Integrality Properties of EPT Hypergraphs

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Abstract

An Edge Path Tree (EPT for shortness) hypergraph is a hypergraph whose hyperedges can be realized as edge sets of simple paths in a tree. In this paper we give a characterization of Ideal, Mengerian and Normal EPT hypergraphs. In particular we show that, within the class of EPT hypergraphs, a hypergraph is Normal if and only if it is Unimodular and hence Balanced. Therefore, Totally Balanced EPT hypergraphs form a proper subclass of Unimodular EPT hypergraphs. Dually, an EPT hypergraph is Ideal if and only if it is Mengerian. Furthermore, Normal EPT hypergraphs form a proper subclass of Ideal EPT hypergraphs. If, in addition, the EPT hypergraph is uniform, then it is Ideal if and only if it is Normal. The latter (somehow strange) phenomenon already occurs among 2-uniform hypergraphs, i.e., graphs. Indeed, since minimal non Ideal and minimal non Normal graphs are precisely the odd circuits, a graph is Ideal if and only if it is Normal and, in that case, bipartite and hence Balanced and Unimodular. Moreover, since among graphs the only Totally Balanced hypergraphs are forests, it follows that Totally Balanced 2-uniform hypergraphs form a subclass of Unimodular 2-uniform hypergraphs. Since graphs are 2-uniform EPT hypergraphs it follows that EPT hypergraphs are sharp generalizations of graphs when the integrality of their fractional covering and matching polyhedra is concerned. Some more advanced issues about the cycle structure and the Helly property are also investigated for EPT hypergraphs. Finally we give an application of our results to max multicommodity flow on trees.

1 Introduction

Given a hypergraph $(V, \mathcal{E})$ and functions $p : \mathcal{E} \to \mathbb{Z}_+$, $w : V \to \mathbb{Z}_+$ consider the following pair of polyhedra:

$$Q_A(p) = \{x \in \mathbb{R}_+^V \mid Ax \geq p\} \tag{1}$$

$$P_A(w) = \{y \in \mathbb{R}_+^{\mathcal{E}} \mid yA \leq w\} \tag{2}$$

where A is the incidence matrix of the hypergraph.

The above polyhedra are two of the most well studied polyhedra in polyhedral combinatorics and combinatorial optimization [1, 2, 4, 7, 17, 24, 26]. Hypergraphs fit into different (and well structured) classes according to the integrality properties of $P_A(w)$ and $Q_A(p)$ (or even the TDI properties of the corresponding defining systems) for special choices of w and p (see Section 1.2). The main aim of this work is to study $P_A(w)$ and $Q_A(p)$ when $(V, \mathcal{E})$ is an Edge Path Tree hypergraph.

1.1 EPT hypergraphs

A hypergraph H, is an Edge Path Tree (EPT for short) hypergraph if there exists a tree T–called the *supporting tree* of H–having as edge set the vertex set of H and such that every hyperedge of H is the set of edges of a path in T. One of the nicest features of these hypergraphs is that they are closed under taking subhypergraphs and minors. A first example of EPT hypergraph is given by any simple undirected graph [10, 23][1]: take as supporting tree of G a star T; since in T all edges, i.e., vertices of G, are pairwise adjacent, those pairs corresponding to adjacent vertices of G are also adjacent edges in T. EPT hypergraphs are characterized by the following Theorem of Fournier [10]. Given a hypergraph

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[1] In the early 80's, Bruno Simeone used this idea to show that finding a minimum size transversal in an EPT hypergraph is an NP-hard problem. The result was published in [23] but it was credited earlier in [15].

$H = (V, \mathcal{E})$, where $\mathcal{E} = \{F_i, i \in I\}$ for some finite set I, we can associate with H a binary matroid M_H on $V \cup I$: M_H is the matroid whose circuits are the (inclusionwise) minimal non-empty sets in

$$\{\Delta_{i \in J} F_i \cup \{i\} | J \subseteq I\},$$

where Δ denotes symmetric difference. In such a matroid V is a base (as a maximal subset not containing any circuit) and a canonical system of fundamental circuits is given by $\{F_i \cup \{i\} | i \in I\}$. Fournier proved the following.

Theorem 1 *A hypergraph H is an EPT hypergraph, if and only if its associated binary matroid M_H is graphic.*

The problem of recognizing EPT hypergraphs, known as the Graph Realization problem, is then the problem of deciding whether a given collection of sets, is the family of fundamental circuits of some graphic matroid. Efficient algorithms, that also explicitly output a supporting tree, can be found in [3, 13]. If H is an EPT hypergraph, it follows by Fournier's Theorem that its vertex edge incidence matrix is regular, i.e., there exists a *signing* of the nonzero components such the resulting signed matrix is Totally Unimodular (see [4]). Recall that a signing of a matrix A consists of multiplying any subset of entries by -1. When EPT's are concerned the signed matrix is not only a Totally Unimodular matrix but it is also a Network Matrix. A Directed Path Tree (DPT for shortness) hypergraph is an EPT hypergraph whose supporting tree can be oriented in such a way that every hyperedge is the set of arcs of a directed path in this directed tree. DPT hypergraphs are Unimodular: their incidence matrix is a submatrix of a nonnegative Network matrix.

1.2 Our result

A hypergraph is *Ideal* if $Q_A(\mathbf{1})$ is integral. A hypergraph is *Mengerian* if the defining system of $Q_A(\mathbf{1})$ is Totally Dual Integral. Thus the class of Mengerian hypergraphs is a subclass of the class of Ideal hypergraphs. The inclusion is strict as the famous Q_6 hypergraph shows.

A hypergraph is *Normal* (its dual is *Perfect*) if the defining system of $P_A(\mathbf{1})$ is Total Dual Integral. In general the classes of Ideal and Normal hypergraphs are incomparable and so are the classes of Mengerian and Normal hypergraphs. Nevertheless, Mengerian and Normal hypergraphs have a nontrivial intersection. In particular *Balanced*, *Unimodular* and *Totally Balanced* hypergraphs are both Normal and Mengerian. A hypergraph is *Balanced* if the defining system of $Q_{A'}(\mathbf{1})$ (equivalently the defining system of $P_{A'}(\mathbf{1})$) is TDI for every submatrix A' of A. Balanced hypergraphs can be equivalently defined as those hypergraphs whose incidence matrix does not contain (as submatrix) the incidence matrix of an odd circuit. The *Totally Balanced* hypergraphs form a proper subclass of Balanced hypergraphs as they are defined as those hypergraphs whose incidence matrix does not contain (as submatrix) the incidence matrix of a circuit. The Unimodular hypergraphs form a proper subclass of Balanced hypergraphs which is, in general, incomparable with the class of Totally Balanced Hypergraphs. A hypergraph is Unimodular if $Q_A(p)$ (equivalently, $P_A(w)$) is integral for each $p \in \mathbb{Z}_+^{\mathcal{E}}$ (for each $w \in \mathbb{Z}_+^V$). Unimodular hypergraphs can be equivalently defined as those hypergraphs not containing any submatrix A' with an even number of nonzero entries per row and per column whose sum of entries is not a multiple of four.

In this paper we show that, within the class of EPT hypergraphs, the following chain of inclusions holds true.

$$\mathbf{I} = \mathbf{M} \supseteq \mathbf{N} = \mathbf{B} = \mathbf{U} \supseteq \mathbf{TB}$$

where, **I**, **M**, **N**, **B**, **U** and **TB** denote, respectively, the classes of Ideal, Mengerian, Normal, Balanced, Unimodular and Totally Balanced hypergraphs. Moreover, within the class of uniform EPT hypergraphs:

$$\mathbf{I} = \mathbf{M} = \mathbf{N} = \mathbf{B} = \mathbf{U} \supseteq \mathbf{TB}.$$

The latter phenomenon occurs already for graphs (i.e., 2-uniform hypergraphs). It is well known (see, e.g., [26], Vol. A, Theorem 18.3) that the vertex cover polyhedron of a graph—i.e., the polyhedron $Q_A(\mathbf{1})$, where A is the edge-vertex incidence matrix of a graph—is integral if and only if the graph is bipartite. In that case A is a Totally Unimodular matrix hence, G is a 2-uniform Unimodular hypergraph. Moreover, since among graphs the only Totally Balanced hypergraphs are forests, it follows that Totally Balanced 2-uniform hypergraphs form a subclass of Unimodular 2-uniform hypergraphs.

In a graph the only "substructures" responsible for the non-integrality of $Q_A(\mathbf{1})$ and $P_A(\mathbf{1})$, are the odd circuits. The role of odd circuits in graphs will be played in EPT hypergraphs by the so-called odd pies.

Informally, pies are special almost 2-regular EPT hypergraphs (see Definition 1), such that the vertices of the hypergraph occurring in exactly two hyperedges span a subtree—of the supporting tree—isomorphic to a subdivision of a star. This (remarkably powerful) notion was introduced by Golumbic and Jamison in [15]. They showed (in their terminology) that an EPT graph, i.e., the line graph of an EPT hypergraph, contains a hole of length at least four if the corresponding EPT hypergraph contains a pie. We will see in Theorem 4, that odd pies are indeed the hypergraph theoretical counterparts of odd circuits in graphs.

Forbidding (in the appropriate way) pies, ensures Total Dual Integrality of the defining systems of matching and covering polyhedra. Thus pies may be viewed as obstructions to the König Property. The way of forbidding such obstructions (in order to guarantees the above TDI-ness) depends on the property we are looking for: normality is preserved under taking partial hypergraphs, while idealness and mengerianity, are preserved under taking minors. Thus, given an EPT hypergraph, we have to distinguish between pies that arise as partial hypergraphs (called later N-pies) and pies that arise as minors (called later M-pies). We will show that an EPT hypergraph is Normal (and in that case Unimodular) if and only if it is odd N-pie free. Analogously, we will show, that an EPT hypergraph is Ideal (and in that case Mengerian) if an only if it is odd M-pie free. In general an N-pie need not be an M-pie but if an EPT hypergraph contains an odd M-pie it contains an odd N-pie. Thus $\mathbf{N} \subseteq \mathbf{I}$. However, as for graphs, a uniform EPT hypergraph contains an odd M-pie if and only if it contains an N-pie. Thus $\mathbf{I} = \mathbf{N}$ for uniform EPT hypergraphs.

Pursuing the analogy with graphs further, we investigate Helly EPT hypergraphs (see Section 2 for definitions). Helly hypergraphs form a large superclass of the class of Normal hypergraphs. Triangle free graphs are Helly 2-uniform hypergraphs. Analogously, an EPT hypergraph is Helly if and only if it does not contain a pie (as partial hypergraph) of size three. Moreover, a result of Ryser (Section 3.2) implies that if an EPT hypergraph is Helly each of its subhypergraphs is Helly as well, that is, an EPT hypergraph is Helly if and only if it is strong Helly.

Integer packing and covering of paths in graphs is strongly related with multicommodity flow problems. These problems are very hard to solve even for very special classes of graphs (including trees, and series parallel graphs). Due to the equivalence between integral w-matchings of paths and multiflows on the one hand, and the equivalence between integral transversals of paths and multicuts on the other hand, we use our results to show that a multicommodity version of the max-flow min-cut Theorem holds for special classes of graphs and special choices of the demand graph.

1.3 Organization

The rest of the paper goes as follows. In Section 2, we give some (almost standard) background notation and terminology. We refer to [5] and Vol. C. of [26], for graph theoretical and hypergraph theoretical undefined terminology. The section is split into small paragraphs to facilitate reading (and skipping). In Section 2.1 we state basic properties of EPT hypergraphs that we need for later purposes. In Section 3, we characterize Normal hypergraphs within the class of EPT hypergraphs and we show that they are Unimodular. Alternative proofs of this fact can be found in Section 3.1, where the "cycle structure" of EPT hypergraphs, is closely investigated. Section 3.2 is devoted to the study of Helly hypergraphs. In Section 4 we give a characterization of Ideal and Mengerian EPT hypergraphs showing that both classes coincide with the class of odd M-pie EPT hypergraphs (Section 4.2). The latter minors are studied in Section 4.1. Finally in Section 5, we give an application of our results to some max multiflows problems.

2 Preliminaries

The difference and the symmetric difference between two sets A and B will be denoted by $A - B$ and $A \Delta B$, respectively. We will not distinguish between singletons and the only element they contain. A hypergraph is *uniform*, if all of its hyperedges have the same cardinality. An r-*uniform* hypergraph is a uniform hypergraph whose hyperedges have cardinality r. Thus a 2-uniform hypergraph is a graph. The edge e of G having u and v as end-vertices will be denoted either by uv (as customary) or by $\{u, v\}$, the latter notation being more appropriate when G is regarded as a 2-uniform hypergraph. If G is a graph, we denote by $V(G)$ and $E(G)$ its set of vertices and its set of edges, respectively. If G is a (hyper)graph, $\delta_G(v)$ and $\deg_G(v)$ (*the degree* of v) will denote, respectively, the set of (hyper)edges containing v and the number of (hyper)edges containing v. The symbol $\Delta(G)$ will denote the maximum of the degrees of the vertices of the (hyper)graph G. For a graph G and $F \subseteq E(G)$, we denote by $V_G(F)$ the set of vertices in the subgraph spanned by F and by $G - F$ the subgraph $(V(G), E(G) - F)$. In a graph a connected component is *trivial* if it reduces to a single vertex. We also do not distinguish between connected components and their set of vertices.

Indexed sets. Especially dealing with minors, we need a shorthand notation for sequences and indexed families. For $h \in \mathbb{N}$, let $[h]$ be the set $\{1, \ldots, h\} \subseteq \mathbb{Z}$. Let $\langle h\rangle$ denote the set of integers $[h]$ cyclically ordered, i.e., two elements i and j are *consecutive in* $\langle h\rangle$ if $|i - j| = 1, k - 1$. If $i, j \in \langle h\rangle$, we write $i < j$ in $\langle h\rangle$ if $i < j$ in $[h]$. Any two elements $i, j \in \langle h\rangle$, $i < j$, define two *intervals* in $\langle h\rangle$: $[i, j]$ is the interval $\{i, \ldots, j\}$, while $[j, i]$ is the interval $\{j, \ldots, k, 1, \ldots, i\}$. An *interval* in $\langle h\rangle$ is a subset of $\langle h\rangle$ which is either of the form $[i, j]$ or $[j, i]$ for some two non consecutive indices i and j, $i < j$. Given a finite indexed family of objects $\{b_1, \ldots, b_h\}$, we say that two members b_i and b_j of the family are *consecutive* if i and j are consecutive in $\langle h\rangle$.

Subhypergraphs and Minors. The terminology used throughout is mostly standard. Let $\mathcal{E}$ be a finite family of (not necessarily distinct) subsets of some underlying ground set V. $\mathcal{E}$ is a *clutter* if its members are pairwise incomparable w.r.t. set inclusion. A hypergraph is a clutter if so is its hyperedge set. Let $V(\mathcal{E}) = \cup(F \mid F \in \mathcal{E})$ (the use of the same symbol V for both the vertex set of a (hyper)graph and the union of the members in some family, will cause no confusion). For $U \subseteq V(\mathcal{E})$, let $\mathcal{E}[U]$ be the family of nonempty members in $\{F \cap U, F \in \mathcal{E}\}$ with repetitions allowed. A subhypergraph of a hypergraph $H = (V, \mathcal{E})$ is a hypergraph $(U, \mathcal{F})$ such that $U \subseteq V$ and $\mathcal{F} \subseteq \mathcal{E}[U]$ (notice that, by this definition, isolated vertices are allowed while empty hyperedges are not). A *partial hypergraph* of H is a hypergraph $H' = (V(\mathcal{F}), \mathcal{F})$ where $\mathcal{F} \subseteq \mathcal{E}$. We also say that H' is spanned by $\mathcal{F}$. For $U, W \subseteq V(\mathcal{E})$ (possibly empty) and $U \cap W = \emptyset$ let $H \setminus U/W = (V(\mathcal{E}) - (U \cup W), \mathcal{E} \setminus U/W)$, where $\mathcal{E} \setminus U/W$ is the family of the (inclusionwise) minimal members in $\{F - W \mid F \cap U = \emptyset, F \in \mathcal{E}\}$. The hypergraph $H \setminus U/W$ is called a *contraction-deletion minor* of H. If H is a clutter so is $H \setminus U/W$. It is well known that $H \setminus U/W = H/W \setminus U$. When $W = \emptyset$ or $U = \emptyset$ the notation will be abridged to $H \setminus U$ (*deletion minor*) and H/W (*contraction minor*), respectively. Also $H \setminus U$ and H/U will be referred to as the hypergraph obtained deleting U and the hypergraph obtained contracting U, respectively. The *restriction* of $\mathcal{E}$ to U is defined as the family $\{F \in \mathcal{E} \mid F \subseteq U\}$. The restriction of H to U is the hypergraph $H|U = (U, \mathcal{E}|U)$. Notice that $H|U = H \setminus (V(H) - U)$. *Duplicating* a vertex v in a hypergraph $H = (V, \mathcal{E})$ means replacing H by $(V \cup v', \mathcal{E} \cup \{(F - v) \cup v' \mid v \in F\})$, where $v' \notin V$. For $w : V \to \mathbb{Z}_+$, let $U = \{v \in V \mid w(v) = 0\}$; H^w is the hypergraph obtained duplicating $w(v) - 1$ times each vertex v of H with $w(v) \geq 2$ and then deleting U. In the rest of the paper we will use the terms *N-minor* and *M-minor* instead of partial hypergraphs and contraction-deletion minor. Also, if K is a given hypergraph we say that H *contains* an N-K if K is N-minor of H and we say that H *contains* an M-K if K is an M-minor of H. Notice that N-minors and M-minors of H are subhypergraphs of H.

Cycles and Circuits. A *simple cycle* (see [9, 20] and [22]–in the latter being called unbalanced circuit–) in a hypergraph is a sequence $C = a_1F_1a_2F_2 \ldots a_kF_ka_1$ of distinct vertices and distinct hyperedges such that $a_i \in F_{i-1} \cap F_i$ and $F_i \cap \{a_1, \ldots, a_k\} = \{a_i, a_{i+1}\}$, $i \in \langle k\rangle$. The sets $\{a_1, \ldots, a_k\}$ and $\{F_1, \ldots, F_k\}$ are the *vertex set* and the *edge set* of the simple cycle. Two vertices of the cycle are *consecutive* if they are consecutive in the sequence of the vertices. Two hyperedges of the cycle are *consecutive* if they are consecutive in the sequence of the hyperedges. Restricting the incidence matrix of the hypergraph to the rows and columns corresponding, respectively, to the vertices and hyperedges of the cycle, gives the incidence matrix of a circuit. In [1, 2] cycles in hypergraphs are more generally defined as alternating sequences of vertices and hyperedges (as above) where the hyperedges are allowed to contain more than two vertices of the sequence. We do not need such a notion here. For a simple cycle C in a hypergraph H, let H_C be the hypergraph spanned by the hyperedges of the cycle. We say that C is a *circuit* if H_C does not contain simple cycles shorter than C.

Helly Property. A family $\mathcal{E}$ has the *Helly Property*, if for each subfamily $\mathcal{E}' \subseteq \mathcal{E}$ of pairwise intersecting edges, one has $\cap(F \mid F \in \mathcal{E}') \neq \emptyset$. A hypergraph is a *Helly hypergraph* if its set of hyperedges has the Helly property. A hypergraph is a *Strong Helly* hypergraph if each of its subhypergraphs is Helly.

Packing, covering and the König property. Let $H = (V, \mathcal{E})$ be a hypergraph and let $w : V \to \mathbb{Z}_+$. A *w-matching* is a function $\lambda : \mathcal{E} \to \mathbb{Z}_+$ such that $\sum\{\lambda_F \mid F \ni v\} \leq w(v)$ for all $v \in V$. A *w-matching* is thus an integral point in the polyhedron (1). The number $\sum\{\lambda_F \mid F \in \mathcal{E}\}$ is called the *size* of λ and the maximum size of a w-matching of H is denoted by $\nu_w(H)$. A *matching* is just a $\mathbf{1}$-matching, $\mathbf{1}$ being the function identically equal to 1 on V (a matching can be viewed as a collection of disjoint hyperedges). The maximum size of matching of H is denoted by $\nu(H)$. A *transversal* is a function $t : V \to \mathbb{Z}_+$ such that $\sum\{t(v) \mid v \in F\} \geq 1$ for all $F \in \mathcal{E}$. A transversal is thus an integral point in the polyhedron (2), with $p = \mathbf{1}$. A transversal can be also viewed as a subset of V intersecting every hyperedge. The *w-size* of a transversal is the number $\sum_{v \in V} w(v)t(v)$. The maximum w-size of transversal of H is denoted by $\tau_w(H)$. When $w = \mathbf{1}$, the w-size of transversal is simply referred to as its size and it is denoted by $\tau(H)$. Moreover, $\nu_w(H) = \nu(H^w)$ and $\tau_w(H) = \tau(H^w)$. A hypergraph H has the *König Property* if $\nu(H) = \tau(H)$.

2.1 Basic Facts about EPT Hypergraphs

Recall that a minor of a graph G is the graph G' resulting by a sequence of edge deletions and edge contractions (in the graph theoretical sense [21]). The following fact is a folklore statement about the supporting tree of an EPT hypergraph (see also Proposition 3.1 page 548 in [24]). We will often make use of the fact that a subtree T' spanned by a subset U of the edges of some tree T, is isomorphic to the minor T'' of T, obtained contracting $E(T) - U$. Strictly speaking, $E(T')$ and $E(T'')$ are different sets. Nevertheless, with a slightly abuse of notation, we identify such sets.

Fact 1 *Let $H = (V, \mathcal{E})$ be an EPT hypergraph supported by a tree T. If $H' = (U, \mathcal{F})$ is a subhypergraph of H then H' is an EPT hypergraph supported by a minor T' of T. In particular, if H' is a connected N-minor of H then T' is the subtree of T spanned by vertices of H' or, equivalently, the minor obtained contracting the vertices of H in $V - U$. If H' is connected M-minor obtained deleting Y and contracting W, then T' is the minor of T obtained contracting the edges of W in the subtree spanned by $E(T) - Y$, or, equivalently, the minor obtained contracting the vertices in $Y \cup W$.*

Proof. Observe that if T supports $\mathcal{E}$ then T supports any subfamily $\mathcal{E}' \subseteq \mathcal{E}$. Moreover, if e is an edge of T that does not occur in any member of $\mathcal{E}'$, then the subtree resulting from T after the contraction of e still supports $\mathcal{E}'$. Thus, to prove the first part of the statement, we can assume, without loss of generality, that $V = V(\mathcal{E})$, $\mathcal{F} = \mathcal{E}[U]$ and $U = V(\mathcal{F}) = V - e$, e being any edge of T. Under these assumptions the thesis follows easily: if F is the edge set of any path in T containing e then $F - e$ is the edge set of path in the tree T' resulting from T by contraction of e. Thus contracting any edge of a supporting tree preserves connectedness. Suppose now that H' is the N-minor $(U, \mathcal{F})$. Recall that a hypergraph is connected if it cannot be written as the union of two vertex disjoint hypergraphs. Let T' be the subforest of T spanned by U. If T' has $s \geq 2$ connected components, then no $F \in \mathcal{F}$ can meet more than one such components. Therefore H' cannot be connected. Finally, if $H' = H \setminus Y/W$ is connected then $H'' = H \setminus Y$ is supported by the subtree T'' spanned by $E(T) - Y$. Therefore, contracting the edges of W in T'' results in a tree T' supporting H''/W. □

From now on, unless otherwise stated, we assume that if $H = (V, \mathcal{E})$ is a hypergraph, then $V = V(\mathcal{E})$. Moreover, if H is an EPT hypergraph we assume that $E(T) = V(\mathcal{E})$, for every supporting tree T of H. If H' is a subhypergraph of an EPT hypergraph H supported by a tree T, we denote by $T(H')$ the supporting tree of H' yield by Fact 1.

The following notion is undoubtedly the cornerstone of our investigation.

Definition 1 *A k-pie is a hypergraph $\Pi = (V, \{F_i, i \in \langle k \rangle\})$, $k \geq 3$, such that $F_i \cap F_j = \emptyset$ if and only if F_i and F_j are not consecutive. Moreover, if $k = 3$ then $F_1 \cap F_2 \cap F_3 = \emptyset$. The number k is said to be the* size *of the pie. A pie is* odd *or* even *according to the parity of k. A pie is a hypergraph Π which is a k-pie for some integer $k \geq 3$.*

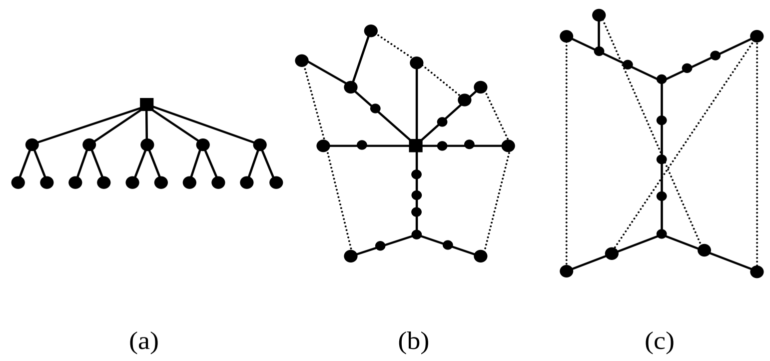


Figure 1: (a) a $T(5)$; (b) a 5-pie supported by a 5-pie tree; (c) a 4-bipie. The dotted lines join the end-vertices of the paths spanned by the hyperedges of an EPT hypergraph.

Fact 2 *A 2-uniform hypergraph is odd N-pie free if and only if it is a bipartite graph.*

Proof. Just observe that a pie in a 2-uniform hypergraph, i.e., in a graph, is a 2-regular connected subgraph, that is a circuit. □

Remark 1 The line graph of a k-pie Π—i.e., the graph on the hyperedges of Π two vertices being joined if the corresponding hyperedges intersect—is a hole of length k.

For $k \in \mathbb{N}$, a $T(k)$ is a tree on $3k+1$ vertices having exactly one vertex of degree k, exactly k vertices of degree three and exactly $2k$ vertices of degree one. Figure 1 (b) shows a $T(5)$. *A k-pie-tree centered at π* is a tree arising from a $T(k)$ after performing the following two operations: delete some pending edge; subdivide some of the remaining edges.

As a direct consequence of the definition of k-pie and k-pie tree one has the following. See Figure 1 (b) for an illustration.

Fact 3 *Let $\Pi = (V, \{F_i, i \in \langle k\rangle\})$ be an EPT hypergraph supported by a tree T. If*

(i) $\cap_{i=1}^{k} V_T(F_i)$ *contains a unique vertex π of T,*

(ii) *the edges of T incident in π can be numbered as $a_1, \ldots, a_k$ such that, $F_i \cap \{a_1, \ldots a_k\} = \{a_i, a_{i+1}\}$, $i \in \langle k\rangle$,*

then Π is k-pie and T is a k pie tree. In particular, if Π and T satisfy (i) and (ii), then $F_{i-1} \cap F_i$ is entirely contained in the unique connected component of T containing π, after the removal of $\{a_j, j \neq i\}$, $i \in \langle k\rangle$.

More strongly, the converse to Fact 3 also holds.

Lemma 1 (Golumbic and Jamison) *Let $\Pi = (V, \{F_i, i \in \langle k\rangle\})$ be a k-pie. Then Π is an EPT hypergraph. Moreover, any of its supporting trees fulfils* (i) *and* (ii) *in Fact* 3. *Therefore, any such supporting tree is a k-pie tree.*

Proof. For $i \in \langle k\rangle$, take a path P_i such that $E(P_i) = R_i \cup (F_{i-1} \cap F_i)$, where R_i is the set of vertices of Π occurring only in F_i. Let a_i and b_i be the endpoints of P_i. We can always arrange the edges of P_i in such a way that those in $F_{i-1} \cap F_i$ come first, moving from a_i to b_i. By definition of pie, the P_i's are edge disjoint. After identifying $a_1, \ldots, a_k$ with a unique vertex π, one gets a k-pie tree (in fact a subdivision of a star) that supports Π. Thus Π is an EPT hypergraph.

Golumbic and Jamison proved (via the Helly property) the second part of the statement for $k \geq 4$. The proof of the case $k = 3$ (not needed for their purposes) is merely a specialization of their arguments.

We give here a proof for completeness. Recall that any collection of vertex sets of paths in a tree satisfies the Helly property (see, e.g., [16]). For $k = 3$, the Helly property and the definition of pie directly imply that $V_T(F_1)$, $V_T(F_2)$ and $V_T(F_3)$ share a vertex of T. Since $F_1 \cap F_2 \cap F_3 = \emptyset$, the corresponding vertex sets have exactly one vertex in common. Let such vertex be π. Now, for $i \in \langle 3 \rangle$, let e_i be any edge in $F_{i-1} \cap F_i$ (for instance the farthest one from π) and let P_i be the unique path in T containing e_i and π. Since $E(P_i) \subseteq F_{i-1} \cap F_i$, no e_j with $i \neq j$ can belong to P_i, because $\Delta(\Pi) = 2$. Therefore, if a_i denote the unique edge of P_i incident in π ($i = 1, 2, 3$), then Π and T satisfy (i) and (ii) of Fact 3. Hence T is a k-pie tree. □

After Lemma 1, we see that pies have a forced representation as EPT hypergraphs: they must be supported by a k-pie tree in such a way that (i) and (ii) are satisfied. It follows that we can speak of the tree $T(\Pi)$ as the k-pie tree supporting Π and of π *as the center of the pie*. More precisely, if $\Pi = (U, \mathcal{F})$ is a N-pie of $H = (V, \mathcal{E})$ then the k-pie tree $T(\Pi)$ is the subtree spanned by U in any tree supporting H and the center π of Π is the unique vertex in any such tree that belongs to the vertex set of each path spanned by the members of $\mathcal{F}$. Analogously, if Π is an M-pie, then $T(\Pi)$ is the minor obtained contracting $V - U$ in any tree supporting H.

For an EPT hypergraph $H = (V, \mathcal{E})$ supported by a tree T call the EPT hypergraph $\tilde{H} = (V, \tilde{\mathcal{E}})$ where

$$\tilde{\mathcal{E}} := \{\{e, f\} \mid e \text{ and } f \text{are adjacent edges of } T \text{ and } \{e, f\} \subseteq F \text{ for some } F \in \mathcal{E}\}$$

the *reduction of* H w.r.t. T.

Fact 4 *An EPT hypergraph is odd N-pie free if and only if so are its reductions.*

Proof. Let $H = (V, \mathcal{E})$ be an EPT hypergraph supported by a tree T. If $\Pi = (U, \{F_1, \dots, F_k\})$ is an N-pie of H centered at π then $\{\{a_i, a_{i+1}\}, i \in \langle k \rangle\}$, spans an N-pie in $\tilde{H}$, where $\{a_1, \dots, a_k\}$ is the set of edges of $T(\Pi)$ incident in π. Conversely, if $\tilde{\Pi} = (\{a_1, \dots, a_k\}, \{\tilde{F}_1, \dots, \tilde{F}_k\})$ is an N-pie in $\tilde{\Pi}$ then there is a collection $\{F_1, \dots, F_k\}$ of hyperedges of Π such that $F_i \supseteq \tilde{F}_i$. Since Π is an EPT hypergraph, by Fact 3, $\{F_1, \dots, F_k\}$ spans an N-pie. □

Although Fact 4 looks somehow an innocent statement, it has a number of interesting consequences. Recall that an orientation of a graph G is a mapping $\phi : V(G) \times E(G) \to \{-1, 0, 1\}$ such that, $\phi(u, e) + \phi(v, e) = 0$, for $e = uv$ and $\phi(v, e) = 0$ if and only if e is not incident in v. Given an EPT hypergraph $H = (V, \mathcal{E})$ supported by a tree T, an orientation ϕ of its supporting tree is a *Directed Path Tree* (*DPT* for short) *orientation*, if for all $F \in \mathcal{E}$ and for each inner vertex v of the path spanned by F in T, one has

$$\sum_{e \in F} \phi(v, e) = 0.$$

An EPT hypergraph whose supporting tree has a DPT orientation is called a DPT hypergraph. Therefore an EPT hypergraph H is a DPT hypergraph if its supporting tree can be oriented in such a way that $\mathcal{E}$ is the collection of the arc sets of a family of directed paths in a directed tree. A straightforward consequence of the definition of DPT orientation is that

Fact 5 *An EPT hypergraph is a DPT hypergraph if and only if so are its reductions.*

Fact 6 *A bipartite graph is a DPT hypergraph.*

Proof. Let the bipartite graph G have color classes A and B. As in Section 1.1, choose as supporting tree of G a star T centered at π. Orient all edges of T belonging to A toward π, and those belonging to B outward π. Thus if uv is an edge of G, then $\{u, v\}$ is the set of arcs of a directed path in T. □

3 Normal EPT Hypergraphs

In this section we characterize Normal EPT hypergraphs. The only obstructions to the property of being Normal are odd N-pies and, interestingly, odd N-pies are also the only obstruction to unimodularity. In this respect, odd N-pies play essentially the same role played in graphs by odd cycles. The latter statement will be made precise in Theorem 4.

Proposition 1 *Odd N-pie free EPT hypergraphs are DPT hypergraphs.*

Proof. Let $H = (V, \mathcal{E})$ be an EPT hypergraph. First observe that the presence of singletons in $\mathcal{E}$ does not affect the property of being a DPT hypergraph. Therefore, we may assume w.l.o.g. that members of $\mathcal{E}$ have at least two elements. By Fact 4 and Fact 5, it suffices to prove the statement when H is 2-uniform (i.e., a graph). Let T be a supporting tree of H. Observe that if ϕ is a DPT orientation for H so is the orientation $-\phi$. Thus, if there exists some DPT orientation ϕ for H, there exists one such that $\phi(v, e) = 1$ where v is a prescribed leaf of T and e is the unique edge of T incident in that leaf. In view of the latter remark we can prove the statement by induction on the diameter $\mathrm{diam}(T)$ of T (i.e., the length of its longest path). If $\mathrm{diam}(T) = 2$ then T is a star. By Fact 2 and Fact 6, H is a DPT hypergraph. Suppose now that every EPT hypergraph supported by a tree of diameter at most $l - 1$ is a DPT hypergraph. Let H be an EPT hypergraph supported by a tree T with $\mathrm{diam}(T) = l$ and let P be a path of length l in T. Suppose first that l is even and let π the middle vertex of P. Let $v_1, \ldots, v_s$ be the neighbors of π in T. For $i = 1, \ldots, s$, let T_i be the unique connected component containing v_i after the removal of edge $v_i\pi$ and let T_i' be the subtree spanned by $E(T_i) \cup v_i\pi$. Finally let T_0' denote the subtree induced by $\{\pi, v_1, \ldots, v_s\}$ (i.e., the star of π). One has, $\mathcal{E} = \cup_{i=0}^{s}(\mathcal{E}|E(T_i'))$ and $\mathrm{diam}(T_i') \le l - 1$, $(i = 0, \ldots, s)$. By the induction hypothesis, $H|E(T_i')$ is a DPT hypergraph $(i = 0, \ldots, s)$. Therefore, for $i = 1, \ldots, s$, we can choose a DPT orientation of $H|E(T_i')$ such that $\phi_0(\pi, u_i\pi) = -\phi_i(u_i, u_i\pi)$, where ϕ_0 is a DPT orientation of $H|E(T_0')$. Thus the orientation ϕ obtained by gluing together $\phi_0, \ldots, \phi_s$, is a DPT orientation of H. If l is odd, let $a = uv$ be the middle edge of P. Let T_u and T_v be connected components of T after the removal of a, containing u and v respectively and let T_u' and T_v' be the subtrees of T spanned by $E(T_u) \cup a$ and $E(T_v) \cup a$, respectively. The maximum length of a path in T_u' and T_v' is bounded above by $l - 1$. Consequently, by the induction hypothesis, both $H|E(T_u')$ and $H|E(T_v')$ are DPT hypergraphs. It follows that there exist DPT orientations ϕ' and ϕ'' such that $\phi'(u, a) = -\phi''(v, a)$. Therefore, the orientation ϕ obtained by gluing ϕ' and ϕ'' is a DPT orientation of H.

□

Theorem 2 *Let H be an EPT hypergraph. The following statements are equivalent.*

(i) *H is Normal;*

(ii) *H is odd N-pie free;*

(iii) *H is Unimodular.*

Proof. No Normal hypergraph can contain an odd N-pie, because $\nu(\Pi) < \tau(\Pi)$ for any such odd pie Π. By Proposition 1, if H is odd N-pie free, then H is a DPT hypergraph and hence Unimodular. □

In Section 3.1 we will obtain another proof of Theorem 2.

Since among 2-uniform hypergraphs the only Unimodular hypergraphs are the bipartite graphs, the following result is a direct consequence of Theorem 2 and it establishes, as a byproduct, the converse to the easy Fact 6, namely a 2-uniform EPT hypergraph is a DPT hypergraph if and only if it bicolorable.

Corollary 1 *An EPT hypergraph is Normal if and only if its reduction is a bipartite graph.*

Corollary 2 *Within the class of EPT hypergraphs the following chain of inclusions holds true,*

$$\mathbf{N} = \mathbf{B} = \mathbf{U} \subseteq \mathbf{TB}. \tag{3}$$

Proof. Directly from Theorem 2 and the fact that class $\mathbf{TB}$ is a proper subclass of $\mathbf{B}$. Actually the inclusion is strict: any 2-uniform even pie is a 2-uniform Normal EPT hypergraph isomorphic to an even circuit; thus $\mathbf{B} \neq \mathbf{TB}$. □

3.1 The cycle structure of EPT Hypergraphs

Chain (3) in Corollary 2 restricted to 2-uniform EPT hypergraphs, reduces to the known statement that *a graph (regarded as a 2-uniform hypergraph) is Normal if and only if it is bicolorable* i.e., if and only if it does not contain odd cycles. Interestingly enough, the natural extension of the previous result to EPT hypergraphs namely, *an EPT hypergraph is Normal if and only if does not contain any odd simple cycle*, turns out to hold showing once more how EPT hypergraphs are close to graphs. This will be implied, after Theorem 2, by Theorem 4 below. Notice that forbidding odd simple cycles in hypergraphs leads already (and in general) to the nice class of Balanced hypergraphs. Therefore, the above statement immediately implies balancedness of Normal EPT hypergraphs[2] (see Corollary 4). The latter fact is not true in general for hypergraphs: the hypergraph having as hyperedges the vertex sets of all paths of length two in a claw is a Normal hypergraph which not Balanced. Indeed restricting the three hyperedges to the leaves of the claw gives an odd simple cycle.

Clearly, the fact that any simple cycle in a hypergraph contains a circuit is a trivial consequence of the definition of simple cycle. Nonetheless, the seemingly obvious fact that an odd simple cycle should contain an odd circuit, does not hold true in general for hypergraphs as the following example shows.

Example 1 Let $H = (V(G), \mathcal{K}(G))$ be the hypergraph of the maximal cliques of a rank 5-wheel with one spoke missing. Such a hypergraph contains five hyperedges: the vertex sets of three triangles and two edges. There is only one vertex that has degree four in G, this vertex being contained in three hyperedges. All other vertices of G are contained in exactly two hyperedges. Restricting the hyperedges of H to the vertices of degree smaller than four gives the vertex set of a pentagon. Thus H contains an odd simple cycle C and $H_C = H$. Observe now that H does not contain any odd simple cycle C' of length three (otherwise restricting the edges of C' to its vertices would give the vertex set of a triangle not in H). On the other hand H contains a simple cycle of length four. Indeed let x_0 and x_2 be the vertices of degree four and degree two in G, respectively, and let x_1 and x_3 be the neighbors of x_2. These vertices are neighbors of x_0 as well, and there is no hyperedge of H containing both. Moreover, $\{x_0, x_1\} \subseteq F'$ and $\{x_0, x_3\} \subseteq F''$ where F' and F'' are hyperedges of H that induce triangles. Furthermore, $\{x_1, x_2\}, \{x_2, x_3\} \in \mathcal{K}(G)$. Therefore, x_0, x_1, x_2, x_3 and $F', F'', \{x_1, x_2\}, \{x_2, x_3\}$ are the vertices and the hyperedges of a simple cycle C' of H_C of length four. Hence C is an odd simple cycle which is not a circuit and which does not contain any odd circuit.

As for graphs, a stronger statement can be made for EPT hypergraphs, namely:

Theorem 3 *Any odd simple cycle $C = e_1F_1 \dots e_kF_k$ in an EPT hypergraph $H = (V, \mathcal{E})$ contains an odd circuit.*

Proof. To prove the statement it suffices to show that if C is not a circuit then $H_C = (U, \{F_i, i \in \langle k\rangle\})$ contains some strictly shorter odd simple cycle. Thus $k \geq 5$. Let us show the following fact first (which holds in general).

(4) If C is not a circuit then $F_q \cap F_r \neq \emptyset$ for some two nonconsecutive hyperedges.

Proof of (4). If C is not a circuit then H_C contains some strictly shorter simple cycle $C' = a_1F_{i_1} \dots a_hF_{i_h}$, $h < k$, $\{i_1, \dots, i_h\} \subseteq \langle k\rangle$. Thus there is at least one subindex $l \in \langle h\rangle$, such that i_l and i_{l+1} are not consecutive in $\langle k\rangle$. Hence $q = i_l$ and $r = i_{l+1}$, for some $q, r \in \langle k\rangle$ and $a_{l+1} \in F_q \cap F_r$. Thus $F_q \cap F_r$ is non empty as required. $\Diamond$

Let $a \in U - \{e_1, \dots, e_k\}$. For $e, f \in \{e_1, \dots, e_k\}$ write $e \sim f$ when e and f belong to the same component of $T(H_C) - a$. Clearly, $\sim$ is an equivalence relation on $U - a$ (hence on $\{e_1, \dots, e_k\}$) with exactly two classes, namely, the edge sets of the two components of $T(H_C) - a$.

(5) Let $a \in U - \{e_1, \dots, e_k\}$. If $a \in F_i \cap F_j - F_{i+1} \cup \dots \cup F_{j-1}$ for some $i, j \in \langle k\rangle$, then $e_i \sim e_{j+1}$, $e_h \sim e_l$, for $h, l \in \{i+1, \dots, j\}$ and $e_i \not\sim e_{i+1}$. If $a \in F_i \cap F_{i+1} \cap \dots, \cap F_j$ then, for $h \in \{i, \dots, j+1\}$ $e_h \sim e_l$ if and only if h and l have the same parity.

[2]Recall that the class of Balanced hypergraphs can be equivalently defined as the class of those hypergraphs all of whose subhypergraphs are bicolorable—a bicoloring of a hypergraph is a partition of its vertex set such that no hyperedge having at least two vertices is contained in a class of the partition—.

Proof of (5). If one had $e_i \sim e_{i+1}$, then every path in $T(H_C)$ containing e_i and e_{i+1} would not contain a. In particular, a would not belong to F_i. Thus $e_i \not\sim e_{i+1}$. By the same reason $e_j \not\sim e_{j+1}$. Moreover, for $h \in \{i+1, \ldots, j-1\}$, $e_h \sim e_{h+1}$ must hold for otherwise F_h would contain a. The second part of the statement follows by the same arguments. $\Diamond$

We are now in position to prove the statement of the theorem. If C is not a circuit, then by (4) there is some $a \in U - \{e_1, \ldots, e_k\}$ such that $a \in F_q \cap F_r \neq \emptyset$ for some two nonconsecutive indices q and r of $\langle k \rangle$. Thus the set $A = \{j \in \langle k \rangle \mid a \in F_j\}$ is non empty and contains at least two elements. $A \neq \langle k \rangle$, otherwise, by (5), e_1 and e_k would belong to a same component of $T(H_C) - a$ contradicting that $a \in F_k$. Therefore, $A = I_1 \cup \ldots \cup I_t$, for some disjoint intervals of $\langle k \rangle$. Let $I_i = \{j \in \langle k \rangle \mid \alpha_i \leq j \leq \beta_i\}$, $i = 1, \ldots, t$, where, possibly after renumbering, $1 = \alpha_1 \leq \beta_1 \leq \alpha_2 \ldots \leq \beta_t \leq k$. A subset of $\langle k \rangle$ will be referred to as even or odd according with the parity of its cardinality. Let $J_i := \{j \in \langle k \rangle \mid \beta_i + 1 \leq j \leq \alpha_{i+1} - 1\})$.

We claim that

(6) J_i is odd for at least one $i \in \{1, \ldots, t\}$.

Proof of (6). Since $a \in F_{\beta_i} \cap F_{\alpha_{i+1}} - \cap_{h \in J_i} F_h$, by (5) one has $e_{\beta_i} \not\sim e_{\alpha_{i+1}}$, for $i \in \langle k \rangle$. Since $a \in \cap_{h \in I_i} F_h$ still by (5) one has $e_{\alpha_i} \not\sim e_{\beta_i}$ if and only if α_i and β_i have different parity, that is if and only if I_i is even. Thus $e_{\alpha_i} \sim e_{\alpha_{i+1}}$ if and only if I_i is even. Therefore, for $2 \leq s \leq t$ one has $e_{\alpha_1} \sim e_{\alpha_s}$ if and only if the number of odd I_h's among $I_1, \ldots, I_{s-1}$ is even. By symmetry, $e_{\alpha_s} \sim e_{\alpha_1}$ if and only if the number of odd I_h's among $I_{s+1}, \ldots, I_t$ is even. Since $e_{\alpha_1} \sim e_{\alpha_1}$, it follows that the number of odd I_i's is even. Hence J_i must be odd for at least one $i \in \{1, \ldots, t\}$, k being odd. $\Diamond$

We are almost done. By (6), there exists $i \in \langle k \rangle$ such that $J_i := \{j \in \langle k \rangle \mid \beta_i + 1 \leq j \leq \alpha_{i+1} - 1\}$ has odd cardinality. By construction $a \notin F_j$, for $j \in J_i$ and $a \in F_{\beta_i} \cap F_{\alpha_{i+1}}$. It follows that

$$C' = aF_{\beta_i} e_{\beta_i+1} \ldots e_{\alpha_{i+1}-1} F_{\alpha_{i+1}-1} e_{\alpha_{i+1}} F_{\alpha_{i+1}}$$

is an odd cycle in H_C of length $\alpha_{i+1} - \beta_i + 1$. Indeed, β_i and α_{i+1} are not consecutive in $\langle k \rangle$. Thus $3 \leq \alpha_{i+1} - \beta_i + 1 \leq k - 1$, that is the length of C' is odd and strictly shorter than k. As required. $\square$

Remark 2 It is interesting to relate Theorem 3 to the graphic matroid M_C associated with H_C in Fournier's Theorem 1. M_C is the cycle matroid of the graph $G = (V(T'), E(T') \cup \{f_i, i = 1, \ldots, k\})$ where, $T' := T(U)$, and $f_i \notin E(T')$ joins the end-vertices of F_i in T', $(i = 1, \ldots, k)$. Thus $F_i \cup f_i$ is a fundamental circuit (w.r.t. to the basis $E(T')$) of M_C (see Figure 1). Therefore, $X = \Delta_{i \in \langle k \rangle}(F_i \cup f_i)$ is a cycle that can be written as a disjoint union of circuits of G. Since $e_{i+1} \in F_i \cap F_{i+1}$, none of these circuits can contain exactly one non-tree edge. Moreover, since f_i occurs in exactly one fundamental circuit $(i = 1, \ldots, k)$, it follows that $X \cap \{f_1, \ldots, f_k\}$ has odd cardinality, k being odd. Hence an odd number of circuits of X meet $\{f_1, \ldots, f_k\}$ in an odd number of elements. By Theorem 3, X contains an odd cycle C' such that $C' \cap \{f_1, \ldots, f_k\} = \{f_i, i \in I\}$, where I is an interval of $\langle k \rangle$. Furthermore, by (6), if $a \in E(T') - \{e_1, \ldots, e_k\}$ belongs to at least two nonconsecutive hyperedges of H_C, then a has even degree in H_C (observe that $a \cup \delta_{H_C}(a)$ is a fundamental co-circuit in M_C).

For a simple cycle C in a hypergraph H, let $\deg_C(x)$ and $\Delta(C)$ denote the degree of a vertex x in H_C and the maximum degree of H_C, respectively.

Corollary 3 *Let $C = e_1F_1 \ldots e_kF_k$ be a circuit of length k in an EPT hypergraph. Then $\Delta(C) > 2$ if and only if there is some vertex a of H_C that belongs to two nonconsecutive hyperedges of C. Moreover, if $\Delta(C) > 2$ then $\Delta(C) = k$ and k is even. Consequently, if k is odd, then $\Delta(C) = 2$ and each vertex of H_C that belongs to two hyperedges of C belongs to two consecutive hyperedges of C.*

Proof. Let U be the vertex set of H_C. If $\Delta(C) > 2$ there is some $a \in U - \{e_1, \ldots, e_k\}$ that belongs to more than two hyperedges of C. By (5), if $k = 3$, then a cannot be in more than two hyperedges of C. Thus, if a is in more than two hyperedges of C, then a is in at least two nonconsecutive such hyperedges. Conversely, suppose that there is some a that belongs to two nonconsecutive hyperedges of C. Thus $a \in U - \{e_1, \ldots, e_k\}$ and the set $A = \{j \in \langle k \rangle \mid a \in F_j\}$ is non empty and contains at least two elements.

By Theorem 3, A must coincide with $\langle k\rangle$, otherwise using the I_i's and the J_i's one could find a shorter cycle in H_C. Therefore $\Delta(C) = k > 2$. Moreover, again by (5), e_1 and e_k cannot have the same parity, otherwise they would belong to the same component of $T(H_C) - a$ contradicting that $a \in F_k$. Therefore, k is even. □

A *k-bipie* is a hypergraph $\Sigma = (U, \{F_1, \dots F_k\})$ such that: k is even and greater than two; $Y := \cap_{i=1}^{k} F_i \neq \emptyset$ and $(F_i - Y) \cap (F_j - Y) = \emptyset$ if and only if F_i and F_j are not consecutive. In other words, the subhypergraph of Σ induced by $U - Y$ is a k-pie. A *bipie* is a hypergraph Σ which is a k-bipie for some integer $k \geq 4$.

Lemma 2 *A k-bipie is an EPT hypergraph. Moreover, if T is any of its supporting trees then T fulfils the following conditions.*

(i) *Y spans a path of T whose endpoints π_1 and π_2 are called the centers of the bipie.*

(ii) *The edges of $E(T) - Y$ incident in π_1 can be numbered as $a_1, a_3, \dots, a_{k-1}$ and those incident in π_2 can be numbered as $a_2, a_4, \dots, a_k$ in such a way that, for $i \in \langle k\rangle$, $\{a_i\} \cup Y \cup \{a_{i+i}\} \subseteq F_i$, and a_i belongs to F_{i-1} and F_i and to no other F_j.*

Proof. By Lemma 1, the subhypergraph Σ' induced by $U - Y$ is an EPT hypergraph, Σ' being a k-pie. Thus Σ' is centered at π and it is supported by an k-pie tree T'. Let $\{a_1, \dots, a_k\} \subseteq U$ be the set of edges of T' incident in π. Let T_1' be the unique connected component of T' containing π after the removal of $a_2, a_4, \dots, a_k$ and let T_2' be the unique connected component of T' containing π after the removal of $a_1, a_3 \dots a_{k-1}$. Replace π in T_i' by π_i and let T_i'' be the resulting tree, $i = 1, 2$. In this way T_1'' and T_2'' are vertex disjoint (but we identify $E(T_i'')$ with $E(T_i')$, $i = 1, 2$). Now join π_1 and π_2 by a path P such that $E(P) = Y$ and let T'' be the resulting tree. Call any tree arising in this way a *k-bipie tree.* Thus T'' supports Σ and fulfils the conditions required by lemma. Hence Σ is an EPT hypergraph. Therefore, Y spans a path in every supporting tree T_0 of Σ. It follows that contracting the edge of Y in T_0 must yield a k-pie tree T_0' fulfilling (i) and (ii) of Lemma 1. If one applies the procedure described above to T_0' one gets a tree which can be identified, without loss of generality, with T_0. By what we have already proved, this k-bipie tree satisfies conditions (i) and (ii) required by the lemma (Figure 1(c)). □

Corollary 3 and definitions of pies and bipies, immediately imply the following result.

Theorem 4 *The edge set of a circuit in an EPT hypergraph spans either an N-pie or an N-bipie. If the circuit is odd then its edge set spans an odd N-pie. Consequently, the hypergraph spanned by the set of edges of any odd circuit has maximum degree two.*

Remark 3 By Theorem 4, the set of edges of any odd circuit spans an odd N-pie and, hence, an almost two regular N-minor. This explains why Normal EPT hypergraphs are Balanced. Indeed, Normal hypergraphs can be equivalently characterized as those hypergraphs whose N-minors have the *edge-coloring* property. Recall that an *edge coloring* of a hypergraph $H = (V, \mathcal{E})$ is a partition of $\mathcal{E}$ into matchings called *colors*. The *chromatic index* $\gamma(H)$ is the minimum number of colors needed for edge-coloring H. H has the *edge coloring property* if $\Delta(H) = \gamma(H)$. A hypergraph is Normal if and only if $\Delta(\Pi) = \gamma(\Pi)$ for every N-minor Π. Thus if C is an odd circuit in a hypergraph, then $\Delta(C) = 2 \Rightarrow \gamma(C) > 2$. Since in an EPT hypergraph $\Delta(C) = 2$, for every odd circuit, it follows that a Normal EPT hypergraph cannot contain odd circuits.

Corollary 4 *An EPT hypergraph is Normal if and only if it is Balanced.*

Theorem 4 and Fact 4, directly imply the following.

Corollary 5 *An EPT hypergraph is Totally Balanced if and only if it is N-pie free, that is, if and only if any of its reductions is a tree.*

Using Theorem 4, an alternative (though inspired by similar ideas) proof of Theorem 2 can be given. To this aim recall Ghouila-Houri's-characterization of Unimodular hypergraphs (see [2]): a hypergraph is Unimodular if and only if each of its subhypergraph has an *equitable* 2-*coloring*. An equitable 2-*coloring* of a hypergraph $H = (V, \mathcal{E})$ is a partition of V into sets U and W such that $||U \cap F| - |W \cap F|| \leq 1$ holds for every $F \in \mathcal{E}$. We need the following facts first.

Fact 7 *If an EPT hypergraph $H = (V, \mathcal{E})$, contains an odd pie as subhypergraph then it contains an odd N-pie. In particular if H contains an odd M-pie, then it contains an odd N-pie.*

Proof. Let $\Pi = (U, \{L_1, \dots, L_k\})$ be an odd k-pie such that $\{L_1, \dots, L_k\} \subseteq \mathcal{E}[U]$. Then Π is supported by $T' := T(\Pi))$, where T supports H. Let $\{a_1, \dots, a_k\}$ be the edges of T' incident in the center, say π of Π. By definition of subhypergraph, $L_i = F \cap U$, $(i = 1, \dots, k)$, for some (possibly not unique) $F \in \mathcal{E}$. Let $F_i \in \mathcal{E}$ be such that $F_i \cap U = L_i$, $(i = 1, \dots, k)$, and let H'' be the hypergraph spanned by $\{F_1, \dots, F_k\}$. Thus $C = a_1 F_1 \dots a_k F_k$ is an odd cycle in H''. Indeed $\deg_C(a_i) = 2$. To see this observe that if $\deg_C(a_i)$ were greater than two, then a_i would belong to some F_j with $j \notin \{i-1, i\}$, $i, j \in \langle k \rangle$. But then $\{a_i, a_j, a_{j+1}\}$ would be contained in F_j implying that $|\delta_{T'}(\pi) \cap F_j| = 3$. A contradiction to Fact 1. Therefore H''_C contains some odd circuit $C' = e_1 F_{i_1} \dots e_h F_{i_h}$, $h \leq k$. By Theorem 4, $\{F_{i_1} \dots F_{i_h}\}$ spans an odd N-pie in H and the thesis follows. Notice in passing that, the subhypergraph of $H_{C'}$ induced by $\{a_1 \dots, a_h\}$ is a subhypergraph of Π. $\square$

Fact 7 directly implies:

Fact 8 *A subhypergraph of an odd N-pie free EPT hypergraph is odd N-pie free.*

Corollary 6 *An EPT hypergraph is Normal if and only if it is Unimodular.*

Proof. One direction is trivial. Let us prove that a Normal EPT hypergraph H is always Unimodular. By Fact 8, if the hypergraph is N-pie-free so are all of its subhypergraphs. Therefore, to prove the statement it suffices to show that H has an equitable 2-coloring whenever it is odd N-pie free. Let $\tilde{H}$ be the reduction of H. By Fact 4 $\tilde{H}$ is a bipartite graph whose color classes define an equitable 2-coloring. Such an equitable 2-coloring is an equitable 2-coloring of H as well. $\square$

3.2 Helly EPT hypergraphs

We have seen that requiring for an EPT hypergraph a "nice" property like e.g., normality, causes H to have other additional stronger properties. This is true even for Helly hypergraphs. We need a little more background first. The *dual hypergraph* of hypergraph $H = (V, \mathcal{E})$ is the hypergraph $H^* = (\mathcal{E}, \{\delta_H(x), x \in V\})$. A result of Ryser (see [19]), states that H is Strong Helly if and only if its incidence matrix does not contain as a square submatrix the vertex edge incidence matrix of a triangle (equivalently if and only if H does not contain circuits of size three). Therefore, the following result follows directly by Ryser's Theorem and by Theorem 4.

Proposition 2 *For a given EPT hypergraph H the following statements are equivalent.*

- *H is Helly.*
- *H does not contain any N-3-pie.*
- *H is Strong Helly.*

Helly (and Strong Helly) hypergraphs are also characterized by means of the *Gilmore Criterion* as in the following theorem (see [2],[19]). We follow Lehel's notation [19].
Let $H = (V, \mathcal{E})$ be a hypergraph and x and y be any two vertices in V. Define $I(x, y)$ as follows

$$I(x,y) = \begin{cases} V, & \text{if } \cap \{F \in \mathcal{E} : F \supseteq \{x,y\}\} = \emptyset \\ \cap \{F \in \mathcal{E} : F \supseteq \{x,y\}\} & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 5 (See [2],[19]) *A hypergraph $H = (V, \mathcal{E})$ is a Helly hypergraph if and only if, for any three vertices x,y and z, one has:*

$$I(x,y) \cap I(x,z) \cap I(z,y) \neq \emptyset \tag{8}$$

A hypergraph $H = (V, \mathcal{E})$ is a Strong Helly hypergraph if and only if, for any three vertices x,y and z, the following holds:

$$I(x,y) \cap I(x,z) \cap I(z,y) \cap \{x,y,z\} \neq \emptyset \ . \tag{9}$$

Remark 4 If H is a Helly (strong Helly, resp.) hypergraph, the map I: $V \times V \to 2^E$ defined by (7) is said to be an *interval structure* (*strong interval structure*, resp.) on V, [19].

It is worth mentioning that Proposition 2 could have been proved using Theorem 5 (which is in fact equivalent to Ryser's result). Indeed, let $H = (V, \mathcal{E})$ be an EPT hypergraph supported by a tree T and let $e, f \in V = E(T)$. If $I(e,f) \neq V$, then $I(e,f)$ spans a path of T (possibly not in $\mathcal{E}$) containing as subpath the path having e and f as end-edges. By this observation one has

$$I(e,f) \cap I(f,g) \cap I(e,g) \neq \emptyset \Rightarrow I(e,f) \cap I(f,g) \cap I(e,g) \cap \{e,f,g\} \neq \emptyset.$$

Indeed let H' be the subhypergraph of H spanned by $\{I(e,f), I(e,g), I(f,g)\}$. In H' consider the sequence $C = eI(e,f)fI(f,g)gI(g,e)$. If $I(e,f) \cap I(f,g) \cap I(e,g) \cap \{e,f,g\} \neq \emptyset$ held true, then C would be an odd circuit in the EPT hypergraph H' with $H'_C = H'$. By Corollary 3, one would have $\Delta(C) = \Delta(H') = 2$ contradicting that $I(e,f) \cap I(f,g) \cap I(e,g) \neq \emptyset$. The same arguments show that $I(e,f) \cap I(f,g) \cap I(e,g) = \emptyset$ if and only if H contains some N-3-pie.

Using Theorem 5 and folklore arguments, we can prove a little bit more about Helly EPT hypergraphs.

Proposition 3 *An EPT hypergraph H is Helly if and only if it is conformal.*

Proof. If $H = (V, \mathcal{E})$ is Helly, by Proposition 2, H is strong Helly. By Ryser's Theorem, H is strong Helly if and only if H^* is strong Helly. In particular H^* is Helly. Hence H is conformal. Conversely, let H be conformal. Then H^* is Helly. Given any three edges e, f and g of T, we show that (8) holds. Since (8) is satisfied whenever at least one of the intervals is $V = E(T)$, we may suppose that all of them are strictly included in V. Therefore, $I(e,f) \neq E(T) \Rightarrow \delta_H(e) \cap \delta_H(f) \neq \emptyset$. Thus $\delta_H(e)$, $\delta_H(f)$ and $\delta_H(g)$ are three hyperedges of H^* that pairwise meet. Since H^* is Helly, $\delta_H(e) \cap \delta_H(f) \cap \delta_H(g) \neq \emptyset$. Let $F \in \delta_H(e) \cap \delta_H(f) \cap \delta_H(g)$. Thus e, f and g all lie on F. Consequently, $I(e,f)$, $I(e,g)$ and $I(f,g)$ are subpaths of F. Hence they span an EPT hypergraph supported by a path. Such a hypergraph is clearly Helly (e.g., it is pie-free). Hence (8) (in fact (9)) follows. □

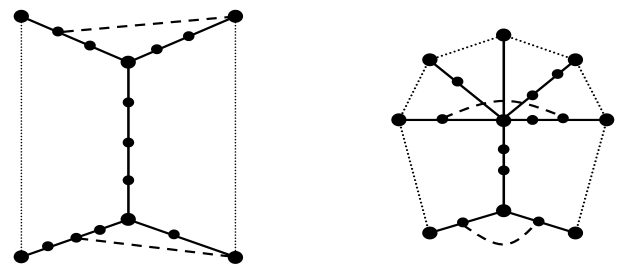
4 Ideal and Mengerian EPT hypergraphs

In this section we characterize Ideal and Mengerian EPT hypergraphs. We can assume, without loss of generality, that such hypergraphs are clutters. As in the Normal case, the only obstructions to both properties are odd M-pies. Therefore, both classes of clutters coincide with the class of odd M-pie free EPT clutters.

4.1 Properties of M-pies

M-pies are pies that arise as deletion-contraction minors in an EPT clutter. By Fact 7, we already know that if a clutter contains an odd M-pie it contains an odd N-pie. However, as the following example shows, the converse statement is not true in general. Therefore, the class of odd N-pie free clutters is a proper subclass of the class of odd M-pie free clutters.

Example 2 The EPT hypergraph in Figure 2(a) is odd M-pie free but contains two odd N-pies: those spanned by $\{L, F, F'\}$ and $\{L', F, F'\}$, where F and F' are the edge sets of paths "closed" by the dotted lines, while L and L' are the edge sets of the paths closed by dashed lines. Giving one more look to Figure 2(a) reveals an important property of odd N-pies in odd M-pie free clutters, namely, L and L' are disjoint and both are contained in $F \Delta F'$. Moreover, looking at Figure 2(b), one sees that a necessary



(a) (b)

Figure 2: (a) an odd M-pie EPT clutter containing two N-3-pies; (b) Not every path in the supporting tree of the 5-pie goes through the center of the pie. Dashed and dotted lines join the end-vertices of the paths spanned by the hyperedges of an EPT clutter.

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condition for an odd N-pie Π in H to be not an odd M-pie is that not all paths of H contained in $T(\Pi)$ go through the center of π.

The above discussion motivates the foregoing definition.

Definition 2 *Let H be a clutter and let F and F' two hyperedges of H. We say that $\{F, F'\}$ is* a pair of mates in *H if $F' \cap F'' \neq \emptyset$ and there are two disjoint hyperedges L and L' of H such that L and L' are both subsets of $F \Delta F'$.*

Remark 5 One can show easily—using, e.g., Fact 9—that any pair of mates has a forced representation as an EPT clutter. For instance any tree supporting a pair of mates $\{F, F'\}$ is such that, $F \Delta F'$ spans exactly two non trivial connected components whose edge sets contain L and L', respectively.

Example 3 After Definition 2 one sees that $\{F, F'\}$ in Figure 2(a), is a pair of mates. A pair of mates can be found also in Figure 2(b). One further consequence might be drawn from the example, namely, *each odd N-pie in the EPT clutters of Figure 2 contains a pair of mates.*

Proposition 4 below will generalize the content of Example 3 to arbitrary odd N-pie in an odd M-pie free EPT clutter. We need some preliminary results, first.

Let $\Pi = (U, \{F_1, \ldots, F_k\})$ be an odd N-pie centered at π in an EPT clutter H supported by a tree T. For $i \in \langle k \rangle$, let $Y_i = F_i \cap F_{i+1}$. Thus Y_i spans a subpath of $T(\Pi)$ having π as one of its endpoints. Let $\pi_i \neq \pi$ be the other endpoint of Y_i. Let $R_{0,i}$ and R_i be the edge sets of the connected components of the forest spanned by $F_i \Delta F_{i+1} = (F_i \cup F_{i+1}) - Y_i$ containing π and π_i, respectively.

Fact 9 *If $L \subseteq (F_i \Delta F_{i+1}) \cap R_i$ then $V_T(L)$ contains π_i as inner vertex. Similarly, if $L \subseteq (F_i \Delta F_{i+1}) \cap R_{0,i}$ then $V_T(L)$ contains π as inner vertex.*

Proof. Since H is a clutter, L must intersect both $F_i - F_{i+1}$ and $F_{i+1} - F_i$. Thus both sets are nonempty. In particular there is an edge $a \in F_i - F_{i+1}$ and edge $b \in F_{i+1} - F_i$ such that a and b are both incident in π_i. Since T is tree, $\{a, b\} \subseteq L$ and π_i is an inner vertex in $V_T(L)$. □

Throughout the rest of the present section we will assume that $H = (V, \mathcal{E})$ is a given odd M-pie free EPT clutter supported by a tree T.

Lemma 3 *Let $\Pi = (U, \{F_i, i \in \langle k \rangle\})$ be an odd N-pie in H. If, for every $F \in \mathcal{E}|U$, $V_T(F)$ contains the center π of Π as inner vertex, then $H|U$ contains an odd M-pie.*

Proof. Suppose that π is an inner vertex of $V_T(F)$ for all $F \in \mathcal{E}|U$. The set of odd N-pies contained in $H|U$ is nonempty, because Π is an N-minor of $H|U$. Thus $H|U$ contains some minimum size odd N-pie $\Omega = (W, \{L_1, \ldots, L_h\})$, $h \leq k$. Since $T(\Omega)$ is a subtree $T(H|U) = T(\Pi)$, Ω is still centered at π. Let $A = \{a_1, \ldots, a_h\}$ be the set of edges of $T(\Omega)$ incident in the center π and let $\Gamma = H|W$. Clearly Γ is a deletion minor of $H|U$. Hence the vertex set of every hyperedge of Γ still contains π as inner vertex. No hyperedge of Γ can contain two nonconsecutive edges of A, otherwise if, say $F \cap A = \{a_i, a_j\}$ for some hyperedge F of Γ and some nonconsecutive $i, j \in \langle h \rangle$, $i < j$, we would have that either $\{F, L_i, \ldots, L_j\}$ or $\{L_j, \ldots, L_h, \ldots L_i, F\}$ is an odd N-pie of $H|U$ of size smaller than h, contradicting the minimality of h. Thus $F \cap A = \{a_i, a_{i+1}\}$ for some $i \in \langle h \rangle$ and all hyperedges F of Γ. Therefore, $\Gamma/(W - A) = (A, \{\{a_i, a_{i+1}\}), i \in \langle k \rangle\})$ is a 2-uniform odd pie centered at π. □

Lemma 4 *Let $\Pi = (U, \{F_i, i \in \langle k \rangle\})$ be an odd N-pie in H If, for some $L \in \mathcal{E}|U$, $V_T(L)$ does not contain the center π of Π as inner vertex, then there is an index $i \in \langle k \rangle$ such that $L \subseteq (F_i \Delta F_{i+1}) \cap R_i$ and $\pi \notin V_T(L)$. Furthermore, $\{L, F_i, F_{i+1}\}$ is an N-3-pie centered at π_i and $L \Delta F_j$, contains no member of $\mathcal{E}$, $j = 1, 2$.*

Proof. Let $\{a_1, \ldots a_k\}$ be the set of edges of $T(\Pi)$ incident in π, where $a_i \in F_{i-1} \cap F_i$. For $i \in \langle k \rangle$, denote by Q_i the edge set of the unique connected component of $T(\Pi)$ containing π after the removal of $\{a_j, j \neq i+1\}$. Notice that $F_i \cap F_{i+1} \subseteq Q_i \subseteq F_i \cup F_{i+1}$. If the vertex set of L does not contain the center of Π as inner vertex then L must be contained in Q_i, for some $i \in \langle k \rangle$. Since H is clutter, L cannot intersect $F_i \cap F_{i+1}$, otherwise it would be contained in at least one among F_i and F_{i+1}. Thus L is contained in R_i. Therefore, $L \subseteq (F_i \Delta F_{i+1}) \cap R_i$, $\pi \notin V_T(L)$ and π_i is an inner vertex in $V_T(L)$ (by Fact 9). Consequently, $\{L, F_i, F_{i+1}\}$ spans an N-3-pie Π' centered at π_i. Moreover, the components of $T(\Pi')$ not containing π_i after the removal of $L \cap F_j$ are (possibly trivial) subpaths of F_j, $j = 1, 2$. Hence, $L \Delta F_j$, contains no member of $\mathcal{E}$, $j = 1, 2$. □

Lemma 5 *Let $\Pi = (U, \{F_1, F_2, F_3\})$ be an odd N-3-pie in H. If $F_3 \subseteq F_1 \Delta F_2$ then either $V_T(F)$ contains the center π of Π as inner vertex $\forall F \in \mathcal{E}|U$ or $\{F_1, F_2\}$ is a pair of mates.*

Proof. By Lemma 4, if, for some $L \in \mathcal{E}|U$, $V_T(L)$ does not contain the center π of Π as inner vertex, then there is an index $i \in \langle 3 \rangle$ such that: $L \subseteq (F_i \Delta F_{i+1}) \cap R_i$, $\pi \notin V_T(L)$ and $\{L, F_i, F_{i+1}\}$ spans an N-3-pie Π' centered at π_i. Since $F_3 \subseteq F_1 \Delta F_2$, L cannot be contained in $F_h \Delta F_3$, $h = 1, 2$. Thus $i = 1$, and L and F_3 are disjoint. Therefore, $\{F_1, F_2\}$ is a pair of mates in Π. □

Proposition 4 *Any odd N-pie in an odd M-pie-free EPT clutter contains a pair of mates.*

Proof. Let $\Pi = (U, \{F_i, i \in \langle k \rangle\})$ be an odd N-pie in H. There must exist some $L \in \mathcal{E}|U$ such that $V_T(L)$ does not contain the center π of Π as inner vertex, otherwise by Lemma 3, $H|U$ (and hence H) would contain an odd M-pie. By Lemma 4, there exists some $i \in \langle k \rangle$ such that $L \subseteq (F_i \Delta F_{i+1}) \cap R_i$ and $\Pi' = (W, \{L, F_i, F_{i+1}\})$ is a N-3-pie centered at π_i. Let us set $F_1' = F_i$, $F_2' = F_{i+1}$, $F_3' = L$ and $\pi' = \pi_i$. We are thus in the hypothesis of Lemma 5 and reasoning exactly as before, not all members of $H|W$ contain π' as inner vertex (otherwise $H|W$ could be contracted to an odd M-pie). Therefore, by the lemma, $\{F_1', F_2'\} = \{F_i, F_{i+1}\}$ is a pair of mates. □

By Definition 2, if an EPT clutter H contains a pair of mates $\{F_1, F_2\}$, then it contains two disjoint hyperedges L_0 and L_1 such that $L_0 \cup L_1 \subseteq F_1 \Delta F_2$. There are two interesting cases where H does not contain any pair of mates: when H is r-uniform or H is Helly. In the former case H cannot contain any pair of mates otherwise, one would get the contradiction: $2r = |F_1| + |F_2| = |F_1 \Delta F_2| + 2|F_1 \cap F_2| > |L_0| + |L_1| = 2r$. In the latter case no pair of mates F_1 and F_2 can exist in H because otherwise H would contain at least two N-3-pies: those spanned by $\{L_0, F_1, F_2\}$ and $\{L_1, F_1, F_2\}$. In view of Proposition 2 this is impossible. Thus, in view of Fact 7, we have proved:

Corollary 7 *Let H be an EPT clutter. If H is either Helly or uniform, then H contains an odd M-pie if and only if it contains and odd N-pie.*

4.2 Characterizing Ideal and Mengerian Clutters

Before proving the main result of the section we need the following theorem due to Lovász (see e.g., [26]) that provides a general characterization of Mengerian clutters. Such a result has been recently proved useful in characterizing other classes of Mengerian hypergraphs (see [6]).

Theorem 6 (Lovász) *A clutter $H = (V, \mathcal{E})$ is Mengerian if and only if*

$$\nu_2(H^w) = 2\nu(H^w) \quad \text{for each } w \in \mathbb{Z}_+^V. \tag{10}$$

Theorem 7 *Let $H = (V, \mathcal{E})$ be an EPT Clutter supported by a tree T and let $E = E(T)$. The following statements are equivalent.*

(i) *H is Ideal;*

(ii) *H is odd M-pie free;*

(iii) *H is Mengerian.*

Proof. ((iii)$\Rightarrow$ (i)). Trivial. ((i)$\Rightarrow$ (ii)). No Ideal clutter can contain an odd M-pie otherwise, by Fact 7, it would contain a 2-uniform odd M-pie. Since 2-uniform odd M-pies are isomorphic to odd circuits in graphs and such clutters are not Ideal, the statement follows. ((ii)$\Rightarrow$ (iii)) Suppose H is odd M-pie free but it is not Mengerian. In particular for some $w \in \mathbb{Z}_+^E$ one has $\nu_2(H^w) > 2\nu(H^w)$. Let w be chosen so as to minimize $\sum_{e \in E} w(e)$ and let $E^* := \{e \in E \mid w(e) \geq 1\}$ be the support of w. Therefore, for $e \in E^*$, $\nu_2(H^{w-\chi_e}) = 2\nu(H^{w-\chi_e})$, $\chi_e \in \mathbb{Z}_+^E$, being the incidence vector of edge e over E. Let $\lambda \in \mathbb{Z}_+^{\mathcal{E}}$ be a $2w$-matching of size $\nu_2(H^w)$ and let $\mathcal{M} = \{F \in \mathcal{E} \mid \lambda_F \geq 1\}$ be its support. The clutter M spanned by $\mathcal{M}$ must contain some odd N-pie otherwise M would be Unimodular and we would have $\nu_2(H^w) = \nu_2(M^w) = 2\nu(M^w) \leq 2\nu(H^w)$. Let $\Pi = (U, \{F_i, i \in \langle k \rangle\})$ be any odd N-pie in M and hence in H. Notice that $U \subseteq V(\mathcal{M}) \subseteq E^*$. By Proposition 4, Π contains a pair of mates $\{F_i, F_{i+1}\}$ for some $i \in \langle k \rangle$. Therefore, there are disjoint members L_0 and L_1 of $\mathcal{E}$ such that $L_j \subseteq F_i \Delta F_{i+1}$, $j = 0, 1$. In particular all of the members of

$$\{F_i \cap L_0, F_i \cap L_1, F_{i+1} \cap L_0, F_{i+1} \cap L_1, F_i \cap F_{i+1}\}$$

are pairwise disjoint. Define $\overline{\lambda}$ as follows:

$$\overline{\lambda}_F = \begin{cases} \lambda_F - 1 & \text{if } F \in \{F_i, F_{i+1}\} \\ \lambda_F + 1 & \text{if } F \in \{L_0, L_1\} \\ \lambda_F & \text{otherwise.} \end{cases}$$

By construction,

$$\sum_{F \ni e} \overline{\lambda}_F = \begin{cases} \sum_{F \ni e} \lambda_F - 1 & \text{if } e \in F_i \cup F_{i+1} \setminus ((L_0 \cup L_1) \cup (F_i \cap F_{i+1})) \\ \sum_{F \ni e} \lambda_F - 2 & \text{if } e \in F_i \cap F_{i+1} \\ \sum_{F \ni e} \lambda_F & \text{otherwise.} \end{cases}$$

Since $F_i \cap F_{i+1}$ is nonempty (because it contains at least the edge $a_{i+1} \in E^*$ incident in the center of Π), it follows that $\overline{\lambda}$ is a $2(w - \chi_{a_{i+1}})$-matching of size

$$\sum_{F \in \mathcal{M} \cup \{P_0, P_1\}} \overline{\lambda}_F = \sum_{F \in \mathcal{M}} \lambda_F,$$

contradicting the minimality of w. $\square$

Putting together the results of the present section and those of the previous sections, we have the following couple of corollaries. In particular, Corollary 9 shows that EPT hypergraphs are genuine generalization of bipartite graphs when the integrality properties of matching and covering polyhedra are concerned.

Corollary 8 *Within the class of EPT hypergraphs* $\mathbf{N} \subseteq \mathbf{M} = \mathbf{I}$.

Corollary 9 *Let H be an EPT hypergraph. If H is either uniform or Helly then the following statements are equivalent.*

- *H is Ideal;*
- *H is Normal;*
- *H is Mengerian;*
- *H is Balanced;*
- *H is Unimodular;*

Proof. By Corollary 7, If H is Helly or H is uniform then H contains an odd M-pie if and only if it contains and odd N-pie. Therefore the thesis follows directly by Theorem 2 and Theorem 7. $\square$

5 Max-Multicommodity flows on trees

Let $G = (V, E)$ and $K = (S, R)$ be two undirected graphs such that $S \subseteq V$ and (for the sake of simplicity) $R \cap E = \emptyset$ and write $G+K$ for the graph $(V, E \cup R)$. Following [12] a path of G will be called K-admissible if it connects two vertices s, t of S and $st \in R$. In the context of multiflow problems, graph G is usually called a supply graph whereas K is usually called a demand graph. The set of edges of K is usually written as $\{s_i t_i, i = 1, \ldots, k\}$ and the endpoints s_i and t_i are thought of as terminals to be connected by a flow of some commodity (the pair $\{s_i, t_i\}$ is in fact called a net or a commodity). The vertices in S are thus referred to as terminals while the vertices in $V - S$ are inner vertices. Let $\mathcal{F}_K$ denote the family of all K-admissible paths of G and let $\mathcal{F}_{K,r} \subseteq \mathcal{F}_K$ be the family of those K-admissible paths connecting the endpoints s_r,t_r of edge $r \in R$. A multiflow (see e.g. [17, 26]), is a function $\lambda : \mathcal{F}_K \to \mathbb{R}_+$. The multiflow is integer if λ is integer valued. The value of the multiflow on the commodity r is $\phi_r = \sum_{F \in \mathcal{F}_{K,r}} \lambda_F$. The total value of the multiflow is the number $\phi = \sum_{r \in R} \phi_r$. Let $w : E \to \mathbb{Z}_+$ be a function to be thought of as a capacity function. A multiflow subject to w in $G + K$ is a multiflow such that,

$$\sum_{F \in \mathcal{F}_K : F \ni e} \lambda_F = \sum_{r \in R} \sum_{F \in \mathcal{F}_{K,r} : F \ni e} \lambda_F \leq w(e), \quad (e \in E) \tag{11}$$

When $w(e) = 1$ for all $e \in E$, an integer multiflow is simply a collection of edge-disjoint K-admissible paths of G. The Max- Multiflow Problem is the problem of finding, for a given capacity function w, a multiflow subject to w of maximum total value. It follows by (11) and the definition of multiflow that a multiflow is a fractional w-matching of K-admissible paths. The blocker of this family (i.e., the family of all of its minimal transversals) is the collection of all subsets of E whose removal from G separates the terminals of every net. Any such set of edges will be called a multicut. The capacity of the multicut B is the number $\sum_{e \in B} w(e)$. For a given demand function $d : R \to \mathbb{Z}_+$, the Feasible-Integer-Multiflow Problem is the problem of finding a multiflow subject to a given capacity function w such that,

$$\sum_{F \in \mathcal{F}_{K,r}} \lambda_F = d(r).$$

When $d(r) = 1$ for all $r \in R$ and $w(e) = 1$ for all $e \in E$, the Integer Multiflow Problem reduces o the well known Edge-Disjoint-Multicommodity Path Problem, namely, the problem of finding edge disjoint paths connecting the terminals of every net.

5.1 The case of Trees

Multiflow Problems are very difficult problems (see [11], [12] and Vol. C, Chapter 70 in [26]). In [14] it has been shown that the Max-Multiflow Problem is NP-hard even for trees and even for $\{1, 2\}$-valued capacity functions. The problem though, is shown to be polynomial time solvable for constant capacity functions by a dynamic programming approach. However, even for constant functions, the problem of maximizing the value of the multiflow over the system of linear inequalities (11) has not even, in general, half-integral optimal solutions. Recently, in [25], the NP-completeness of the Edge-Disjoint-Multicommodity Path Problem for series parallel graph (and partial 2-trees) has been established while, previously in [27], the polynomial time solvability of the same problem for partial 2-trees was proved under some restriction either on the number of the commodities (required to be a logarithmic function of the order of the graph) or on the location of the nets.

When G is a tree T (hence K is a co-tree of $G+K$), $\mathcal{F}_K$ spans an EPT hypergraph and $\mathcal{F}_{K,r}$ reduces to the unique path F_r connecting s_r and t_r in G. Using our results we give a further contribution to the above mentioned problems.

Proposition 5 *Let G be a series parallel graph. For every spanning tree T of G, the maximum total value of a multiflow in $T+K$ equals the minimum capacity of a multicut, K being the co-tree of T. Furthermore, both problems can be solved in strongly polynomial time.*

Proof. By a classical theorem of Dirac (see e.g., [21]), G is a series parallel if and only if it does not contain a K_4 minor. Let T be a spanning tree of G and let $\mathcal{F}_K$ the corresponding family of K-admissible paths, K being the co-tree of T. We claim that $H = (E(G), \mathcal{F}_K)$ is N-pie free, that is, $\mathcal{F}_K = \{F_r, r \in R\}$ spans a Totally Balanced hypergraph. Indeed if $\{F_{r_1}, \ldots, F_{r_k}\}$ spans an N-pie Π in H, then $G' = T(\Pi) + \{r_1, \ldots, r_k\}$ is homeomorphic to a rank k-wheel. Thus G' would contain an homeomorphic copy of the K_4 as subgraph. Using the greedy algorithm by Hoffman, Kolen and Sakarovitch presented in [18] (Farber's Algorithm in [8] is also available to this purpose), both a maximum size integer w-matching of K-admissible paths (i.e., a maximum total value multiflow) and a minimum cost transversal of H (i.e., a minimum capacity multicut) can be found in strongly polynomial time. □

Clearly Proposition 5 is a particular case of the following Fact.

Proposition 6 *Let $G = T + K$, where T is a spanning tree of T and K is the corresponding co-tree. If $(E(G), \mathcal{F}_K)$ is odd-M-pie free then the maximum total value of a multiflow in $T+K$ equals the minimum capacity of a multicut. Furthermore, both problems can be solved in strongly polynomial time.*

6 Conclusion

Let us stress here a point that might have escaped reader's attention. While by Corollary 1 the problem of recognizing Normal EPT hypergraphs is reduced to testing the bicolorability of the corresponding reductions (provided that a supporting tree is given), the problem of recognizing Ideal EPT hypergraphs is left open by this paper. Neither we have a model—like, e.g., the DPT model for Normal hypergraphs—for this class of EPT hypergraphs. In our opinion, the study of these problems might give further insights on the structure of "nice" EPT hypergraphs and may give substance to Proposition 6.

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