

$$\begin{cases} \frac{d}{dt}p_k(t) = \mu_{k+1}p_{k+1}(t) - \mu_k p_k(t), & 0 \le k \le n_0, \\ p_k(0) = \begin{cases} 1, & k = n_0, \\ 0 & 0 \le k < n_0. \end{cases} \end{cases} \tag{1.4}$$

In this paper we consider fractional versions of the processes described above, where fractionality is obtained by substitution of the integer-order derivatives appearing in (1.2) and (1.4), with the fractional derivative called Caputo or Dzhrbashyan–Caputo derivative, defined as follows

$$\begin{cases} \frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f'(s)}{(t-s)^\nu}\, ds, & 0 < \nu < 1, \\ f'(t), & \nu = 1. \end{cases} \tag{1.5}$$

The main advantage of the Dzhrbashyan–Caputo fractional derivative over the usual Riemann–Liouville fractional derivatives is that the former requires

only integer-order derivatives in the initial conditions. The population size is governed by

$$\begin{cases} \frac{d^\nu}{dt^\nu} p_k(t) = \mu_{k+1} p_{k+1}(t) - \mu_k p_k(t), \quad 0 \le k \le n_0, \\ p_k^\nu(0) = \begin{cases} 1, & k = n_0, \\ 0 & 0 \le k < n_0, \end{cases} \end{cases} \tag{1.6}$$

and is denoted by $M^\nu(t)$, $t > 0$. The distribution

$$p_k^\nu(t) = \Pr\left\{M^\nu(t) = k \mid M^\nu(0) = n_0\right\}, \qquad 0 \le k \le 0, \tag{1.7}$$

is obtained explicitly and reads

$$p_k^\nu(t) = \begin{cases} E_{\nu,1}(-\mu_{n_0} t^\nu), & k = n_0, \\ \prod\limits_{j=k+1}^{n_0} \mu_j \sum\limits_{m=k}^{n_0} \dfrac{E_{\nu,1}(-\mu_m t^\nu)}{\prod\limits_{\substack{h=k \\ h \ne m}}^{n_0} (\mu_h - \mu_m)}, & 0 < k < n_0, \\ 1 - \sum\limits_{m=1}^{n_0} \prod\limits_{\substack{h=1 \\ h \ne m}}^{n_0} \left(\dfrac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu), & k = 0,\ n_0 > 1. \end{cases} \tag{1.8}$$

Obviously, for $k = 0$, $n_0 = 1$,

$$p_0^\nu(t) = 1 - E_{\nu,1}(-\mu_1 t^\nu). \tag{1.9}$$

The Mittag–Leffler functions appearing in (1.8) are defined as

$$E_{\nu,\gamma}(x) = \sum_{h=0}^{\infty} \frac{x^h}{\Gamma(\nu h + \gamma)}, \qquad x \in \mathbb{R}, \quad \nu, \gamma > 0. \tag{1.10}$$

For $\nu = \gamma = 1$, $E_{1,1}(x) = e^x$ and formulae (1.8) provide the explicit distribution of the classical non-linear death process.

For $\mu_k = k\mu$ the distribution of the fractional linear death process can be obtained either directly by solving the Cauchy problem (1.6) with $\mu_k = k \cdot \mu$, or by specialising (1.8) resulting in the following form

$$p_k^\nu(t) = \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0 - k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu). \tag{1.11}$$

A technical tool necessary for our manipulations is the Laplace transform of Mittag–Leffler functions which we write here for the sake of completeness:

$$\int_0^\infty e^{-zt} t^{\gamma-1} E_{\nu,\gamma}(\pm\vartheta t^\nu) dt = \frac{z^{\nu-\gamma}}{z^\nu \mp \vartheta}, \qquad \Re(z) > |\vartheta|^{\frac{1}{\nu}}. \tag{1.12}$$

Another special case is the so-called fractional sublinear death process (for sublinear birth processes consult [8]) where the death rates have the form $\mu_k = \mu(n_0 + 1 - k)$. In the sublinear process, the annihilation of particles or individuals accelerates with decreasing population size.

The distribution $\mathfrak{p}_k^\nu(t)$, $0 \leq k \leq n_0$ of the fractional sublinear death process $\mathfrak{M}^\nu(t)$, $t > 0$, is strictly related to that of the fractional linear birth process $N^\nu(t)$, $t > 0$:

$$\Pr\{\mathfrak{M}^\nu(t) = 0 \mid \mathfrak{M}^\nu(0) = n_0\} = \Pr\{N^\nu(t) > n_0 \mid N^\nu(0) = 1\}. \tag{1.13}$$

In general, the connection between the fractional sublinear death process and the fractional linear death process is expressed by the relation

$$\begin{aligned} &\Pr\{\mathfrak{M}^\nu(t) = n_0 - (k-1) \mid \mathfrak{M}^\nu(0) = n_0\} \\ &= \Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\}, \qquad 1 \leq k \leq n_0. \end{aligned} \tag{1.14}$$

This shows a sort of symmetry in the evolution of fractional linear birth and fractional sublinear death processes.

For all fractional processes considered in this paper, a subordination relationship holds. In particular, for the fractional linear death process we can write that

$$M^\nu(t) = M(T_{2\nu}(t)), \qquad 0 < \nu < 1,\ t > 0, \tag{1.15}$$

where $T_{2\nu}(t)$ is a process for which

$$\Pr\{T_{2\nu}(t) \in ds\} = ds\, q(s,t), \tag{1.16}$$

is a solution to the following Cauchy problem (see [5])

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^{2\nu}} q(s,t) = \frac{\partial^2}{\partial s^2} q(s,t), & t > 0,\ s > 0, \\ \frac{\partial}{\partial t} q(s,t)\big|_{s=0} = 0, \\ q(s,0) = \delta(s), & 0 < \nu \leq 1, \end{cases} \tag{1.17}$$

with the additional initial condition

$$q_t(s,0) = 0, \qquad 1/2 < \nu \leq 1. \tag{1.18}$$

In equation (1.15), $M(t)$, $t > 0$, represents the classical linear death process.

We also show that all the fractional death processes considered below can be viewed as classical death processes with rate $\mu \cdot \Xi$, where Ξ is a Wright-distributed random variable.

2 The fractional linear death process and its properties

In this section we derive the distribution of the fractional linear death process as well as some interesting related properties and interpretations.

Theorem 1 *The distribution of the fractional linear death process $M^\nu(t)$, $t > 0$ with n_0 initial individuals and death rates $\mu_k = \mu \cdot k$, is given by*

$$\begin{aligned} p_k^\nu(t) &= Pr\{M^\nu(t) = k \mid M^\nu(0) = n_0\} \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu), \end{aligned} \tag{2.1}$$

where $0 \leq k \leq n_0$, $t > 0$ and $\nu \in (0,1]$. The function $E_{\nu,1}(x)$ is the Mittag–Leffler function previously defined in (1.10).

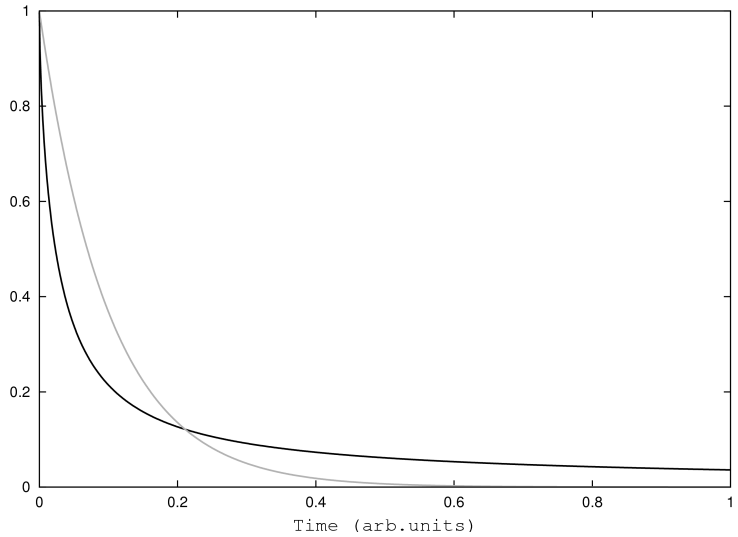

Fig. 1 Plot of $p^{0.7}_{n_0}(t)$ (in black) and $p^{1}_{n_0}(t)$ (in grey), both with $n_0 = 10$.

Proof *The state probability $p^{\nu}_{n_0}(t)$, $t > 0$ is readily obtained by applying the Laplace transform to equation* (1.6)*, with $\mu_k = \mu \cdot k$, and then transforming back the results, thus yielding*

$$p^{\nu}_{n_0}(t) = E_{\nu,1}(-n_0 \mu t^{\nu}), \qquad t > 0,\ \nu \in (0, 1]. \tag{2.2}$$

When $k = n_0 - 1$, in order to solve the related differential equation, we can write

$$\begin{aligned} z^{\nu}\mathcal{L}\left\{p_{n_0-1}\right\}(z) &= \mu n_0 \frac{z^{\nu-1}}{z^{\nu} + n_0\mu} - \mu(n_0 - 1)\mathcal{L}\left\{p_{n_0-1}\right\}(z) \qquad (2.3)\\ \Leftrightarrow \quad \mathcal{L}\left\{p_{n_0-1}\right\}(z) &= \mu n_0 z^{\nu-1} \frac{1}{z^{\nu} + n_0\mu} \cdot \frac{1}{z^{\nu} + (n_0 - 1)\mu}\\ \Leftrightarrow \quad \mathcal{L}\left\{p_{n_0-1}\right\}(z) &= n_0 z^{\nu-1}\left(\frac{1}{z^{\nu} + (n_0 + 1)\mu} - \frac{1}{z^{\nu} + n_0\mu}\right). \end{aligned}$$

By inverting equation (2.3)*, we readily obtain that*

$$p^{\nu}_{n_0-1}(t) = n_0\left(E_{\nu,1}(-(n_0 - 1)\mu t^{\nu}) - E_{\nu,1}(-n_0\mu t^{\nu})\right). \tag{2.4}$$

For general values of k, with $0 \le k < n_0$, we must solve the following Cauchy problem:

$$\begin{aligned} \frac{d^{\nu}}{dt^{\nu}} p_k(t) &= \mu(k+1)\binom{n_0}{k+1} \qquad (2.5)\\ &\times \sum_{r=0}^{n_0-k-1} \binom{n_0 - k - 1}{r}(-1)^r E_{\nu,1}(-(k + 1 + r)\mu t^{\nu}) - \mu k p_k(t), \end{aligned}$$

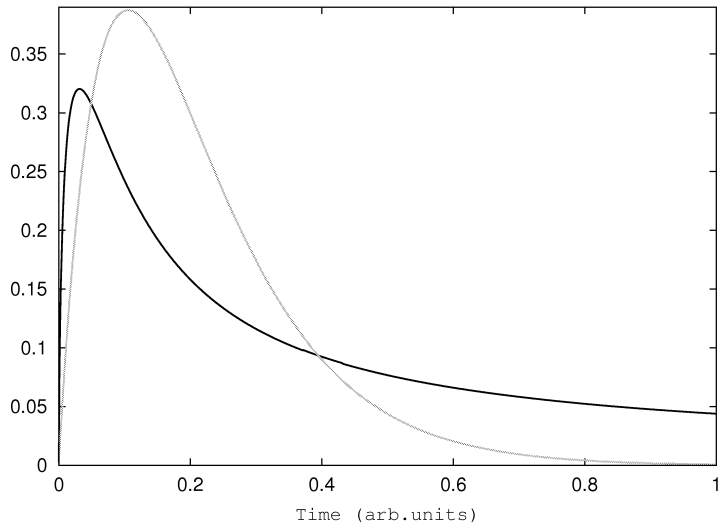

Fig. 2 Plot of $p^{0.7}_{n_0-1}(t)$ (in black) and $p^{1}_{n_0-1}(t)$ (in grey). Here $n_0 = 10$.

subject to the initial condition $p_k(0) = 0$ *and with* $\nu \in (0, 1]$. *The solution can be found by resorting to the Laplace transform, as we see in the following.*

$$z^\nu \mathcal{L}\left\{p_k\right\}(z) = \mu(k+1)\binom{n_0}{k+1} \times \sum_{r=0}^{n_0-k-1} \binom{n_0-k-1}{r}(-1)^r \frac{z^{\nu-1}}{z^\nu + (k+1+r)\mu} - \mu k \mathcal{L}\left\{p_k\right\}(z). \tag{2.6}$$

The Laplace transform $\mathcal{L}\left\{p_k\right\}(z)$ *can thus be written as*

$$\begin{aligned}
&\mathcal{L}\left\{p_k\right\}(z) \\
&= \mu(k+1)\binom{n_0}{k+1} \sum_{r=0}^{n_0-k-1} \binom{n_0-k-1}{r}(-1)^r \frac{z^{\nu-1}}{z^\nu + (k+1+r)\mu} \cdot \frac{1}{z^\nu + k\mu} \\
&= \binom{n_0}{k} \sum_{r=0}^{n_0-k-1} \binom{n_0-k}{r+1}(-1)^r z^{\nu-1} \left(\frac{1}{z^\nu + k\mu} - \frac{1}{z^\nu + (k+1+r)\mu}\right) \\
&= \binom{n_0}{k} \sum_{j=1}^{n_0-k} \binom{n_0-k}{j}(-1)^{j-1} z^{\nu-1} \left(\frac{1}{z^\nu + k\mu} - \frac{1}{z^\nu + (k+j)\mu}\right) \\
&= \binom{n_0}{k} \sum_{j=1}^{n_0-k} \binom{n_0-k}{j}(-1)^j \frac{z^{\nu-1}}{z^\nu + (k+j)\mu} \\
&\quad - \binom{n_0}{k} \frac{z^{\nu-1}}{z^\nu + k\mu} \sum_{j=1}^{n_0-k} \binom{n_0-k}{j}(-1)^j
\end{aligned} \tag{2.7}$$

$$= \binom{n_0}{k} \sum_{j=1}^{n_0-k} \binom{n_0-k}{j} (-1)^j \frac{z^{\nu-1}}{z^\nu + (k+j)\mu} + \binom{n_0}{k} \frac{z^{\nu-1}}{z^\nu + k\mu}$$

$$= \binom{n_0}{k} \sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j \frac{z^{\nu-1}}{z^\nu + (k+j)\mu}.$$

By taking now the inverse Laplace transform of (2.7)*, we obtain the claimed result* (1.11)*.*

Remark 1 When $\nu = 1$, equation (1.11) easily reduces to the distribution of the classical linear death process, i.e.

$$p_k(t) = \binom{n_0}{k} e^{-k\mu t} \left(1 - e^{-\mu t}\right)^{n_0-k}, \qquad t > 0,\ 0 \le k \le n_0. \tag{2.8}$$

In the following theorem we give a proof of an interesting subordination relation.

Theorem 2 *The fractional linear death process $M^\nu(t)$, $t > 0$ can be represented as*

$$M^\nu(t) \stackrel{i.d.}{=} M(T_{2\nu}(t)), \qquad t > 0,\ \nu \in (0, 1], \tag{2.9}$$

where $M(t)$, $t > 0$ is the classical linear death process (see e.g. [1]*, page 90) and $T_{2\nu}(t)$, $t > 0$, is a random time process whose one-dimensional distribution coincides with the solution to the following fractional diffusion equation*

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^\nu} q(s,t) = \frac{\partial^2}{\partial s^2} q(s,t), \quad s \in \mathbb{R}^+,\ t > 0,\ \nu \in (0,1], \\ u_s(0,t) = 0, \\ u(s,0) = \delta(s), \end{cases} \tag{2.10}$$

with the additional condition $u_t(s,0) = 0$ if $\nu \in (1/2, 1]$ (see [5]*).*

Proof By evaluating the Laplace transform of the generating function of the fractional linear death process $M^\nu(t)$, $t > 0$, we obtain that

$$\int_0^\infty e^{-zt} G^\nu(u,t) dt \tag{2.11}$$

$$= \int_0^\infty e^{-zt} \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r E_{\nu,1}(-(k+r)\mu t^\nu) dt$$

$$= \sum_0^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{z^{\nu-1}}{z^\nu + (k+r)\mu}$$

$$= \int_0^\infty \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r z^{\nu-1} e^{-s(z^\nu + (k+r)\mu)} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r e^{-s(k+r)\mu} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} e^{-\mu s k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r e^{-sr\mu} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} \left\{ \sum_{k=0}^{n_0} u^k \binom{n_0}{k} e^{-\mu s k} (1-e^{-\mu s})^{n_0-k} \right\} ds$$

$$= \int_0^\infty e^{-sz^\nu} z^{\nu-1} G(u,s) ds$$

$$= \int_0^\infty e^{-zt} \int_0^\infty \sum_{k=0}^{n_0} u^k Pr\{M(s)=k\} f_{T_{2\nu}}(s,t) ds\, dt$$

$$= \int_0^\infty e^{-zt} \left\{ \sum_{k=0}^{\infty} u^k Pr\{M(T_{2\nu}(t))=k\} \right\} dt,$$

and this is sufficient to prove that (2.9) *holds. Note that we used two facts. The first one is that*

$$\int_0^\infty e^{-zt} f_{T_{2\nu}}(s,t) dt = z^{\nu-1} e^{-sz^\nu}, \qquad s>0,\ z>0, \tag{2.12}$$

is the Laplace transform of the solution to (2.10)*. The second fact is that the Laplace transform of the Mittag–Leffler function is*

$$\int_0^\infty e^{-zt} E_{\nu,1}(-\vartheta t^\nu) dt = \frac{z^{\nu-1}}{z^\nu + \vartheta}. \tag{2.13}$$

In figures 1 and 2, we compare the behaviour of the fractional probabilities $p^{0.7}_{n_0}(t)$ and $p^{0.7}_{n_0-1}(t)$ with their classical counterparts $p^1_{n_0}(t)$ and $p^1_{n_0-1}(t)$, $t>0$. What emerges from the inspection of both figures is that, for large values of t, the probabilities, in the fractional case, decrease more slowly than $p^1_{n_0}(t)$ and $p^1_{n_0-1}(t)$. The probability $p^{0.7}_{n_0-1}(t)$, increases initially faster than $p^1_{n_0-1}(t)$, but after a certain time lapse, $p^1_{n_0-1}(t)$ dominates $p^{0.7}_{n_0-1}(t)$.

Remark 2 For $\nu = 1/2$, in view of the integral representation

$$E_{\frac{1}{2},1}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2+2xw} dw, \qquad x \in \mathbb{R}, \tag{2.14}$$

we extract from (1.11) that

$$p^{\frac{1}{2}}_k(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r e^{-2w(k+r)\mu t^{\frac{1}{2}}} \tag{2.15}$$

$$= \int_0^\infty \frac{e^{-y^2}_{4t}}{\sqrt{\pi t}} p^1_k(y) dy = \Pr\left\{M(|B(t)|)=k\right\},$$

where $B(t)$, $t>0$ is a Brownian motion with volatility equal to 2.

Remark 3 We can interpret formula (1.11) in an alternative way, as follows. For each integer $k \in [0, n_0]$ we have that

$$\begin{aligned} p_k^\nu(t) &= \Pr\{M^\nu(t) = k \mid M^\nu(0) = n_0\} \qquad (2.16)\\ &= \int_0^\infty p_k(s) \Pr\{T_{2\nu}(t) \in ds\} \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \int_0^\infty e^{-\mu(k+r)s} \Pr\{T_{2\nu}(t) \in ds\} \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \int_0^\infty e^{-\mu(k+r)s} t^{-\nu} W_{-\nu,1-\nu}(-st^{-\nu}) ds \\ &= \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \int_0^\infty e^{-\xi\mu(k+r)t^\nu} W_{-\nu,1-\nu}(-\xi) d\xi \\ &= \int_0^\infty W_{-\nu,1-\nu}(-\xi) \Pr\{M_\xi(t^\nu) = k \mid M_\xi(0) = n_0\} \, d\xi, \end{aligned}$$

where $W_{-\nu,1-\nu}(-\xi)$ is the Wright function defined as

$$W_{-\nu,1-\nu}(-\xi) = \sum_{r=0}^{\infty} \frac{(-\xi)^r}{r! \Gamma\left(1 - \nu(r+1)\right)}, \qquad 0 < \nu \leq 1. \qquad (2.17)$$

We therefore obtain an interpretation in terms of a classical linear death process $M_\Xi(t)$, $t > 0$ evaluated on a new time scale and with random rate $\mu \cdot \Xi$, where Ξ is a random variable, $\xi \in \mathbb{R}^+$, with Wright density

$$f_\Xi(\xi) = W_{-\nu,1-\nu}(-\xi), \qquad \xi \in \mathbb{R}^+. \qquad (2.18)$$

From equation (1.6) with $\mu_k = k \cdot \mu$, the related fractional differential equation governing the probability generating function, can be easily obtained, leading to

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} G^\nu(u,t) = -\mu u(u-1) \frac{\partial}{\partial u} G^\nu(u,t), \quad \nu \in (0,1], \\ G^\nu(u,0) = u^{n_0}. \end{cases} \qquad (2.19)$$

From this, and by considering that $\mathbb{E}M^\nu(t) = \frac{\partial}{\partial u} G^\nu(u,t)\big|_{u=1}$, we obtain that

$$\begin{cases} \frac{d^\nu}{dt^\nu} \mathbb{E}M^\nu(t) = -\mu \mathbb{E}M^\nu(t), \quad \nu \in (0,1], \\ \mathbb{E}M^\nu(t) = n_0. \end{cases} \qquad (2.20)$$

Equation (2.20) is easily solved by means of the Laplace transforms, yielding

$$\mathbb{E}M^\nu(t) = n_0 E_{\nu,1}(-\mu t^\nu), \qquad t > 0, \ \nu \in (0,1]. \qquad (2.21)$$

Remark 4 The mean value $\mathbb{E}M^\nu(t)$ can also be directly calculated.

$$\begin{aligned}
\mathbb{E}M^\nu(t) &= \sum_{k=0}^{n_0} k p_k^\nu(t) \qquad (2.22)\\
&= \sum_{k=0}^{n_0} k \binom{n_0}{k} \sum_{r=k}^{n_0} \binom{n_0-k}{r-k} (-1)^{r-k} E_{\nu,1}(-r\mu t^\nu)\\
&= \sum_{r=0}^{n_0} E_{\nu,1}(-r\mu t^\nu)(-1)^r \sum_{k=1}^{r} k \binom{n_0}{k} \binom{n_0-k}{r-k} (-1)^k\\
&= \sum_{r=1}^{n_0} E_{\nu,1}(-r\mu t^\nu)(-1)^r n_0 \binom{n_0-1}{r-1} \sum_{k=1}^{r} \binom{r-1}{k-1} (-1)^k\\
&= n_0 E_{\nu,1}(-\mu t^\nu).
\end{aligned}$$

This last step in (2.22) holds because

$$\sum_{k=1}^{r} \binom{r-1}{k-1} (-1)^k = \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{k+1} = \begin{cases} -1, & r = 1,\\ 0, & r > 0. \end{cases} \qquad (2.23)$$

3 Related models

In this section we present two models which are related to the fractional linear death process. The first one is its natural generalisation to the non-linear case i.e. we consider death rates in the form $\mu_k > 0$, $0 \le k \le n_0$. The second one is a sublinear process (see [8]), namely with death rates in the form $\mu_k = \mu(n_0 + 1 - k)$; the death rates are thus an increasing sequence as the number of individuals in the population decreases towards zero.

3.1 Generalisation to the non-linear case

Let us denote by $\mathcal{M}^\nu(t)$, $t > 0$ the random number of components of a non-linear fractional death process with death rates $\mu_k > 0$, $0 \le k \le n_0$.

The state probabilities $p_k^\nu(t) = \Pr\{\mathcal{M}^\nu(t) = k \mid \mathcal{M}^\nu(0) = n_0\}$, $t > 0$, $0 \le k \le n_0$, $\nu \in (0, 1]$ are governed by the following difference-differential equations

$$\begin{cases} \frac{d^\nu}{dt^\nu} p_k(t) = \mu_{k+1} p_{k+1}(t) - \mu_k p_k(t), & 0 < k < n_0,\\ \frac{d^\nu}{dt^\nu} p_0(t) = \mu_1 p_1(t), & k = 0,\\ \frac{d^\nu}{dt^\nu} p_{n_0}(t) = -\mu_{n_0} p_{n_0}(t), & k = n_0,\\ p_k(0) = \begin{cases} 0, & 0 \le k < n_0,\\ 1, & k = n_0. \end{cases} \end{cases} \qquad (3.1)$$

The fractional derivatives appearing in (3.1) provide the system with a global memory; i.e. the evolution of the state probabilities $p_k^\nu(t)$, $t > 0$, is influenced by the past, as definition (1.5) shows. This is a major difference with

the classical non-linear (and, of course, linear and sublinear) death processes, and reverberates in the slowly decaying structure of probabilities extracted from (3.1).

In the non-linear process, the dependence of death rates from the size of the population is arbitrary, and this explains the complicated structure of the probabilities obtained. Further generalisation can be considered by assuming that the death rates depend on t (non-homogeneous, non-linear death process).

We outline here the evaluation of the probabilities $p_k^\nu(t)$, $t > 0$, $0 \leq k \leq n_0$, which can be obtained, as in the linear case, by means of a recursive procedure (similar to that implemented in [7] for the fractional linear birth process).

Let $k = n_0$. By means of the Laplace transform applied to equation (3.1) we immediately obtain that

$$p_{n_0}^\nu(t) = E_{\nu,1}(-\mu_{n_0} t^\nu). \tag{3.2}$$

When $k = n_0 - 1$ we get

$$\begin{aligned} & z^\nu \mathcal{L}\left\{p_{n_0-1}^\nu\right\}(z) = -\mu_{n_0-1}\mathcal{L}\left\{p_{n_0-1}^\nu\right\}(z) + \mu_{n_0}\frac{z^{\nu-1}}{z^\nu+\mu_{n_0}} \qquad (3.3)\\ & \Leftrightarrow \mathcal{L}\left\{p_{n_0-1}^\nu\right\}(z) = \mu_{n_0}\frac{z^{\nu-1}}{z^\nu+\mu_{n_0}}\cdot\frac{1}{z^\nu+\mu_{n_0-1}}\\ & \Leftrightarrow \mathcal{L}\left\{p_{n_0-1}^\nu\right\}(z) = \mu_{n_0} z^{\nu-1}\left[\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-1}}\right]\frac{1}{\mu_{n_0-1}-\mu_{n_0}}\\ & \Leftrightarrow p_{n_0-1}^\nu(t) = \frac{\mu_{n_0}}{\mu_{n_0-1}-\mu_{n_0}}\left\{E_{\nu,1}(-\mu_{n_0}t^\nu) - E_{\nu,1}(-\mu_{n_0-1}t^\nu)\right\}. \end{aligned}$$

For $k = n_0 - 2$ we obtain in the same way that

$$\begin{aligned} & z^\nu \mathcal{L}\left\{p_{n_0-2}^\nu\right\}(z) \qquad (3.4)\\ & = -\mu_{n_0-2}\mathcal{L}\left\{p_{n_0-2}^\nu\right\}(z) + \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}}\left[\frac{z^{\nu-1}}{z^\nu+\mu_{n_0}} - \frac{z^{\nu-1}}{z^\nu+\mu_{n_0-1}}\right], \end{aligned}$$

so that

$$\begin{aligned} & \mathcal{L}\left\{p_{n_0-2}^\nu\right\}(z) \qquad (3.5)\\ & = \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}} z^{\nu-1}\left[\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-1}}\right]\frac{1}{z^\nu+\mu_{n_0-2}}\\ & = \frac{\mu_{n_0}\mu_{n_0-1}}{\mu_{n_0-1}-\mu_{n_0}} z^{\nu-1}\left[\left(\frac{1}{z^\nu+\mu_{n_0}} - \frac{1}{z^\nu+\mu_{n_0-2}}\right)\frac{1}{\mu_{n_0-2}-\mu_{n_0}}\right.\\ & \quad \left. - \left(\frac{1}{z^\nu+\mu_{n_0-1}} - \frac{1}{z^\nu+\mu_{n_0-2}}\right)\frac{1}{\mu_{n_0-2}-\mu_{n_0-1}}\right]. \end{aligned}$$

By inverting the Laplace transform we readily arrive at the following result

$$p_{n_0-2}^\nu(t) = \mu_{n_0}\mu_{n_0-1}\left[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}\right. \tag{3.6}$$

$$
\begin{aligned}
&- \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})} - \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})} \\
&+ \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\Bigg] \\
&= \mu_{n_0}\mu_{n_0-1}\Bigg[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})} \\
&+ \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})}\left(\frac{1}{\mu_{n_0-2}-\mu_{n_0-1}} - \frac{1}{\mu_{n_0-2}-\mu_{n_0}}\right) \\
&- \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\Bigg] \\
&= \mu_{n_0}\mu_{n_0-1}\Bigg[\frac{E_{\nu,1}(-\mu_{n_0}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})} \\
&+ \frac{E_{\nu,1}(-\mu_{n_0-2}t^\nu)}{(\mu_{n_0-2}-\mu_{n_0-1})(\mu_{n_0-2}-\mu_{n_0})} - \frac{E_{\nu,1}(-\mu_{n_0-1}t^\nu)}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0-1})}\Bigg].
\end{aligned}
$$

The structure of the state probabilities for arbitrary values of $k = n_0 - l$, $0 \le l < n_0$, can now be easily obtained. The proof follows the lines of the derivation of the state probabilities for the fractional non-linear pure birth process adopted in Theorem 2.1 in [7]. We have that

$$
p^\nu_{n_0-l}(t) = \begin{cases} \displaystyle\prod_{j=0}^{l-1}\mu_{n_0-j}\sum_{m=0}^{l}\frac{E_{\nu,1}(-\mu_{n_0-m}t^\nu)}{\displaystyle\prod_{\substack{h=0\\h\neq m}}^{l}(\mu_{n_0-h}-\mu_{n_0-m})}, & 1 \le l < n_0, \\ E_{\nu,1}(-\mu_{n_0}t^\nu), & l = 0. \end{cases} \tag{3.7}
$$

By means of a simple change of indices, formula (3.7) can also be written as

$$
p^\nu_k(t) = \begin{cases} \displaystyle\prod_{j=k+1}^{n_0}\mu_j\sum_{m=k}^{n_0}\frac{E_{\nu,1}(-\mu_m t^\nu)}{\displaystyle\prod_{\substack{h=k\\h\neq m}}^{n_0}(\mu_h-\mu_m)}, & 0 < k < n_0, \\ E_{\nu,1}(-\mu_{n_0}t^\nu), & k = n_0. \end{cases} \tag{3.8}
$$

For the extinction probability, we have to solve the following initial value problem:

$$
\begin{cases} \frac{d^\nu}{dt^\nu}p_0(t) = \mu_1\displaystyle\prod_{j=2}^{n_0}\mu_j\sum_{m=1}^{n_0}\frac{E_{\nu,1}(-\mu_m t^\nu)}{\displaystyle\prod_{\substack{h=1\\h\neq m}}^{n_0}(\mu_h-\mu_m)}, & n_0 > 1, \\ \frac{d^\nu}{dt^\nu}p_0(t) = \mu_1 E_{\nu,1}(-\mu_1 t^\nu), & n_0 = 1, \\ p_0(0) = 0, & n_0 \ge 1. \end{cases} \tag{3.9}
$$

When $n_0 > 1$, starting from (3.9) and by resorting to the Laplace transform once again, we have that

$$\mathcal{L}\left\{p_0^\nu\right\}(z) = \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} \cdot \frac{z^{-1}}{z^\nu + \mu_m}. \tag{3.10}$$

The inverse Laplace transform of (3.10) leads to

$$\begin{aligned} p_0^\nu(t) &= \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} t^\nu E_{\nu,\nu+1}(-\mu_m t^\nu) \qquad (3.11) \\ &= \prod_{j=1}^{n_0} \mu_j \sum_{m=1}^{n_0} \frac{1}{\prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} (\mu_h - \mu_m)} \cdot \frac{1}{\mu_m} \left[1 - E_{\nu,1}(-\mu_m t^\nu)\right] \\ &= \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) - \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu) \\ &= 1 - \sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) E_{\nu,1}(-\mu_m t^\nu). \end{aligned}$$

Note that, in the last step, we used the following fact:

$$\sum_{m=1}^{n_0} \prod_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) \equiv 1. \tag{3.12}$$

This can be ascertained by observing that

$$\prod_{1\leq h<l\leq n_0} (\mu_h - \mu_l) = \det \boldsymbol{A} = \sum_{j=1}^{n_0} A_{1,j}(-1)^{j+1}\mathrm{Min}_{1,j} \tag{3.13}$$

where

$$\boldsymbol{A} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \mu_1 & \mu_2 & \dots & \mu_{n_0} \\ \mu_1^2 & \mu_2^2 & \dots & \mu_{n_0}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{n_0-1} & \mu_2^{n_0-1} & \dots & \mu_{n_0}^{n_0-1} \end{vmatrix}, \tag{3.14}$$

is a Vandermonde matrix and $\mathrm{Min}_{1,j}$ is the determinant of the matrix resulting from $\boldsymbol{A}$ by removing the first row and the j-th column.

When $n_0 = 1$ we obtain

$$\mathcal{L}\left\{p_0^\nu\right\}(z) = \mu_1 \frac{z^{-1}}{z^\nu + \mu_1}, \tag{3.15}$$

so that the inverse Laplace transform can be written as

$$\begin{aligned} p_0^\nu(t) &= \mu_1 t^\nu E_{\nu,\nu+1}(-\mu_1 t^\nu) \\ &= 1 - E_{\nu,1}(-\mu_1 t^\nu). \end{aligned} \tag{3.16}$$

We can therefore summarise the results obtained as follows:

$$p_k^\nu(t) = \begin{cases} \prod\limits_{j=k+1}^{n_0} \mu_j \sum\limits_{m=k}^{n_0} \dfrac{E_{\nu,1}(-\mu_m t^\nu)}{\prod\limits_{\substack{h=k \\ h\neq m}}^{n_0} (\mu_h - \mu_m)}, & 0 < k < n_0,\ n_0 > 1, \\ E_{\nu,1}(-\mu_{n_0} t^\nu), & k = n_0,\ n_0 \geq 1, \end{cases} \tag{3.17}$$

and

$$p_0^\nu(t) = \begin{cases} 1 - \sum\limits_{m=1}^{n_0} \prod\limits_{\substack{h=1 \\ h\neq m}}^{n_0} \left(\dfrac{\mu_h}{\mu_h - \mu_m} \right) E_{\nu,1}(-\mu_m t^\nu), & n_0 > 1, \\ 1 - E_{\nu,1}(-\mu_1 t^\nu), & n_0 = 1. \end{cases} \tag{3.18}$$

3.2 A fractional sublinear death process

We consider in this section the process where the infinitesimal death probabilities have the form

$$\Pr\{\mathfrak{M}(t, t+dt] = -1 \mid \mathfrak{M}(t) = k\} = \mu(n_0 + 1 - k)dt + o(dt), \tag{3.19}$$

where n_0 is the initial number of individuals in the population. The state probabilities

$$\mathfrak{p}_k(t) = \Pr\{\mathfrak{M}(t) = k \mid \mathfrak{M}(0) = n_0\}, \qquad 0 \leq k \leq n_0, \tag{3.20}$$

satisfy the equations

$$\begin{cases} \frac{d}{dt}\mathfrak{p}_k(t) = -\mu(n_0 + 1 - k)\mathfrak{p}_k(t) + \mu(n_0 - k)\mathfrak{p}_{k+1}(t), & 1 \leq k \leq n_0, \\ \frac{d}{dt}\mathfrak{p}_0(t) = \mu n_0 \mathfrak{p}_1(t), & k = 0, \\ \mathfrak{p}_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \leq k < n_0. \end{cases} \end{cases} \tag{3.21}$$

In this model the death rate increases with decreasing population size.

The probabilities $\mathfrak{p}_k^\nu(t) = \Pr\{\mathfrak{M}^\nu(t) = k \mid \mathfrak{M}^\nu(0) = n_0\}$ of the fractional version of this process are governed by the equations

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathfrak{p}_k(t) = -\mu(n_0 + 1 - k)\mathfrak{p}_k(t) + \mu(n_0 - k)\mathfrak{p}_{k+1}(t), & 1 \leq k \leq n_0, \\ \frac{d^\nu}{dt^\nu}\mathfrak{p}_0(t) = \mu n_0 \mathfrak{p}_1(t), & k = 0, \\ \mathfrak{p}_k(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \leq k < n_0. \end{cases} \end{cases} \tag{3.22}$$

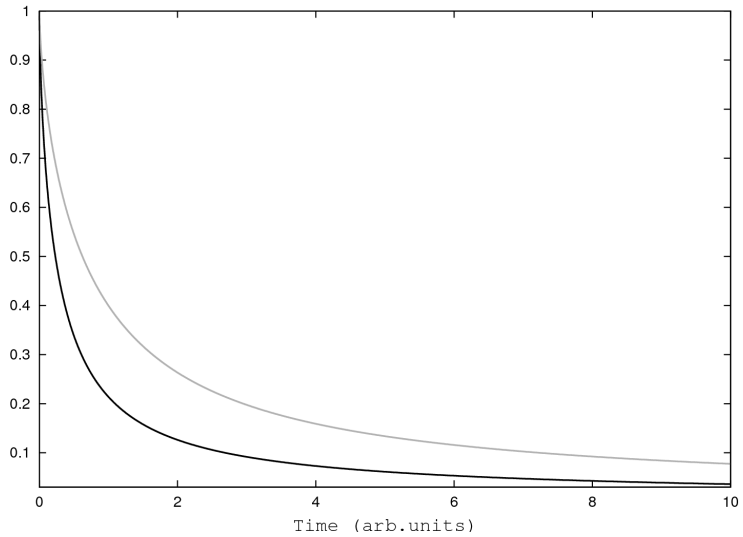

Fig. 3 Plot of $p^{0.7}_{n_0}(t)$ (in black) and $\mathfrak{p}^{0.7}_{n_0}(t)$ (in grey), with $n_0 = 2$.

We first observe that the solution to the Cauchy problem

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathfrak{p}_{n_0}(t) = -\mu\mathfrak{p}_{n_0}(t), \\ \mathfrak{p}_{n_0}(0) = 1, \end{cases} \tag{3.23}$$

is $\mathfrak{p}^\nu_{n_0}(t) = E_{\nu,1}(-\mu t^\nu)$, $t > 0$.

In order to solve the equation

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathfrak{p}_{n_0-1}(t) = -2\mu\mathfrak{p}_{n_0-1}(t) + \mu E_{\nu,1}(-\mu t^\nu), \\ \mathfrak{p}_{n_0-1}(0) = 0, \end{cases} \tag{3.24}$$

we resort to the Laplace transform and obtain that

$$\begin{aligned} \mathcal{L}\left\{\mathfrak{p}^\nu_{n_0-1}\right\}(z) &= \mu z^{\nu-1}\frac{1}{z^\nu+\mu}\cdot\frac{1}{z^\nu+2\mu} \\ &= z^{\nu-1}\left(\frac{1}{z^\nu+\mu} - \frac{1}{z^\nu+2\mu}\right). \end{aligned} \tag{3.25}$$

By inverting (3.25) we extract the following result

$$\mathfrak{p}^\nu_{n_0-1}(t) = E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu). \tag{3.26}$$

By the same technique we solve

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathfrak{p}_{n_0-2}(t) = -3\mu\mathfrak{p}_{n_0-2}(t) + 2\mu\left[E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu)\right], \\ \mathfrak{p}_{n_0-2}(0) = 0, \end{cases} \tag{3.27}$$

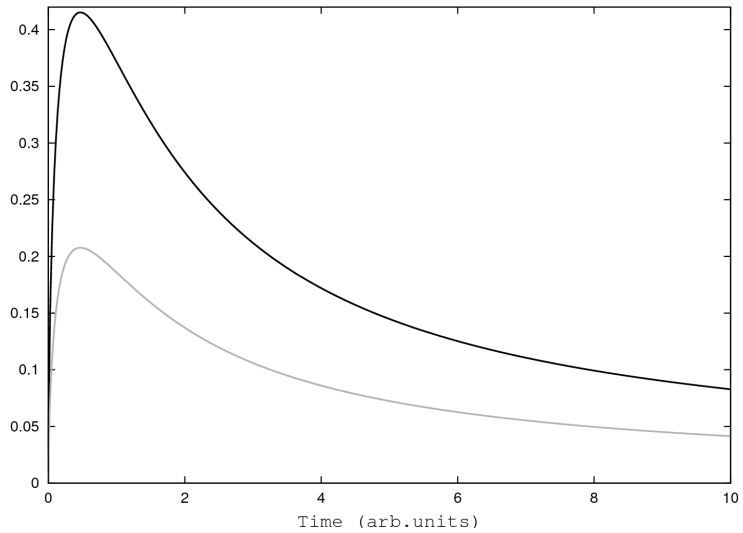

Fig. 4 Plot of $p^{0.7}_{n_0-1}(t)$ (in black) and $\mathfrak{p}^{0.7}_{n_0-1}(t)$ (in grey), with $n_0 = 2$.

thus obtaining

$$\begin{aligned}\mathcal{L}\left\{\mathfrak{p}^{\nu}_{n_0-2}\right\}(z) &= 2\mu z^{\nu-1}\left[\frac{1}{z^{\nu}+\mu}-\frac{1}{z^{\nu}+2\mu}\right]\frac{1}{z^{\nu}+3\mu} \qquad (3.28)\\ &= 2\mu z^{\nu-1}\left[\left(\frac{1}{z^{\nu}+\mu}-\frac{1}{z^{\nu}+3\mu}\right)\frac{1}{2\mu}-\left(\frac{1}{z^{\nu}+2\mu}-\frac{1}{z^{\nu}+3\mu}\right)\frac{1}{\mu}\right]\\ &= \frac{z^{\nu-1}}{z^{\nu}+\mu}-2\frac{z^{\nu-1}}{z^{\nu}+2\mu}+\frac{z^{\nu-1}}{z^{\nu}+3\mu}.\end{aligned}$$

In light of (3.28), we infer that

$$\mathfrak{p}^{\nu}_{n_0-2}(t) = E_{\nu,1}(-\mu t^{\nu}) - 2E_{\nu,1}(-2\mu t^{\nu}) + E_{\nu,1}(-3\mu t^{\nu}). \qquad (3.29)$$

For all $1 \leq n_0 - m \leq n_0$, by similar calculations, we arrive at the general result

$$\mathfrak{p}^{\nu}_{n_0-m} = \sum_{l=0}^{m}\binom{m}{l}(-1)^l E_{\nu,1}\left(-\left(l+1\right)\mu t^{\nu}\right), \qquad 1 \leq n_0 - m \leq n_0. \quad (3.30)$$

Introducing the notation $n_0 - m = k$, we rewrite the state probabilities (3.30) in the following manner

$$\mathfrak{p}^{\nu}_{k} = \sum_{l=0}^{n_0-k}\binom{n_0-k}{l}(-1)^l E_{\nu,1}\left(-\left(l+1\right)\mu t^{\nu}\right), \qquad 1 \leq k \leq n_0. \qquad (3.31)$$

For the extinction probability we must solve the following Cauchy problem

$$\begin{cases} \frac{d^\nu}{dt^\nu}\mathfrak{p}_0(t) = \mu n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l}(-1)^l E_{\nu,1}\left(-\left(l+1\right)\mu t^\nu\right), \\ \mathfrak{p}_0(t) = 0. \end{cases} \tag{3.32}$$

The Laplace transform of (3.32) yields

$$z^\nu \mathcal{L}\left\{\mathfrak{p}_0^\nu\right\}(z) = \mu n_0 \sum_{l=0}^{n_0-1} \binom{n_0-1}{l}(-1)^l \frac{z^{\nu-1}}{z^\nu + \mu(l+1)}. \tag{3.33}$$

The inverse Laplace transform can be written down as

$$\mathfrak{p}_0^\nu(t) = \mu n_0 \sum_{l=0}^{n_0-1} (-1)^l \frac{1}{\Gamma(\nu)} \int_0^t E_{\nu,1}\left(-(l+1)\mu s^\nu\right)(t-s)^{\nu-1} ds. \tag{3.34}$$

The integral appearing in (3.34) can be suitably evaluated as follows

$$\begin{aligned} &\int_0^t E_{\nu,1}\left(-(l+1)\mu s^\nu\right)(t-s)^{\nu-1} \\ &= \sum_{m=0}^{\infty} \frac{(-(l+1)\mu)^m}{\Gamma(\nu m+1)} \int_0^t s^{\nu m}(t-s)^{\nu-1} ds \\ &= \sum_{m=0}^{\infty} \frac{(-(l+1)\mu)^m}{\Gamma(\nu m+1)} \frac{t^{\nu(m+1)}\Gamma(\nu)\Gamma(\nu m+1)}{\Gamma(\nu m+\nu+1)} \\ &= \frac{\Gamma(\nu)}{(-\mu(l+1))} \sum_{m=0}^{\infty} \frac{(-(l+1)\mu t^\nu)^{m+1}}{\Gamma(\nu(m+1)+1)} \\ &= \frac{\Gamma(\nu)}{(-\mu(l+1))} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right]. \end{aligned} \tag{3.35}$$

By inserting result (3.35) into (3.34), we obtain

$$\begin{aligned} \mathfrak{p}_0^\nu(t) &= n_0 \sum_{l=0}^{n_0-1} \frac{(-1)^{l+1}}{l+1} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right] \\ &= \sum_{l=0}^{n_0-1} \binom{n_0}{l+1}(-1)^{l+1} \left[E_{\nu,1}(-(l+1)\mu t^\nu) - 1\right] \\ &= \sum_{l=1}^{n_0} \binom{n_0}{l}(-1)^l E_{\nu,1}(-l\mu t^\nu) - \sum_{l=1}^{n_0} \binom{n_0}{l}(-1)^l \\ &= 1 + \sum_{l=1}^{n_0} \binom{n_0}{l}(-1)^l E_{\nu,1}(-l\mu t^\nu) \\ &= \sum_{l=0}^{n_0} \binom{n_0}{l}(-1)^l E_{\nu,1}(-l\mu t^\nu). \end{aligned} \tag{3.36}$$

Remark 5 We check that the probabilities (3.31) and (3.36) sum up to unity. We start by analysing the following sum:

$$\sum_{k=1}^{n_0} \mathfrak{p}_k^{\nu}(t) = \sum_{k=1}^{n_0} \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}(-(l+1)\mu t^{\nu}). \tag{3.37}$$

In order to evaluate (3.37), we resort to the Laplace transform

$$\sum_{k=1}^{n_0} \mathcal{L}\left\{\mathfrak{p}_k^{\nu}\right\}(z) = \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l \frac{1}{\frac{z^{\nu}}{\mu}+1+l}. \tag{3.38}$$

By using formula (6) of [3] (see also [2], formula (5.41), page 188), we obtain that

$$\begin{aligned}
\sum_{k=1}^{n_0} \mathcal{L}\left\{\mathfrak{p}_k^{\nu}\right\}(z) &= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \frac{\Gamma(n_0-k+1)}{\left(\frac{z^{\nu}}{\mu}+1\right)\left(\frac{z^{\nu}}{\mu}+2\right)\ldots\left(\frac{z^{\nu}}{\mu}+1+n_0-k\right)} \qquad (3.39)\\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \frac{\Gamma\left(\frac{z^{\nu}}{\mu}+1\right)\Gamma(n_0-k+1)}{\Gamma\left(\frac{z^{\nu}}{\mu}+1+n_0-k\right)}\\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \int_0^1 x^{\frac{z^{\nu}}{\mu}} (1-x)^{n_0-k} dx\\
&= \frac{z^{\nu-1}}{\mu} \int_0^1 x^{\frac{z^{\nu}}{\mu}-1} \left[1-(1-x)^{n_0}\right] dx\\
&= \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \frac{\Gamma\left(\frac{z^{\nu}}{\mu}\right)\Gamma(n_0+1)}{\Gamma\left(\frac{z^{\nu}}{\mu}+n_0+1\right)}\\
&= \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^1 x^{\frac{z^{\nu}}{\mu}-1}(1-x)^{n_0} dx\\
&\overset{(-\ln x=y)}{=} \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^{\infty} e^{-y\frac{z^{\nu}}{\mu}} \left(1-e^{-y}\right)^{n_0} dy\\
&\overset{(y/\mu=w)}{=} \frac{1}{z} z^{\nu-1} \int_0^{\infty} e^{-wz^{\nu}} \left(1-e^{-\mu w}\right)^{n_0} dw\\
&= \frac{1}{z} - z^{\nu-1} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \int_0^{\infty} e^{-z^{\nu}w-\mu wk} dw\\
&= \frac{1}{z} - z^{\nu-1} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \frac{1}{z^{\nu}+\mu k}.
\end{aligned}$$

The inverse Laplace transform of (3.39) is therefore

$$\sum_{k=1}^{n_0} \mathfrak{p}_k^{\nu}(t) = 1 - \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^{\nu}) \tag{3.40}$$

$$= -\sum_{k=1}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^\nu).$$

By putting (3.36) and (3.40) together, we conclude that

$$\sum_{k=0}^{n_0} \mathfrak{p}_k^\nu(t) = 1, \tag{3.41}$$

as it should be.

Remark 6 We observe that, in the linear and sublinear death processes, the extinction probabilities coincide. This implies that although the state probabilities $p_k^\nu(t)$ and $\mathfrak{p}_k^\nu(t)$ differ for all $1 \leq k \leq n_0$, we have that

$$\sum_{k=1}^{n_0} p_k^\nu(t) = \sum_{k=1}^{n_0} \mathfrak{p}_k^\nu(t). \tag{3.42}$$

This can be checked by performing the following sum

$$\begin{aligned}
&\sum_{k=1}^{n_0} \mathcal{L}\left\{p_k^\nu(t)\right\}(z) \qquad (3.43)\\
&= \sum_{k=1}^{n_0} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{z^{\nu-1}}{z^\nu + \mu(k+r)}\\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \sum_{r=0}^{n_0-k} \binom{n_0-k}{r} (-1)^r \frac{1}{\frac{z^\nu}{\mu} + k + r}\\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \frac{(n_0-k)!}{\left(\frac{z^\nu}{\mu}+k\right)\left(\frac{z^\nu}{\mu}+k+1\right)\dots\left(\frac{z^\nu}{\mu}+n_0\right)}\\
&= \frac{z^{\nu-1}}{\mu} \sum_{k=1}^{n_0} \binom{n_0}{k} \frac{\Gamma(n_0-k+1)\,\Gamma\left(\frac{z^\nu}{\mu}+k\right)}{\Gamma\left(\frac{z^\nu}{\mu}+n_0+1\right)}\\
&= \frac{z^{\nu-1}}{\mu} \int_0^1 (1-x)^{\frac{z^\nu}{\mu}-1} \sum_{k=1}^{n_0} \binom{n_0}{k} x^{n_0-k}(1-x)^k dx\\
&= \frac{z^{\nu-1}}{\mu} \int_0^1 (1-x)^{\frac{z^\nu}{\mu}-1} \left(1-x^{n_0}\right) dx\\
&= \frac{1}{z} - \frac{z^{\nu-1}}{\mu} \int_0^1 x^{n_0}(1-x)^{\frac{z^\nu}{\mu}-1} dx. \qquad (3.44)
\end{aligned}$$

This coincides with the fourth-to-last step of (3.39) and therefore we can conclude that

$$\sum_{k=1}^{n_0} p_k^\nu(t) = -\sum_{k=1}^{n_0} \binom{n_0}{k} (-1)^k E_{\nu,1}(-\mu k t^\nu) = \sum_{k=1}^{n_0} \mathfrak{p}_k^\nu(t). \tag{3.45}$$

3.2.1 Mean value

Theorem 3 *Consider the fractional sublinear death process $\mathfrak{M}^\nu(t)$, $t>0$ defined above. The probability generating function $\mathfrak{G}^\nu(u,t)=\sum_{k=0}^{n_0}u^k\mathfrak{p}_k^\nu(t)$, $t>0$, $|u|\leq 1$, satisfies the following partial differential equation:*

$$\frac{\partial^\nu}{\partial t^\nu}\mathfrak{G}^\nu(u,t)=\mu(n_0+1)\left(\frac{1}{u}-1\right)\left[\mathfrak{G}^\nu(u,t)-\mathfrak{p}_0^\nu(t)\right]+\mu(u-1)\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t). \tag{3.46}$$

subject to the initial condition $\mathfrak{G}^\nu(u,0)=u^{n_0}$, for $|u|\leq 1$, $t>0$.

Proof Starting from (3.22)*, we obtain that*

$$\begin{aligned}&\frac{d^\nu}{dt^\nu}\sum_{k=0}^{n_0}u^k\mathfrak{p}_k^\nu(t) \\ &=-\mu\sum_{k=1}^{n_0}u^k(n_0+1-k)\mathfrak{p}_k^\nu(t)+\mu\sum_{k=0}^{n_0-1}u^k(n_0-k)\mathfrak{p}_{k+1}^\nu(t),\end{aligned} \tag{3.47}$$

so that

$$\begin{aligned}\frac{\partial^\nu}{\partial t^\nu}\mathfrak{G}^\nu(u,t)=&-\mu(n_0+1)\left[\mathfrak{G}^\nu(u,t)-\mathfrak{p}_0^\nu(t)\right]+\mu u\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t) \\ &+\frac{\mu(n_0+1)}{u}\left[\mathfrak{G}^\nu(u,t)-\mathfrak{p}_k^\nu(t)\right]-\mu\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t) \\ =&\,\mu(n_0+1)\left(\frac{1}{u}-1\right)\left[\mathfrak{G}^\nu(u,t)-\mathfrak{p}_0^\nu(t)\right]+\mu(u-1)\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t).\end{aligned} \tag{3.48}$$

Theorem 4 *The mean number of individuals $\mathbb{E}\mathfrak{M}^\nu(t)$, $t>0$ in the fractional sublinear death process, reads*

$$\mathbb{E}\mathfrak{M}^\nu(t)=\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^{k+1}E_{\nu,1}(-\mu kt^\nu),\qquad t>0,\ \nu\in(0,1]. \tag{3.49}$$

Proof From (3.46) *and by considering that $\mathbb{E}\mathfrak{M}^\nu(t)=\frac{\partial}{\partial u}\mathfrak{G}^\nu(u,t)\big|_{u=1}$, we directly arrive at the following initial value problem:*

$$\begin{cases}\frac{d^\nu}{dt^\nu}\mathbb{E}\mathfrak{M}^\nu(t)=-\mu(n_0+1)\left[1-\mathfrak{p}_0^\nu(t)\right]+\mu\mathbb{E}\mathfrak{M}^\nu(t),\\ \mathbb{E}\mathfrak{M}^\nu(0)=n_0,\end{cases} \tag{3.50}$$

which can be solved by resorting to the Laplace transform, as follows:

$$\begin{aligned}\mathcal{L}\left\{\mathbb{E}\mathfrak{M}^\nu(t)\right\}(z)&=n_0\frac{z^{\nu-1}}{z^\nu-\mu}+\mu(n_0+1)\sum_{k=1}^{n_0}\binom{n_0}{k}(-1)^k\frac{z^{\nu-1}}{z^\nu+\mu k}\cdot\frac{1}{z^\nu-\mu} \\ &=n_0\frac{z^{\nu-1}}{z^\nu-\mu}+\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^k\left[\frac{z^{\nu-1}}{z^\nu-\mu}-\frac{z^{\nu-1}}{z^\nu+\mu k}\right].\end{aligned} \tag{3.51}$$

In (3.51)*, formula* (3.36) *must be considered. By inverting the Laplace transform we obtain that*

$$\begin{aligned}
&\mathbb{E}\mathfrak{M}^{\nu}(t) \qquad (3.52)\\
&= n_0 E_{\nu,1}(\mu t^{\nu}) + \sum_{k=1}^{n_0} \binom{n_0+1}{k+1} (-1)^k \left[E_{\nu,1}(\mu t^{\nu}) - E_{\nu,1}(-\mu k t^{\nu})\right]\\
&= n_0 E_{\nu,1}(\mu t^{\nu}) + E_{\nu,1}(\mu t^{\nu}) \sum_{k=1}^{n_0} \binom{n_0+1}{k+1} (-1)^k\\
&\quad - \sum_{k=1}^{n_0} \binom{n_0+1}{k+1} (-1)^k E_{\nu,1}(-\mu k t^{\nu})\\
&= \sum_{k=1}^{n_0} \binom{n_0+1}{k+1} (-1)^{k+1} E_{\nu,1}(-\mu k t^{\nu}),
\end{aligned}$$

as desired.

Remark 7 The mean value (3.49) can also be directly derived as follows.

$$\begin{aligned}
\mathbb{E}\mathfrak{M}^{\nu}(t) &= \sum_{k=0}^{n_0} k \mathfrak{p}_k^{\nu}(t) \qquad (3.53)\\
&= \sum_{k=1}^{n_0} k \sum_{l=0}^{n_0-k} \binom{n_0-k}{l} (-1)^l E_{\nu,1}(-(l+1)\mu t^{\nu})\\
&= \sum_{k=1}^{n_0} k \sum_{l=1}^{n_0+1-k} \binom{n_0-k}{l-1} (-1)^{l-1} E_{\nu,1}(-\mu l t^{\nu})\\
&= \sum_{l=1}^{n_0} (-1)^{l-1} E_{\nu,1}(-\mu l t^{\nu}) \sum_{k=1}^{n_0+1-l} k \binom{n_0-k}{l-1}.
\end{aligned}$$

It is now sufficient to show that

$$\sum_{k=1}^{n_0+1-l} k \binom{n_0-k}{l-1} = \binom{n_0+1}{l+1}. \qquad (3.54)$$

Indeed,

$$\begin{aligned}
\sum_{k=1}^{n_0+1-l} k \binom{n_0-k}{l-1} &= \sum_{k=l-1}^{n_0-1} (n_0-k) \binom{k}{l-1} \qquad (3.55)\\
&= \sum_{k=l-1}^{n_0-1} (n_0+1-k-1) \binom{k}{l-1}\\
&= (n_0+1) \sum_{k=l-1}^{n_0-1} \binom{k}{l-1} - l \sum_{k=l-1}^{n_0-1} \binom{k+1}{l}
\end{aligned}$$

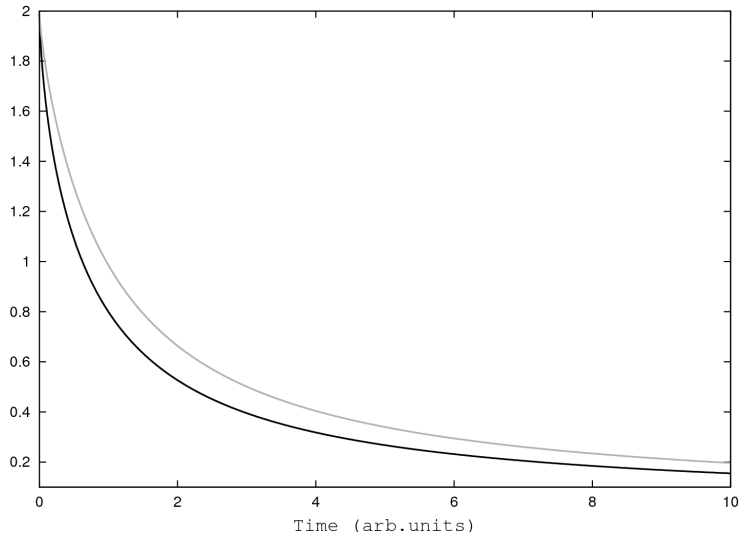

Fig. 5 Plot of $\mathbb{E}M^{0.7}(t)$ (in black) and $\mathbb{E}\mathfrak{M}^{0.7}(t)$ (in grey), $n_0 = 2$.

$$
\begin{aligned}
&= (n_0+1)\sum_{k=l}^{n_0}\binom{k-1}{l-1} - l\sum_{k=l+1}^{n_0+1}\binom{k-1}{l} \\
&= (n_0+1)\binom{n_0}{l} - l\binom{n_0+1}{l+1} \\
&= \binom{n_0+1}{l+1}.
\end{aligned}
$$

The crucial step of (3.55) is justified by the following formula

$$
\sum_{k=j}^{n_0}\binom{k-1}{j-1} = 1 + \binom{j}{j-1} + \cdots + \binom{n_0-1}{j-1} = \binom{n_0}{j}. \tag{3.56}
$$

Figure 5 shows that in the sublinear case, the mean number of individuals in the population, decays more slowly than in the linear case, as expected.

Note that (3.49) satisfies the initial condition $\mathbb{E}\mathfrak{M}^{\nu}(0) = n_0$. In order to check this, it is sufficient to show that

$$
\begin{aligned}
&\sum_{k=1}^{n_0}\binom{n_0+1}{k+1}(-1)^{k+1} = \sum_{r=2}^{n_0+1}\binom{n_0+1}{r}(-1)^r \qquad (3.57)\\
&= \left[\sum_{r=0}^{n_0+1}\binom{n_0+1}{r}(-1)^r\right] - 1 + \binom{n_0+1}{1} = n_0.
\end{aligned}
$$

The details in (3.57) explain also the last step of (3.52).

3.2.2 Comparison of $\mathfrak{M}^\nu(t)$ with the fractional linear death process $M^\nu(t)$ and the fractional linear birth process $N^\nu(t)$

The distributions of the fractional linear and sublinear processes examined above display a behaviour which is illustrated in Table 1.

Table 1 State probabilities $p_k^\nu(t)$ for the fractional linear death process $M^\nu(t)$, $t > 0$, and $\mathfrak{p}_k^\nu(t)$ for the fractional sublinear death process $\mathfrak{M}^\nu(t)$.

State Probabilities
$p_{n_0}^\nu(t) = E_{\nu,1}(-\mu n_0 t^\nu)$
$\mathfrak{p}_{n_0}^\nu(t) = E_{\nu,1}(-\mu t^\nu)$
$p_{n_0-1}^\nu(t) = n_0\left[E_{\nu,1}(-(n_0-1)\mu t^\nu) - E_{\nu,1}(-n_0\mu t^\nu)\right]$
$\mathfrak{p}_{n_0-1}^\nu(t) = E_{\nu,1}(-\mu t^\nu) - E_{\nu,1}(-2\mu t^\nu)$
$\vdots$
$p_k^\nu(t) = \binom{n_0}{k}\sum_{l=0}^{n_0-k}\binom{n_0-k}{l}(-1)^l E_{\nu,1}(-(k+l)\mu t^\nu)$
$\mathfrak{p}_k^\nu(t) = \sum_{l=0}^{n_0-k}\binom{n_0-k}{l}(-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right)$
$\vdots$
$p_1^\nu(t) = n_0\sum_{l=0}^{n_0-1}\binom{n_0-1}{l}(-1)^l E_{\nu,1}(-(1+l)\mu t^\nu)$
$\mathfrak{p}_1^\nu(t) = \sum_{l=0}^{n_0-1}\binom{n_0-1}{l}(-1)^l E_{\nu,1}\left(-(l+1)\mu t^\nu\right)$
$p_0^\nu(t) = \sum_{l=0}^{n_0}\binom{n_0}{l}(-1)^l E_{\nu,1}(-l\mu t^\nu)$
$\mathfrak{p}_0^\nu(t) = \sum_{l=0}^{n_0}\binom{n_0}{l}(-1)^l E_{\nu,1}(-l\mu t^\nu)$

The most striking fact about the models dealt with above, is that the linear probabilities decay faster than the corresponding sublinear ones, for small values of k; whereas, for large values of k, the sublinear probabilities take over and the extinction probabilities in both cases coincide. The reader should also compare the state probabilities of the death models examined here with those of the fractional linear pure birth process (with birth rate λ and one progenitor). These read

$$\hat{p}_k^\nu(t) = \sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda j t^\nu), \qquad k \geq 1. \tag{3.58}$$

Note that $\hat{p}_1^\nu(t) = E_{\nu,1}(-\lambda t^\nu)$ is of the same form as $\mathfrak{p}_{n_0}^\nu(t) = E_{\nu,1}(-\mu t^\nu)$. We now show that

$$\begin{aligned}\sum_{k=n_0+1}^{\infty}\hat{p}_k^\nu(t) &= 1 - \sum_{k=1}^{n_0}\hat{p}_k^\nu(t) \\ &= 1 - \sum_{k=1}^{n_0}\sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda j t^\nu)\end{aligned} \tag{3.59}$$

Table 2 Mean values for the fractional linear birth $N^\nu(t)$, fractional linear death $M^\nu(t)$ and fractional sublinear death $\mathfrak{M}^\nu(t)$ processes.

$\mathbb{E}N^\nu(t) = E_{\nu,1}(\lambda t^\nu)$
$\mathbb{E}M^\nu(t) = n_0 E_{\nu,1}(-\mu t^\nu)$
$\mathbb{E}\mathfrak{M}^\nu(t) = \sum_{k=1}^{n_0} \binom{n_0+1}{k+1}(-1)^{k+1} E_{\nu,1}(-\mu k t^\nu)$

$$
\begin{aligned}
&= 1 - \sum_{j=1}^{n_0} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \sum_{k=j}^{n_0} \binom{k-1}{j-1} \\
&= 1 - \sum_{j=1}^{n_0} (-1)^{j-1} \binom{n_0}{j} E_{\nu,1}(-\lambda j t^\nu) \\
&= (3.36) \text{ with } \lambda \text{ replacing } \mu.
\end{aligned}
$$

Note that in the above step we used formula (3.56).

By comparing formulae (3.4) of [7] and (3.31) above, we arrive at the conclusion that (for $\lambda = \mu$)

$$
\begin{aligned}
&\Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\} \qquad (3.60) \\
&= \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda_j t^\nu) \\
&= \Pr\{\mathfrak{M}^\nu(t) = n_0 + 1 - k \mid \mathfrak{M}(0) = n_0\}, \qquad 1 \le k \le n_0.
\end{aligned}
$$

For $k = 0$ the probability of extinction corresponds to the probability of the event $\{N^\nu(t) > n_0\}$ for the fractional linear birth process.

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