A FUZZY LOGIC APPROACH TO POVERTY ANALYSIS BASED ON THE GINI AND BONFERRONI INEQUALITY INDICES

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Abstract: In the poverty analysis framework, a great deal of attention has been paid to the poverty measurement in terms of monetary variables, such as income or consumption. In this context, a relevant open problem is connected with the distinction between poor and non-poor. In fact, the concept of poverty is rather vague and cannot be defined in a clear way. In this respect, following a fuzzy logic approach, some new poverty measures are proposed. In particular, the fuzzy extension of two existing poverty measures based on the Gini and Bonferroni inequality indices is provided. Some synthetic and real applications are given in order to show how the proposed poverty measures work.

Keywords: Fuzzy logic approach, Gini and Bonferroni inequality indices, Sen poverty index, Poverty measures

1. Introduction

In the last few decades, a great deal of attention has been paid to the analysis and measurement of poverty in order to understand the phenomenon and make the policy makers able to defeat it. Poverty measurement is mainly evaluated in terms of monetary variables (e.g. income, consumption). However, for the sake of completeness, it is worth mentioning that poverty analysis can be addressed considering not only monetary variables but also other features concerning living conditions. See, for instance, Bourguignon and Chakravarty (1999, 2003) and Deutsch and Silber (2005).

In the income (or consumption) poverty framework, a relevant open problem is connected with the distinction between poor and non-poor. This is usually done by distinguishing the observation units according to a pre-specified income threshold called poverty line. If the associated income is lower (higher) than the poverty line, then the observation unit is poor (non-poor or rich). Unfortunately, such a procedure may often lead to questionable results. This is so because the concept of poverty is rather vague. In this connection, a promising line of research is the measurement of poverty following a fuzzy logic approach. The fuzzy logic theory is a precise theory for dealing with imprecision. In the literature, a lot of works involve the use of fuzzy logic for the measurement of poverty. See, for instance, Cerioli and Zani (1990), Dagum et al. (1992), Chiappero Martinetti (1994, 2000), Cheli and Lemmi (1995), Dagum and Costa (2004), Lemmi and Betti (2006), Betti and Verma (2008).

In the present paper, the poverty measurement is investigated following a fuzzy logic approach and some new poverty measures will be derived. A deeper insight into the existing differences between the classical and fuzzy logic approaches is given in the next section, where the potentialities of the latter are clarified and discussed. Then, in Section 3, we recall two classical poverty measures based on the Gini and Bonferroni inequality indices. Section 4 is devoted to the fuzzy logic extension of the above mentioned poverty measures. Section 5 concerns two applications to synthetic and real data. Finally, in Section 6, a discussion about the fuzzy logic approach to poverty is made.

2. Classical and fuzzy logic approaches to poverty measures

Let X be the distribution of a monetary variable concerning a set of n observation units. For instance, without loss of generality, we suppose that X denotes an income distribution. Specifically, we indicate by x_i the income of the i-th observation unit (i=1,…,n) and we assume that $(0\leq)x_1\leq x_2\leq\ldots\leq x_n$. Our primary interest relies in assessing whether a given observation unit can be considered poor or not. The classical approach to poverty mainly consists of introducing a poverty line, say x_P, and concluding that the generic i-th observation unit is poor when $x_i<x_P$ (and rich otherwise). In this respect, a very extensive line of research can be found in the literature. See, for instance, Foster (1984), Atkinson (1987, 1992), Foster and Shorrocks (1988), Shorrocks (1995), Bourguignon and Fields (1997), Chakravarty (1997), Xu and Osberg (2002), Chakravarty and Muliere (2004), Bresson (forthcoming). Unfortunately, this approach can often be limited. For example, suppose to deal with three observation units (x_1, x_2 and x_3) such that $x_1=x_P-\varepsilon$, $x_2=x_P+\varepsilon$ (with $\varepsilon>0$) and $x_3>>x_P$. Following the classical approach to poverty, we conclude that the first observation is poor, whereas the remaining two are rich. However, this contradicts the common human thinking. Let us start with observations 1 and 2. From a mathematical point of view, as $x_1=x_P-\varepsilon$ and $x_2=x_P+\varepsilon$, the former observation is considered poor and the latter rich. Conversely, from a practical point of view, both observations can be evaluated as poor. In fact, also observation 2 is *approximately* poor because her/his income is very close to the poverty line, even if slightly higher than. Also the conclusion about observations 2 and 3 is in contrast with the human common-sense reasoning. In fact, both observations are considered rich, whereas it seems to be more appropriate to conclude that the former is rich *to some very limited extent* and the latter is *definitively* rich.

The above example shows that the classical approach to poverty can be inadequate. The reason why relies in the fact that several real life phenomena, such as poverty, are affected by vagueness. Generally speaking, the vagueness associated to the concept of poverty can be highlighted with respect to at least two characteristics. First, it is not reasonable that, when the income of an observation increases by a very small amount (as is 2ε in the previous example), such an observation moves from the poor status to the rich one (as is for observations 1 and 2). Furthermore, it would be desirable to classify some observations as *borderline* poor (or poor *to some extent*) in order to suitably distinguish different levels of poverty (as is for observations 2 and 3). See also Qizilbash (2006). To this purpose, fuzzy logic (Zadeh, 1965) seems to be a more promising tool of research. In a fuzzy logic setting we assume that there exist a poverty line x_P and a richness line x_R ($x_P<x_R$). The generic i–th observation is considered poor (non-rich) if $x_i<x_P$. In a similar way, we say that the i–th observation is rich (non-poor) when $x_i\geq x_R$. Finally, when $x_P\leq x_i<x_R$, we can conclude that the i–th observation belongs to a limbo corresponding to the non-poor & non-rich case. Hence, since the i–th observation cannot be considered rich, it follows that such an observation is poor *to some extent*. Therefore, we no longer distinguish poor and rich (non-poor) observation units in a dichotomous way. Rather, following the fuzzy logic approach, we assign to every observation unit a degree of poverty ranging from 0 to 1 according to the corresponding income. This can be done by introducing the so-called membership function (of an observation i to the attribute 'poor'). In particular, if $\widetilde{X}$ indicates the attribute 'poor' (the symbol '~' denotes that a fuzzy logic approach is adopted), the membership function $\mu_{\widetilde{X}}:\mathrm{R}^+\rightarrow[0,1]$ allows us to express to which extent the i–th observation is poor. This is done by quantifying the degree according to which x_i belongs to $\widetilde{X}$. The membership function can be defined as

$$\mu_{\tilde{X}}(x_i) = \begin{cases} 1 & x_i < x_P, \\ f(x_i) & x_P \le x_i < x_R, \\ 0 & x_i \ge x_R, \end{cases} \tag{1}$$

where $f(x_i)$ is a decreasing function from $[x_P, x_R)$ to $[0,1]$ such that $f(x_P)=1$, $f(x_i)<1$ $\forall$ $x_i>x_P$, $f(x_i)>0$ $\forall$ $x_i<x_R$, and $\lim_{x_i \to x_R} f(x_i) = 0$. A possible choice for f can be $f(x_i) = 1 - \left(\frac{x_i - x_P}{x_R - x_P} \right)^{\beta}$, where β is a positive parameter tuning the decreasing trend of f. In particular, when β =1 such a trend is linear. Note that, in this case, the membership function in (1) coincides with that proposed by Cerioli and Zani (1990). When β <1, the decreasing trend is more rapid with respect to the linear case. The opposite comment holds when β >1. Thus, the membership function ranges from 0 (complete non-membership: the *i*–th observation is non-poor or rich) to 1 (full membership: the *i*–th observation is poor). See, for further details about fuzzy logic, Dubois and Prade (1980, 1988) and Zimmermann (2001).

Note that rewriting the concept of poverty in terms of membership function according to the classical approach leads to

$$\mu_X(x_i) = \begin{cases} 1 & x_i < x_P, \\ 0 & x_i \ge x_P, \end{cases} \tag{2}$$

from which it should be clear that an observation unit can be considered *strictly* either poor or non-poor (tertium non datur).

To further clarify the fuzzy approach to poverty, let us introduce the following example. Suppose to set β =1, x_R=1,000 and x_P=3,000. By means of (1), if x_i=800 we get that the *i*–th observation unit is poor with membership function equal to 1 (i.e. s(he) is definitively poor), if x_i=3,400 we conclude that (s)he is poor with membership function equal to 0 (i.e. (s)he is rich), if x_i=1,200 we obtain that the *i*–th observation unit is poor with membership function equal to 0.9 (i.e. (s)he is non-poor & non-rich). Thus, her/his income implies that the *i*–th observation can be considered neither completely rich nor completely poor. We can say that s(he) is poor with degree 0.9 and rich (non-poor) with degree 0.1.

3. A review of some poverty measures following the classical approach

In the literature, there is a wide range of works devoted to the development of measures for evaluating the deprivation level of a population. See, for instance, Sen (1976), Kakwani (1980), Chakravarty (1983, 1997), Atkinson (1987), Haagenars (1987), Yaari (1988), Shorrocks (1995), Jenkins and Lambert (1997), Aaberge (2001), Giorgi and Crescenzi (2001). For a detailed review one can refer to Chakravarty and Muliere (2004).

In this paper, we focus our attention to the poverty measures proposed by Sen (1976) and Giorgi and Crescenzi (2001). The former is based on the Gini index (Gini, 1914), whereas the latter on the Bonferroni one (Bonferroni, 1930). In order to recall the above mentioned measures, it is fruitful to define the following "ingredients" to be used in their set-up. First of all, let $p = \sum_{i=1}^{n} U\{x_i < x_P\}$ be the number of poor, where U denotes the indicator function. With regard to the poor observation units, the poverty-gap associated to the *i*–th observation is $g_i = x_P - x_i$. Thus, g_i is the (positive) difference between the poverty threshold x_P and the income x_i (i=1,...,p). Let $\bar{g} = (1/p)\sum_{i=1}^{p} g_i$ be the mean of the poverty-gaps of the poor. The income mean over the p poor observation units is $m_P = (1/p)\sum_{i=1}^{p} x_i$, and the income mean over the i poorest observation units ($i = 1,\dots,p-1$) is $m_i = (1/i)\sum_{j=1}^{i} x_j$. We can now define the following three ratios, which will be used for determining the Sen poverty measure:

$$I = \frac{\sum_{i=1}^{p} g_i}{x_P p} = \frac{\bar{g}}{x_P}, \tag{3}$$

$$G_P = \frac{p-1}{p} R_P = \frac{p-1}{p}\left(1 - \frac{2\sum_{i=1}^{p-1} i m_i}{p(p-1)m_P}\right), \tag{4}$$

$$H = \frac{p}{n}. \tag{5}$$

The ratio in (3) is usually known as the poverty-gap ratio of the poor. It compares the average poverty-gap $\bar{g}$ (at the numerator) and the poverty threshold x_P. It is easy to see that I takes values in [0,1]. When I=0, the income distribution does not contain poor observation units. On the contrary, if I=1, all the p poor are such that $x_1=...=x_p=0$. Thus, the bigger I is, the poorer the observations are. The index in (4) is the Gini coefficient computed among the poor. Note that G_P takes scores in [0, (p-1)/p]. As it is well-known, it represents a measure of inequality. A low level of the Gini coefficient indicates approximately equal values in the income distribution, while the opposite comment holds in the case of a high level of the Gini coefficient. In particular, $G_P=0$ corresponds to perfect equality (all the poor have exactly the same income) and $G_P=(p-1)/p$ to perfect inequality (among the poor, one has all the income, while everyone else has zero income). Note that the Gini coefficient among the poor is equal, up to the constant (p-1)/p, to the Gini concentration index among the poor R_P, implicitly defined in (4). It is worth noticing that R_P takes scores from 0 (perfect equality) to 1 (perfect inequality). Finally, (5) is the head-count ratio. It expresses the relative amount of poor in the income distribution. Obviously, H ranges from 0 (no poor in the distribution) to 1 (all poor).

By taking into account the ratios in (3)-(5), Sen (1976) suggests the following poverty measure:

$$S = H\left\{1-(1-I)\left[1-\frac{p}{p+1}G_P\right]\right\}. \tag{6}$$

It takes values in [0,1]. S is equal to 0 when there are no poor and equal to 1 when all the observation units have no income ($x_1=\ldots=x_n=0$). It is important to note that Sen (1976) proposes (6) according to an axiomatic framework. In particular, Sen (1976) introduces a set of desirable axioms that a deprivation measure must fulfil and proves that S is the only measure satisfying these axioms.

A modification of S has been proposed by Giorgi and Crescenzi (2001), who suggest using the Bonferroni index in place of the Gini one. The Bonferroni index B (see, e.g., Giorgi, 1998; Chakravarty, 2007) is defined as

$$B = \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{m-m_i}{m} = 1 - \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{m_i}{m}, \tag{7}$$

and, among the poor, as

$$B_P = \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{m_p - m_i}{m_P} = 1 - \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{m_i}{m_P}, \qquad (8)$$

where m denotes the income mean computed over the entire distribution. As for the Gini ratio, also B is an inequality index taking scores ranging from 0 (perfect equality) to 1 (perfect inequality). The need for the Bonferroni index instead of the Gini one mainly relies in the fact that the former is more sensitive than the latter to the poorest observation units belonging to the income distribution. Furthermore, it can be shown that, if an amount of income moves from a donor to a recipient, the variation of the Gini ratio depends only on the distance between their ranks, whereas the variation of the Bonferroni index depends also on their exact positions in the income ordering. This provides a more powerful tool for assessing the level of deprivation. As a consequence, from a theoretical point of view, we can conclude that poverty measures based on B_P should be more powerful than those based on G_P for inspecting an income distribution. See, for more details, Giorgi and Crescenzi (2001).

In an axiomatic framework, the Giorgi and Crescenzi (2001) measure is given by

$$S_B = H\left\{1-(1-I)\left[1-\frac{p-1}{p}B_P\right]\right\}. \qquad (9)$$

By comparing (6) and (9), it is easy to see that the main difference relies in the use of G_P (weighted by $p/(p+1)$) in S and that of B_P (weighted by $(p-1)/p$) in S_B.

4. Some proposals of poverty measures following the fuzzy logic approach

In this section, we extend in a fuzzy logic approach the measures recalled in (6) and (9). This will be done by generalizing the above poverty measures making use of the fuzzy logic definition of poverty given in (1). From a practical point of view, this implies the development of suitable extensions of the ratios and indices in (3)-(5) and (8).

The first step towards the fuzzy logic generalization of (6) and (9) involves a different way to assess whether the observations are poor. In fact, this should be done considering not only the (completely) poor observations (those with membership function equal to one) but also the poor to some extent (those with membership function strictly between 0 and 1), already labelled as non-poor & non-rich. As a consequence, the evaluation of the number of poor p can be extended by considering the sum of the membership functions of all the n observation units (the membership function of a rich is zero):

$$\tilde{p} = \sum_{i=1}^{n}\mu_{\tilde{X}}(x_i), \qquad (10)$$

where the symbol 'tilde' is used in order to highlight that the fuzzy logic approach is adopted[1]. It is easy to see that $\tilde{p} \geq p$. Moreover, we denote by r and $\bar{r}$ the numbers of rich ($r = \sum_{i=1}^{n}U\{x_i \geq x_R\}$) and non-rich ($\bar{r} = \sum_{i=1}^{n}U\{x_i < x_R\}$), respectively. The number of non-rich is usually different from that of poor. In particular, it is $\bar{r} \geq p$, taking into account that some observations belong to the non-poor & non-rich limbo ($\bar{r} - p = \sum_{i=1}^{n}U\{x_P \leq x_i < x_R\}$).

Following the fuzzy logic approach the poverty-gap ratio in (3) can be extended as

[1] It is important to remark that the symbol 'tilde' is usually used for denoting a fuzzy set. However, in (10) as well as in the sequel, this symbol will be used for denoting (non-fuzzy) quantities based on the fuzzy logic approach to poverty.

$$\widetilde{I}=\frac{\sum_{i=1}^{\bar{r}}\widetilde{g}_i\mu_{\widetilde{X}}(x_i)}{x_R\widetilde{p}}=\frac{\widetilde{\bar{g}}}{x_R}, \tag{11}$$

with $\tilde{g}_i = x_R - x_i$. Thus, the gap is no longer computed with respect to the poverty line but with respect to the richness line. Therefore, it appears to be more suitable to refer to (11) as the non-richness-gap ratio of the non-rich. This is obtained dividing the weighted mean of the non-richness-gaps (using the values of the membership function in (1) as system of weights) by the threshold value x_R.

The fuzzy logic extensions of G_P and B_P can be developed by suitably modifying the quantities m_P and m_i, $i=1,\ldots,p-1$. First of all, in a fuzzy logic context, attention should be paid to the $\bar{r}$ non-rich (not only to the p poor). Moreover, these average values can be computed considering the system of weights given by the membership function information. We then get $\widetilde{m}_{\bar{R}}=\frac{\sum_{i=1}^{\bar{r}}x_i\mu_{\widetilde{X}}(x_i)}{\sum_{i=1}^{\bar{r}}\mu_{\widetilde{X}}(x_i)}$, which represents the generalization of m_P, and

$\widetilde{m}_i=\frac{\sum_{j=1}^{i}x_j\mu_{\widetilde{X}}(x_j)}{\sum_{j=1}^{i}\mu_{\widetilde{X}}(x_j)}$, $i=1,\ldots,\bar{r}-1$, which is the fuzzy logic counterpart of m_i, $i=1,\ldots,p-1$. It is useful to note that, in $\widetilde{m}_{\bar{R}}$ and $\widetilde{m}_i$, $i=1,\ldots,\bar{r}-1$, the richer observation units play a less relevant role, if compared with the poorer ones, since the membership function decreases when the income increases. On the basis of the above indices, the fuzzy logic generalizations of the Gini coefficient and the Bonferroni index among the non-rich are

$$\widetilde{G}_{\bar{R}}=\frac{\bar{r}-1}{\bar{r}}\widetilde{R}_{\bar{R}}=\frac{\bar{r}-1}{\bar{r}}\left(1-\frac{2\sum_{i=1}^{\bar{r}-1}i\widetilde{m}_i}{\bar{r}(\bar{r}-1)\widetilde{m}_{\bar{R}}}\right), \tag{12}$$

$$\widetilde{B}_{\bar{R}}=\frac{1}{\bar{r}-1}\sum_{i=1}^{\bar{r}-1}\frac{\widetilde{m}_{\bar{R}}-\widetilde{m}_i}{\widetilde{m}_{\bar{R}}}=1-\frac{1}{\bar{r}-1}\sum_{i=1}^{\bar{r}-1}\frac{\widetilde{m}_i}{\widetilde{m}_{\bar{R}}}. \tag{13}$$

Finally, in a fuzzy logic setting, the Head Count ratio in (5) can be generalized considering $\widetilde{p}=\sum_{i=1}^{n}\mu_{\widetilde{X}}(x_i)$ rather than p. We thus obtain

$$\widetilde{H}=\frac{\sum_{i=1}^{n}\mu_{\widetilde{X}}(x_i)}{n}=\frac{\widetilde{p}}{n}. \tag{14}$$

Proposition 1. The ratios in (11), (13) and (14) take values in [0,1], whereas the coefficient in (12) in [$0,(\bar{r}-1)/\bar{r}$].
Proof. See Appendix.

Remark 1. Let us suppose to deal with an income distribution such that $\bar{r}-p=0$. In other words, there do not exist observation units belonging to the non-poor & non-rich limbo. In this case, for each observation unit, the membership function in (1) takes only two values: 0 if $x_i < x_P$ and 1 if $x_i \geq x_R$. One may expect that the values of the extended ratios and indices in

(11)-(14) are the same as for the standard ones in (3)-(5) and (8). In fact, this is so for the Gini ratio and the Bonferroni index among the poor as well as for the head count ratio. On the contrary, the poverty-gap ratio in (3) and the non-richness-gap ratio in (11) usually give different scores even if the non-poor & non-rich limbo is empty. This can be explained and motivated as follows. First of all this depends on the fact that the former is built using x_P and the latter using x_R. In this respect, we used different names for referring to the two ratios. However, it is important to stress that, in the presence of different results, there is no reason to claim that the standard poverty-gap ratio is correct. Rather, since the theory of fuzzy sets offers us a greater capability to model the human common-sense reasoning with respect to the classical one, one may conclude that (11), based on the fuzzy logic approach to poverty, gives more reliable information than (3), based on the classical approach.

In order to derive new poverty measures in a fuzzy logic framework extending the ones by Sen (1976) and Giorgi and Crescenzi (2001), we adopt an axiomatic approach. As already noted, (6) and (9) satisfy a set of desirable axioms. These are the axioms of monotonicity, transfer, poor proportion, normalized poverty value and ordinal rank weights, even if the two measures differ in the definition of the weights. (6) and (9) can be expressed as a normalized weighted sum of the poverty-gaps g_i:

$$M = K(x_P, x)\sum_{i=1}^{p} g_i w_i(x_P, x)\,, \qquad (15)$$

where $K(x_P, x)$ is the normalizing term and $w_i\,(x_P, x)$ is the (non-negative) weight associated to the i-th observation. For simplicity of notation, in the sequel we will refer to the weights as w_i omitting (x_P, x). Sen (1976) proposes to weigh the poverty gap of the generic i-th poor by using the rank order of i in the interpersonal ordering of the poor. We thus get $w_i = p+1-i$. Giorgi and Crescenzi (2001) suggest setting $w_i = \sum_{j=i}^{p}\frac{1}{j}$ under the assumption that, if the position of the i-th poor in the income ordering is low, her/his perception of poverty is high. On the basis of (15), the above mentioned axioms determine one and only one poverty measure. In particular, the poverty measures in (6) or (9) are uniquely obtained according to the chosen weighting system. The normalizing term $K(x_P, x)$ is essential in order to satisfy the axiom of normalized poverty value. It is worth recalling that, according to such an axiom, if all the poor observations have the same income, then the poverty measure must be equal to HI.

In a fuzzy logic context, a similar set-up can be followed. More specifically, we now propose two poverty measures in a fuzzy logic setting starting from an extension of (15). In fact, we no longer consider the poverty-gaps but the non-richness-gaps. Then, following the fuzzy logic approach, the poverty measures can be expressed as a normalized weighted sum of the non-richness-gaps:

$$\widetilde{M} = \widetilde{K}(x_R, x)\sum_{i=1}^{\bar{r}} \widetilde{g}_i \widetilde{w}_i(x_R, x)\,. \qquad (16)$$

By comparing (15) and (16) we can also see that in (16) the normalizing term and the weights are constructed on the basis of the richness level x_R, and the weighted sum is computed over the $\bar{r}$ non-rich.

Let us now consider the extension of the Sen poverty measure S. The following axiom about the weighting system needs to be introduced.

Axiom (Ordinal Rank Weights). The weights being associated with the $\bar{r}$ non-richness-gaps are

$$\widetilde{w}_i(x_R, x) = \widetilde{w}_i = \mu_{\widetilde{X}}(x_i) \sum_{j=i}^{\bar{r}} \frac{j}{\sum_{k=1}^{j} \mu_{\widetilde{X}}(x_k)}, i = 1, \ldots, \bar{r} \quad . \qquad (17)$$

Remark 2. If in the income distribution there do not exist observation units belonging to the non-poor & non-rich limbo, $\bar{r} = p$ and all the poor are such that $\mu_{\widetilde{X}}(x_i) = 1$. In this case the weighting system in (17) coincides with the one proposed by Sen (1976). In fact, it can be proved that $\widetilde{w}_i = p + 1 - i$.

By this axiom, we can see that the weight associated to the *i*-th non-richness-gap depends on the degree of poverty of the observation involved (the poorer the observation is, the higher the weight is) and on the degrees of poverty associated to the non-rich, whose incomes are higher than *i* (the higher the number of richer non-rich is, the higher the weight is).

The fuzzy logic extension of (6) is obtained according to the following theorem.

Theorem 1. The only measure expressed in terms of (16) using (17) as system of weights, which satisfies the axiom of normalized poverty value is

$$\widetilde{S} = \widetilde{H}\left\{1 - (1 - \widetilde{I})\left[1 - \frac{\bar{r}}{\bar{r} + 1}\widetilde{G}_{\bar{R}}\right]\right\}. \qquad (18)$$

Proof. See appendix.

The poverty measure in (18) is the fuzzy logic extension of *S*.

The fuzzy extension of S_B can be obtained in a similar way. First of all, a new system of weights is introduced.

Axiom (Ordinal Rank Weights). The weights being associated with the $\bar{r}$ non-richness-gaps are

$$\widetilde{w}_i(x_R, x) = \widetilde{w}_i = \mu_{\widetilde{X}}(x_i) \sum_{j=i}^{\bar{r}} \frac{1}{\sum_{k=1}^{j} \mu_{\widetilde{X}}(x_k)}, i = 1, \ldots, \bar{r} \quad . \qquad (19)$$

By comparing (17) and (19), one can see that the only difference concerns the numerator in the sum.

Remark 3. When there do not exist observation units belonging to the non-poor & non-rich limbo, the weights in (19) coincide with the ones proposed by Giorgi and Crescenzi (2001). In fact, we have that $\widetilde{w}_i = \sum_{j=i}^{p} \frac{1}{j}$ taking into account that $\bar{r} = p$ and $\mu_{\widetilde{X}}(x_i) = 1$ for all the poor.

The following theorem is useful in order to derive the fuzzy logic extension of (9).

Theorem 2. The only measure expressed in terms of (16) using (19) as system of weights, which satisfies the axiom of normalized poverty value is

$$\widetilde{S}_B = \widetilde{H}\left\{1 - (1 - \widetilde{I})\left[1 - \frac{\bar{r} - 1}{\bar{r}} B_{\bar{R}}\right]\right\}. \qquad (20)$$

Proof. See Appendix.

The poverty measure in (20) is the fuzzy logic extension of S_B.

4.1. Poverty axioms fulfilled by $\tilde{S}$ and $\tilde{S}_B$

The fuzzy poverty measures $\tilde{S}$ and $\tilde{S}_B$ satisfy some desirable poverty axioms. We already saw that such measures fulfil the following two properties.

Axiom NPV (Normalized Poverty Value). If all the poor have the same income then the poverty measures are equal to $\tilde{H}\tilde{I}$.

Axiom ORW (Ordinal Rank Weights). The non-richness-gaps concerning the $\bar{r}$ poorest are weighted by means of the system of weights in (17) for $\tilde{S}$ and (19) for $\tilde{S}_B$.

In a fuzzy logic context, Chakravarty (2006) suggests a set of axioms for multi-dimensional poverty measures. Here, we consider such axioms limited to the univariate case. We get the following results.

Axiom FOC (FOCus). Given the population size *n*, the poverty measures depend only on the non-rich. Thus, the measures do not vary if the income of a rich observation changes. It is straightforward to see that $\tilde{S}$ and $\tilde{S}_B$ fulfil the axiom.

Axiom IMF (Increasing Membership Function). Let *X* and *Y* be two income distributions satisfying $x_1 \le x_2 \le \ldots \le x_n$ and $y_1 \le y_2 \le \ldots \le y_n$ with the same number $\bar{r}$ of non-rich. If, among the non-rich, it is $\mu_{\tilde{X}}(x_i) \le \mu_{\tilde{Y}}(y_i)$, that is the *i*-th non-rich in *X* is equally well-off or richer than the corresponding one in *Y*, then the poverty measure for *X* is lower than that for *Y*. This property can be proved for $\tilde{S}$ and $\tilde{S}_B$ from an empirical point of view.

Axiom MON (MONotonicity). Poverty decreases if the income of a non-rich increases. In fact, it is important to note that Chakravarty (2006) introduces such an axiom in a general form. In this work, we consider two levels of monotonicity, limiting our attention to the case in which the increase of income does not imply that the observation involved becomes rich.

- If the increase of income concerns a poor and such an observation remains poor (thus, (s)he does not becomes non-poor & non-rich), the non-richness-gap associated to the richer observation unit decreases. Since the weights $\tilde{w}_i$'s do not change, on the basis of (16), it is clear that $\tilde{S}$ and $\tilde{S}_B$ decrease. We may refer to this case as Minimal MONotonicity (M-MON).
- If the increase of income concerns a poor becoming non-poor & non-rich or a non-poor & non-rich (without becoming rich), the weights $\tilde{w}_i$'s vary. Therefore, the reasoning adopted for M-MON is no longer applicable. However, on the basis of empirical analyses, we conjecture that $\tilde{S}$ and $\tilde{S}_B$ decrease. We refer to this property as Weak MONotonicity (W-MON).

Axiom NOR (NORmalitation). If all the observation units are rich, then the measures are equal to zero. In this case, it is $\tilde{H} = 0$ and, therefore, $\tilde{S} = \tilde{S}_B = 0$.

Axiom NPG (Non-Poverty Growth). Poverty decreases if a rich is added to the income distribution. In this case, $\tilde{G}_{\bar{R}}$, $\tilde{B}_{\bar{R}}$ and $\tilde{I}$ do not change. On the contrary, $\tilde{H}$ decreases since its denominator increases. It follows that $\tilde{S}$ and $\tilde{S}_B$ decrease.

Axiom SCI (SCale Invariance). The poverty measures are invariant under scale transformations of the income. Let *X* and *Y* be two income distributions with *X*=*cY* with *c*>0. The threshold values are $x_P=cy_P$ and $x_R=cy_R$. It is ${}^X\tilde{g}_i = x_R - x_i = c(y_R - y_i) = c\,{}^Y\tilde{g}_i$ and, hence,

$\sum_{i=1}^{\bar{r}} {}^{X}\widetilde{g}_i\widetilde{w}_i = c\sum_{i=1}^{\bar{r}} {}^{Y}\widetilde{g}_i\widetilde{w}_i$, where the subscripts help us to specify the income distributions. Moreover, it is $\widetilde{K}(x_R, x) = (1/c)\widetilde{K}(y_R, y)$. Therefore, taking into account (16), it follows that ${}^{X}\widetilde{S} = {}^{Y}\widetilde{S}$ and ${}^{X}\widetilde{S}_B = {}^{Y}\widetilde{S}_B$.

Axiom SYM (SYMmetry). Any characteristic other than the income information does not affect the measurement of poverty. $\tilde{S}$ and $\tilde{S}_B$ fulfil the axiom.

Axiom TRP (TRansfer Principle). Poverty increases in the case of an income transfer from a poor to some extent (poor or non-poor & non-rich) to anyone who is richer. As it was done for MON, we limit our attention to some specific cases.

- If the transfer occurs between two poor with no one changing her/his status (we refer to this case as Minimal TRansfer Principle (M-TRP)), the transfer does not change the system of weights, but affects the non-richness-gaps associated to the two observations involved. In particular, if *t* is the amount of income transfer, the non-richness-gap of the donor increases by *t*, whereas the one of the recipient decreases by *t*. As the weight associated to the donor is higher than that of the recipient (taking into account that the donor is poorer than the recipient), from (16) it follows that $\tilde{S}$ and $\tilde{S}_B$ increase. When the transfer occurs between two observations belonging to the non-poor & non-rich limbo, the transfer changes the system of weights. However, also in this case, we conjecture that the M-TRP property still holds.
- If the recipient is rich (we refer to this case as Weak TRansfer Principle (W-TRP)) and the donor is poor, then the transfer increases her/his non-richness-gap without modifying the system of weights. Therefore, it is clear that $\tilde{S}$ and $\tilde{S}_B$ increase. When the donor is non-poor & non-rich, the transfer modifies not only the associated non-richness-gap but also the system of weights. However, on the basis of empirical analyses, we get that $\tilde{S}$ and $\tilde{S}_B$ still fulfil W-TRP.

5. Applications

In this section we compute the poverty measures in (18) and (20) as well as their counterparts according to the classical approach on two data sets. First a synthetic data set is considered. It is composed by two populations of size *n*=10 on which the income is observed. Then a real data set about the Italian population income in 2001 is analyzed.

5.1. Synthetic data

The data refer to two income distributions, each concerning *n*=10 observation units and are reported in Table 1.

Table 1. Income distributions *X* and *Y*

Income distribution *X*					Income distribution *Y*			
Obs. unit *i*	Income x_i (€)	Classical $\mu_X(x_i)$	Fuzzy $\mu_{\tilde{X}}(x_i)$		Obs. unit *i*	Income y_i (€)	Classical $\mu_Y(y_i)$	Fuzzy $\mu_{\tilde{Y}}(y_i)$
1	700	1	1		1	700	1	1
2	800	1	1		2	800	1	1
3	950	1	1		3	950	1	1
4	1,050	0	0.95		4	1,050	0	0.95
5	1,100	0	0.90		5	1,850	0	0.15
6	1,150	0	0.85		6	1,900	0	0.10
7	1,200	0	0.80		7	1,950	0	0.05
8	2,100	0	0		8	2,100	0	0
9	2,200	0	0		9	2,200	0	0
10	2,300	0	0		10	2,300	0	0

We assume that x_P=y_P=€ 1,000 and x_R=y_R=€ 2,000. According to these thresholds, we can see that there are p=3 poor and $\bar{r}=7$ non-rich in both the distributions. Furthermore, the distributions *X* and *Y* have poor and rich with the same income values, whereas those belonging to the non-poor & non-rich limbo ($\bar{r}-p=4$ observations) have different incomes, with the exception of observation 4. Specifically, in the income distribution *X*, observations 5-7 have incomes slightly higher than x_P, while, in the income distribution *Y*, their incomes are slightly lower than y_R. Therefore, even if such observations are classified as non-poor & non-rich, their economical status is very different. In fact, the common human sense suggests considering those from *X* as *approximately poor* and those from *Y* as *approximately rich*. Only the fuzzy logic approach to poverty allows us to suitably handle such a situation. This can be seen in Table 1, which also contains the membership function values according to (1) assuming a linear decreasing trend (i.e. β =1) and those resulting from the classical approach using (2). In fact, observations 1-3 and 8-10 have the same membership function values (1 and 0, respectively) adopting either the fuzzy logic or classical approach. On the contrary, the use of the classical or fuzzy approach leads to different values of the membership function associated to observations 4-7 belonging to the limbo. Following the classical approach they are classified as rich because their incomes are (slightly or remarkably) higher than the poverty line and their membership function values are equal to 0. Following the fuzzy logic approach, the membership function values are higher than 0. In particular, they are poor to a very high (low) extent if their incomes are very close to (far from) x_P.

To evaluate the level of poverty in the two distributions we compute the indices *S* and S_B obtaining S^X=S^Y=0.0675 and $S_B^X=S_B^Y=0.0733$, where the letters *X* and *Y* denote the corresponding distribution. Hence, the classical approach to poverty is inadequate to distinguish the two income distributions. Obviously, this depends on the too rigid assumption according to which the level of deprivation is measured with respect to x_P. On the whole, *X* is poorer than *Y* but this derives from some observations that can be reasonably considered neither definitively poor nor definitively rich. From a computational point of view, this occurs because the measures in (6) and (9) are built only on the basis of the definitively poor observations.

In view of the fuzzy logic approach, we get $\tilde{S}^X=0.3566$ and $\tilde{S}_B^X=0.3721$ for *X* and $\tilde{S}^Y=0.2361$ and $\tilde{S}_B^Y=0.2452$ for *Y*. It is clear that the fuzzy logic approach allows us to highlight the existing differences between the two distributions. In fact, the poverty measures for X are uniformly higher than the corresponding ones for Y. We can conclude that, on the whole, X is poorer than Y.

Summing up, this result shows that the fuzzy logic approach is more powerful than the classical approach since it allows us to consider also the observations poor to some extent when measuring the poverty level of a distribution. In doing so, the flexibility of the fuzzy logic approach plays a crucial role. Unfortunately, this requires setting not only the poverty threshold x_P, but also the richness threshold x_R and the shape of the membership function β. Thus, one may argue that the classical approach makes things easier because it only needs to set x_P. By means of a small sensitive analysis we are going to show that the values of the poverty measures in (18) and (20) are more stable if compared with those in (6) and (9). In other words, we aim at stressing that the 'black or white' nature (poor or non-poor) of the classical approach to poverty implies that a small change of the parameter x_P may lead to conflicting results. By contrast, the 'grey-scale' nature (poor to some extent) of the fuzzy logic approach to poverty implies that small changes of the parameters x_P, x_R and β do not lead to remarkably different conclusions. We thus inspect how S, S_B, $\tilde{S}$ and $\tilde{S}_B$ vary by increasing or decreasing x_P and x_R by € 100 and by choosing β =1/2 or β =2 in (1). Note that if β =2 the membership function strongly decreases when $x>x_P$. Thus, as soon as the income x is higher than x_P the corresponding level of poverty strongly decreases, i.e. the observation is rich with a fairly high degree. The opposite comment holds if β =1/2. The results of the sensitivity analysis are reported in Table 2.

Table 2. Results of the sensitivity analysis

x_P	x_R	β	$S^X = S^Y$	$S_B^X = S_B^Y$	$\tilde{S}^X$	$\tilde{S}_B^X$	$\tilde{S}^Y$	$\tilde{S}_B^Y$
1,000	2,000	1			0.3566	0.3721	0.2361	0.2452
1,000	2,000	1/2			0.3124	0.3247	0.2229	0.2297
1,000	2,000	2			0.3781	0.3952	0.2467	0.2588
1,000	1,900	1			0.3384	0.3546	0.2236	0.2305
1,000	1,900	1/2	0.0675	0.0733	0.2957	0.3083	0.2134	0.2195
1,000	1,900	2			0.3607	0.3787	0.2283	0.2358
1,000	2,100	1			0.3729	0.3879	0.2545	0.2654
1,000	2,100	1/2			0.3277	0.3396	0.2365	0.2443
1,000	2,100	2			0.3937	0.4101	0.2723	0.2879
900	2,000	1			0.3382	0.3525	0.2285	0.2369
900	2,000	1/2			0.2801	0.2904	0.2037	0.2095
900	2,000	2			0.3728	0.3896	0.2443	0.2558
900	1,900	1			0.3196	0.3342	0.2160	0.2224
900	1,900	1/2	0.0370	0.0389	0.2638	0.2742	0.1940	0.1990
900	1,900	2			0.3547	0.3722	0.2267	0.2342
900	2,100	1			0.3551	0.3690	0.2464	0.2566
900	2,100	1/2			0.2951	0.3052	0.2169	0.2236
900	2,100	2			0.3890	0.4051	0.2689	0.2837
1100	2,000	1			0.3733	0.3902	0.2397	0.2492
1100	2,000	1/2			0.3527	0.3680	0.2347	0.2425
1100	2,000	2			0.3811	0.3985	0.2487	0.2614
1100	1,900	1			0.3560	0.3736	0.2266	0.2338
1100	1,900	1/2	0.1036	0.1144	0.3357	0.3517	0.2254	0.2324
1100	1,900	2			0.3642	0.3825	0.2289	0.2366
1100	2,100	1			0.3890	0.4051	0.2588	0.2704
1100	2,100	1/2			0.3681	0.3828	0.2488	0.2576
1100	2,100	2			0.3963	0.4130	0.2755	0.2919
Mean			0.0694	0.0755	0.3489	0.3640	0.2358	0.2450
Standard Deviation			0.0272	0.0309	0.0348	0.0368	0.0195	0.0224
Coefficient of Variation			0.3923	0.4086	0.0998	0.1011	0.0829	0.0915

By inspecting Table 2 we can see that the poverty measures based on the classical approach are highly affected by the choice of x_P. The average values are $\overline{S} = 0.0694$ and $\overline{S}_B = 0.0755$ and the standard deviations $\sigma_S = 0.0272$ and $\sigma_{S_B} = 0.0309$. Thus, the observed standard deviations are noticeably high if compared with the sizes of the corresponding average values. Hence, we get the coefficients of variation $CV_S = 0.3923$ and $CV_{S_B} = 0.4086$. These values are remarkably lower than those observed in the fuzzy logic approach despite varying according to the choices of x_P, x_R and β. Specifically, we can see that $CV_{\tilde{S}^X} = 0.0998$ and $CV_{\tilde{S}_B^X} = 0.1011$ for X and $CV_{\tilde{S}^Y} = 0.0829$ and $CV_{\tilde{S}_B^Y} = 0.0915$ for Y. From Table 2 we also observe that $\tilde{S}$ and $\tilde{S}_B$ slightly increase whenever x_P, x_R or β increase.

All in all, we can conclude that the fuzzy logic approach is less sensitive than the classical approach with respect to the choices of the threshold values and, in the former case, the shape of the membership function. Specifically, it is reasonable to assume that the status of an observation with a given income does not *strongly* vary if x_P (or x_R or β) *slightly* increases or decreases. This is consistent with the common human thinking and, therefore, with the fuzzy logic approach due to its flexibility. By contrast, this is not the case for the classical approach.

5.2. Real data

In this section we determine the values of $\tilde{S}$ and $\tilde{S}_B$ for a real data set. The data here considered come from the 2001 Istat (Italian national institute of statistics) Household Budget Survey. Specifically, to measure poverty in Italy, Istat uses a survey based on a sample of households. A two-stage cluster sample is adopted where the "primary" sampling units are municipalities and the "secondary" sampling units are households. A stratified sampling is used for the "primary" sampling units. The main object of the survey is to record expenditures made for goods and services (in addition to possible self-consumption) by households. However, the survey measures in detail all items relative to consumption expenditures, the main socio-characteristics of the household components, the main characteristics of the home, the ownership of permanent goods and some information on the wage and savings. The available survey database contains all these items relative to monthly consumption expenditures (obtained by means of Istat data processing). The total monthly consumption expenditure (net from expenses on extraordinary home maintenance, loans and premiums paid for life insurance and life annuity that are not part of the economic concept of consumption expenses) is the main variable used for analyzing poverty. In fact, to assess whether a given household is poor, Istat compares the corresponding total monthly consumption expenditure with a (monthly) poverty threshold to be discussed below. Finally, a weight is associated to every household in order to obtain suitable estimates of the population statistics. See, for further details, Istat (2001).

To compute $\tilde{S}$ and $\tilde{S}_B$ the following steps are needed. First, we determined the total monthly consumption expenditure for every household. However, this is not yet the distribution to be analyzed for deriving the values of $\tilde{S}$ and $\tilde{S}_B$. In fact, the standard poverty measure (hereinafter, *spm*) adopted by Istat (following the International Standard of Poverty Line) is equal to the mean per capita total monthly consumption expenditure (€ 814.55 in 2001). Such a poverty threshold is good only for households with two components. Thus it must be modified for households with number of members different from two according to the equivalence scale (Carbonaro, 1985) reported in Table 3.

Table 3. Equivalence scale

Household components	Normalizing factor (*nf*)	Normalized poverty threshold (*nspm*=814.55/*nf*)
1	0.60	488.73
2	1.00	814.55
3	1.33	1,083.35
4	1.63	1,327.72
5	1.90	1,547.65
6	2.16	1,759.43
7	2.40	1,954.92
≥8	2.62	2,134.12

This allows us to assess poverty for different sized households. However, for our purpose, we need to perform the inverse procedure with respect to the one given in Table 3. More specifically, rather than normalizing the poverty threshold *spm* by means of the specific normalizing factor *nf* (and comparing it with the observed total monthly consumption expenditure), we need to normalize every total monthly consumption expenditure. This was done by multiplying it by 1/*nf* according to the household size. For instance, if the observed total monthly consumption expenditure of a household with three components is € 1,500, then the normalized one is 1,500 / 1.33 = 1,127.82.

The next step was how to choose the poverty and richness thresholds. To do it, we decided to follow some suggestions by Istat (Istat, 2002). A household is considered *definitively* poor if its total monthly consumption expenditure is lower than a percentage of 80% of *spm* (0.8 × 814.55 = 651.64). We thus set x_P = € 651.64. Similarly, a household is considered *definitively* non-poor (i.e. *definitively* rich) if its total monthly consumption expenditure is higher than a percentage of 120% of *spm* (1.2 × 814.55 = 977.46). In this way we obtained x_R = € 977.46. Note also that Istat refers to households with total monthly consumption expenditure between € 651.64 (80% of *spm*) and € 814.55 (100% of *spm*) as *nearly* poor and those with total monthly consumption expenditure between € 814.55 (100% of *spm*) and € 977.46 (120% of *spm*) as *almost* poor. Therefore, we can say that Istat implicitly considers the concept of poverty in a vague way (admitting degrees of poverty) but, unfortunately, without managing it by a fuzzy logic approach. In fact, a household considered almost poor would be poor to some extent (i.e. with a membership function value strictly lower than one). Nonetheless, this is fully missed by Istat which introduces some classical (non-fuzzy) sets (definitively poor, nearly poor, almost poor, definitively non-poor) according to the classical approach to poverty rather than a unique fuzzy set allowing us to deal jointly with the different degrees of poverty.

The last step concerned the way to take into account the weights associated to the households, say α_i's. In order to estimate the population values of $\tilde{S}$ and $\tilde{S}_B$, suitable modifications of (10)-(14) are required. First of all, the number of poor $\tilde{p}$ in (10) should be replaced by

$$\tilde{p}_\alpha = \sum_{i=1}^{n} \alpha_i \mu_{\tilde{X}}(x_i), \tag{21}$$

where *n* is the sample size, and the non-richness-gap ratio is then defined as

$$\tilde{I}_\alpha = \frac{\sum_{i=1}^{\bar{r}} \alpha_i \mu_{\tilde{X}}(x_i)}{x_R \tilde{p}_\alpha}, \tag{22}$$

where $\bar{r}$ denotes the number of non-rich observation units in the sample. Furthermore, by taking into account the weights α_i's in the definitions of $\tilde{m}_{\bar{R}}$ and $\tilde{m}_i$, $i = 1, \ldots, \bar{r} - 1$, we obtained $\tilde{m}_{\bar{R}\alpha} = \dfrac{\sum_{i=1}^{\bar{r}} x_i \mu_{\tilde{X}}(x_i)\alpha_i}{\sum_{i=1}^{\bar{r}} \mu_{\tilde{X}}(x_i)\alpha_i}$ and $\tilde{m}_{i\alpha} = \dfrac{\sum_{j=1}^{i} x_j \mu_{\tilde{X}}(x_j)\alpha_j}{\sum_{j=1}^{i} \mu_{\tilde{X}}(x_j)\alpha_j}$, respectively. We thus suggest computing the Gini coefficient and the Bonferroni index among the non-rich as

$$\tilde{G}_{\bar{R}\alpha} = \frac{\left(\sum_{j=1}^{\bar{r}-1} \alpha_j\right)}{\left(\sum_{j=1}^{\bar{r}} \alpha_j\right)} \left[1 - \frac{2 \sum_{i=1}^{\bar{r}-1} \tilde{m}_i \sum_{j=1}^{i} \alpha_j}{\tilde{m}_{\bar{R}} \left(\sum_{j=1}^{\bar{r}} \alpha_j\right)\left(\sum_{j=1}^{\bar{r}-1} \alpha_j\right)} \right], \tag{23}$$

$$\tilde{B}_{\bar{R}\alpha} = 1 - \frac{1}{\sum_{j=1}^{\bar{r}-1} \alpha_j} \sum_{i=1}^{\bar{r}-1} \frac{\tilde{m}_i \alpha_i}{\tilde{m}_{\bar{R}}}. \tag{24}$$

Finally, (14) can be replaced by

$$\widetilde{H}_\alpha = \frac{\widetilde{p}_\alpha}{\sum_{i=1}^{n} \alpha_i}. \tag{25}$$

Using (21)-(25) we got

$$\widetilde{S}_\alpha = \widetilde{H}_\alpha \left\{ 1-(1-\widetilde{I}_\alpha)\left[1-\frac{\sum_{j=1}^{\bar{r}} \alpha_j}{\sum_{j=1}^{\bar{r}+1} \alpha_j} \widetilde{G}_{\bar{R}\alpha} \right] \right\}, \tag{26}$$

$$\widetilde{S}_{B\alpha} = \widetilde{H}_\alpha \left\{ 1-(1-\widetilde{I}_\alpha)\left[1-\frac{\sum_{j=1}^{\bar{r}-1} \alpha_j}{\sum_{j=1}^{\bar{r}} \alpha_j} B_{\bar{R}\alpha} \right] \right\}, \tag{27}$$

and, setting β=1, we found $\widetilde{S}_\alpha = 0.1322$ and $\widetilde{S}_{B\alpha} = 0.0572$.

To sum up, in the case of sample data, the population poverty measures $\tilde{S}$ and $\tilde{S}_B$ can be derived. However, to do it, their ingredients in (10)-(14) must be modified in a suitable way to take into account the weights associated with the sampling units. This involves revisiting $\tilde{S}$ and $\tilde{S}_B$ in (18) and (20) by means of (26) and (27).

6. Discussion

In this paper, we have followed the fuzzy logic approach for measuring the level of poverty in a distribution. This allowed us to evaluate the poverty level of a given observation according to a degree (membership function) ranging from 0 (completely non-poor or completely rich) to 1 (completely poor). For instance, a degree equal to 0.8 means that the observation is poor with a degree equal to 0.8. At the same time, this observation is non-poor (rich) with a degree equal to 0.2. Therefore, in the fuzzy logic approach, an observation can be simultaneously poor and non-poor with. Of course, this contradicts the classical (Aristotelian) logic in which an observation is either completely poor or completely non-poor. Generally speaking, poverty is a vague concept. By examples, we have shown that tools based on the fuzzy logic approach seem to be more fruitful than those based on the classical ones in order to cope with the vagueness associated to the concept of poverty.

According to the classical approach, several indices have been proposed in the literature for measuring the poverty level of a distribution. In this paper, following the fuzzy logic approach, we have generalized the well-known poverty measure S based on the classical approach provided by Sen (1976) and its extension S_B (Giorgi and Crescenzi, 2001) involving the use of the Bonferroni index in place of the Gini ratio. The new poverty measures have been denoted by $\tilde{S}$ and $\tilde{S}_B$, respectively. The most relevant difference between S and S_B, on the one hand, and $\tilde{S}$ and $\tilde{S}_B$, on the other hand, is that the former ones are constructed considering the p poor whereas the latter ones take into account the $\bar{r}$ non-rich, i.e. the completely poor and the poor to some extent. Thus, in the fuzzy logic approach, the indices cover the gray area of the non-rich and non-poor. If this gray area is fairly large in a society, the limitation of using a particular poverty line is obvious. If the distribution contains only poor or rich (when $p=\bar{r}$), then the indices based on the classical and fuzzy logic approaches give almost equal outputs. We have investigated how $\tilde{S}$ and $\tilde{S}_B$ work by means of some applications on real and simulated data and we have found that they work better than their classical counterparts.

In the literature, starting from Sen (1976), a great deal of attention has been paid to the derivation of desirable properties (axioms) that a poverty measure should fulfil. The debate mainly focused on the classical approach to poverty. Nonetheless, Chakravarty (2006) introduced a set of axioms to be fulfilled by a poverty measure according to the fuzzy logic approach. Along the paper, we have shown the axioms satisfied by $\tilde{S}$ and $\tilde{S}_B$. In principle, a poverty measure fulfilling all the possible axioms would be the ideal one. Unfortunately, it is recognized that there does not exist such an ideal index. Nonetheless, several authors tried to improve some existing indices in order to construct more satisfactory measures of poverty. For instance, Shorrocks (1995) noted that *S* is not replication invariant, it is not a continuous function of individual incomes and it fails to satisfy the transfer axioms and, to account for these drawbacks, proposed a modified version of *S*, say S_{Mod}. However, as remarked by Shorrocks (1995), S_{Mod} still fails to be subgroup consistent. In the future, it will be interesting to provide a modified version of $\tilde{S}$ taking into account S_{Mod} (hopefully leading to $\tilde{S}_{Mod}$, the fuzzy logic version of S_{Mod}). It is interesting to see that the replication invariant issue can easily be accommodated by replacing (18) as

$$\tilde{S}_{Mod(RI)} = \tilde{H}\left\{1-(1-\tilde{I})\left[1-\tilde{G}_{\tilde{R}}\right]\right\}. \qquad (21)$$

In (21), the asymptotic approximation of $\tilde{S}$ is considered with $\frac{\bar{r}}{\bar{r}+1} \to 1$, when the population size (and the number of non-rich) is large.

In the income inequality research, several works about graphical devices such as (generalized) Lorenz curve can be found. In other words, the (generalized) Lorenz curve can be used for analyzing the relative income differences (see, for instance, Yaari, 1987; Jenkins and Lambert, 1997; Aaberge, 2001). An interesting line of research is to provide alternative justifications of the proposed indices following the fuzzy logic approach as well as for their 'ingredients', in particular the fuzzy logic versions of the Gini and Bonferroni coefficients. This could be done by developing fuzzy logic extensions of the (generalized) Lorenz curve and looking for possible geometric interpretations of the here-proposed indices and coefficients.

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Appendix

Proof of Proposition 1. We start considering (11). When all the observation units are rich, it is $x_i \geq x_R$ and, from (1), $\mu_{\tilde{X}}(x_i) = 0$. Hence, $\bar{\tilde{g}} = 0$ and, thus, $\tilde{I} = 0$. On the contrary, if all the observation units are such that $x_i = 0$, we have that $\tilde{g}_i = x_R$ and $\mu_{\tilde{X}}(0) = 1$. It follows that

$$\bar{\tilde{g}} = \frac{\sum_{i=1}^{n} x_R \cdot 1}{n} = \frac{n x_R}{n} = x_R$$ and, therefore, $\tilde{I} = 1$.

If all the (poor) observation units have the same income (perfect equality), say x, we obtain $\tilde{m}_i = \frac{\sum_{j=1}^{i} x\mu_{\tilde{X}}(x)}{\sum_{j=1}^{i}\mu_{\tilde{X}}(x)} = x$ and $\tilde{m}_{\bar{r}} = \frac{\sum_{i=1}^{\bar{r}} x\mu_{\tilde{X}}(x)}{\sum_{i=1}^{\bar{r}}\mu_{\tilde{X}}(x)} = x$, which thus coincide ($\tilde{m}_i = \tilde{m}_{\bar{r}} = x$, $i = 1,\ldots,\bar{r}-1$ with $\bar{r} = n$). Using this in (12) yields $\tilde{R}_{\bar{R}} = 1 - \frac{2\sum_{i=1}^{\bar{r}-1} i}{\bar{r}(\bar{r}-1)} = 1 - \frac{2\frac{\bar{r}(\bar{r}-1)}{2}}{\bar{r}(\bar{r}-1)} = 0$. We then get $\tilde{G}_{\bar{R}} = 0$. In the case of perfect inequality, we have $x_1=\ldots=x_{n-1}=0$ and $x_n=x$. Note that, since the Gini coefficient is computed among the non-rich, it is $x<x_R$. It follows that $\tilde{m}_{\bar{r}} = \frac{0\cdot(n-1) + x\mu_{\tilde{X}}(x)}{(n-1)+\mu_{\tilde{X}}(x)} = \frac{x\mu_{\tilde{X}}(x)}{(n-1)+\mu_{\tilde{X}}(x)}$ and $\tilde{m}_i = \frac{\sum_{j=1}^{i} 0\cdot 1}{\sum_{j=1}^{i} 1} = 0$, $i = 1,\ldots,\bar{r}-1$ with $\bar{r} = n$).

From (12) we have $\tilde{R}_{\bar{R}} = 1 - \frac{2\sum_{i=1}^{\bar{r}-1} i\cdot 0}{\bar{r}(\bar{r}-1)\tilde{m}_{\bar{r}}} = 1$. Thus, $\tilde{G}_{\bar{R}} = (\bar{r}-1)/\bar{r}$.

Using the same reasoning adopted for $\tilde{R}_{\bar{R}}$, we obtain the following results for $\tilde{B}_{\bar{R}}$. In the case of perfect equality, we have $\tilde{B}_{\bar{R}} = 1 - \frac{1}{\bar{r}-1}\sum_{i=1}^{\bar{r}-1}\frac{x}{x} = 0$, while, in the vcase of perfect inequality, we have $\tilde{B}_{\bar{R}} = 1 - \frac{1}{\bar{r}-1}\sum_{i=1}^{\bar{r}-1}\frac{0}{\tilde{m}_{\bar{r}}} = 1$.

With regard to the Head-Count ratio in (14), let us start with the case with all rich observations. We have $\tilde{p} = \sum_{i=1}^{n} 0 = 0$ from which $\tilde{H} = 0$. When all the observations are poor and have zero income, we get $\tilde{p} = \sum_{i=1}^{n} 1 = n$ and $\tilde{H} = 1$. □

Proof of Theorem 1. We have that $\sum_{i=1}^{\bar{r}} \tilde{g}_i\tilde{w}_i = \sum_{i=1}^{\bar{r}}(x_R - x_i)\tilde{w}_i = x_R\sum_{i=1}^{\bar{r}}\tilde{w}_i - \sum_{i=1}^{\bar{r}} x_i\tilde{w}_i$. Bearing in mind (17), we also have that

$$\sum_{i=1}^{\bar{r}}\tilde{w}_i = \sum_{i=1}^{\bar{r}}\mu_{\tilde{X}}(x_i)\sum_{j=i}^{\bar{r}}\frac{j}{\sum_{k=1}^{j}\mu_{\tilde{X}}(x_k)} = \frac{\bar{r}(\bar{r}+1)}{2}, \tag{A1}$$

$$\sum_{i=1}^{\bar{r}} i\tilde{m}_i = \sum_{i=1}^{\bar{r}} i\frac{\sum_{j=1}^{i} x_j\mu_{\tilde{X}}(x_j)}{\sum_{j=1}^{i}\mu_{\tilde{X}}(x_j)} = \sum_{i=1}^{\bar{r}} x_i\mu_{\tilde{X}}(x_i)\sum_{j=1}^{\bar{r}}\frac{j}{\sum_{k=1}^{j}\mu_{\tilde{X}}(x_k)} = \sum_{i=1}^{\bar{r}} x_i\tilde{w}_i .$$

(A
2)

In (12), $\tilde{G}_{\bar{R}}$ and, thus, $\tilde{R}_{\bar{R}}$ are based on $\sum_{i=1}^{\bar{r}-1} i\tilde{m}_i$. However, (12) can be rewritten as $\tilde{R}_{\bar{R}} = 1 + \frac{2}{\bar{r}-1} - \frac{2}{\tilde{m}_{\bar{r}}\bar{r}(\bar{r}-1)}\sum_{i=1}^{\bar{r}} i\tilde{m}_i$, from which

$$\sum_{i=1}^{\bar{r}} i\tilde{m}_i = \left(\frac{\bar{r}+1}{\bar{r}-1} - \tilde{R}_{\bar{R}}\right)\frac{\bar{r}(\bar{r}-1)}{2}\tilde{m}_{\bar{r}}\,. \tag{A3}$$

Substituting (A3) into (A2) leads to

$$\sum_{i=1}^{\bar{r}} x_i\tilde{w}_i = \left(\frac{\bar{r}+1}{\bar{r}-1} - \tilde{R}_{\bar{R}}\right)\frac{\bar{r}(\bar{r}-1)}{2}\tilde{m}_{\bar{r}}\,. \tag{A4}$$

By using (A1) and (A4) and bearing in mind the relation between $\tilde{G}_{\bar{R}}$ and $\tilde{R}_{\bar{R}}$ given in (12) we then get

$$\begin{aligned}\sum_{i=1}^{\bar{r}} \tilde{g}_i\tilde{w}_i &= \sum_{i=1}^{\bar{r}} (x_R - x_i)\tilde{w}_i = x_R\frac{\bar{r}(\bar{r}+1)}{2} - \left(\frac{\bar{r}+1}{\bar{r}-1} - \frac{\bar{r}}{\bar{r}-1}\tilde{G}_{\bar{R}}\right)\frac{\bar{r}(\bar{r}-1)}{2}\tilde{m}_{\bar{r}} \\ &= x_R\frac{\bar{r}(\bar{r}+1)}{2}\left[1 - \frac{\tilde{m}_{\bar{r}}}{x_R}\left(1 - \frac{\bar{r}}{\bar{r}+1}\tilde{G}_{\bar{R}}\right)\right].\end{aligned} \tag{A5}$$

Moreover, by exploiting (11), we have that $\tilde{I} = \frac{\sum_{i=1}^{n}(x_R - x_i)\mu_{\tilde{X}}(x_i)}{x_R\sum_{i=1}^{n}\mu_{\tilde{X}}(x_i)} = 1 - \frac{\sum_{i=1}^{n}x_i\mu_{\tilde{X}}(x_i)}{x_R\sum_{i=1}^{n}\mu_{\tilde{X}}(x_i)} = 1 - \frac{\tilde{m}_{\bar{r}}}{x_R}$

and, thus, $\frac{\tilde{m}_{\bar{r}}}{x_R} = 1 - \tilde{I}$. Therefore, (A5) can be rewritten as

$$\sum_{i=1}^{\bar{r}} \tilde{g}_i\tilde{w}_i = x_R\frac{\bar{r}(\bar{r}+1)}{2}\left[1 - (1-\tilde{I})\left(1 - \frac{\bar{r}}{\bar{r}+1}\tilde{G}_{\bar{R}}\right)\right]. \tag{A6}$$

Upon substituting (A6) in (16), we obtain that

$$\tilde{S} = \tilde{K}(x_R, x)x_R\frac{\bar{r}(\bar{r}+1)}{2}\left[1 - (1-\tilde{I})\left(1 - \frac{\bar{r}}{\bar{r}+1}\tilde{G}_{\bar{R}}\right)\right]. \tag{A7}$$

In order to fulfil the axiom of normalized poverty value, taking into account (14), it must be $\tilde{K}(x_R, x) = \frac{2\tilde{p}}{x_R n\bar{r}(\bar{r}+1)}$. We thus get (18). □

Proof of Theorem 2. It is easy to see that

$$\sum_{i=1}^{\bar{r}} \tilde{w}_i = \sum_{i=1}^{\bar{r}} \mu_{\tilde{X}}(x_i)\sum_{j=i}^{\bar{r}}\frac{1}{\sum_{k=1}^{j}\mu_{\tilde{X}}(x_k)} = \bar{r}\,, \tag{A8}$$

$$\sum_{i=1}^{\bar{r}} \tilde{m}_i = \sum_{i=1}^{\bar{r}} i\frac{\sum_{j=1}^{i}x_j\mu_{\tilde{X}}(x_j)}{\sum_{j=1}^{i}\mu_{\tilde{X}}(x_j)} = \sum_{i=1}^{\bar{r}} x_i\mu_{\tilde{X}}(x_i)\sum_{j=1}^{\bar{r}}\frac{1}{\sum_{k=1}^{j}\mu_{\tilde{X}}(x_k)} = \sum_{i=1}^{\bar{r}} x_i\tilde{w}_i\,. \tag{A9}$$

The Bonferroni index among the poor in (13), based on $\sum_{i=1}^{\bar{r}-1}\tilde{m}_i$, can be rewritten as $\tilde{B}_{\bar{R}} = 1 + \frac{1}{\bar{r}-1} - \frac{1}{\tilde{m}_{\bar{r}}(\bar{r}-1)}\sum_{i=1}^{\bar{r}}\tilde{m}_i$, from which

$$\sum_{i=1}^{\bar{r}} \tilde{m}_i = \left(\frac{\bar{r}}{\bar{r}-1} - \tilde{B}_{\bar{R}} \right)(\bar{r}-1)\tilde{m}_{\bar{r}} \,. \tag{A10}$$

Substituting (A10) into (A9) leads to

$$\sum_{i=1}^{\bar{r}} x_i \tilde{w}_i = \left(\frac{\bar{r}}{\bar{r}-1} - \tilde{B}_{\bar{R}} \right)(\bar{r}-1)\tilde{m}_{\bar{r}} \,. \tag{A11}$$

From (A8) and (A11), we have that

$$\begin{aligned} \sum_{i=1}^{\bar{r}} \tilde{g}_i \tilde{w}_i &= \sum_{i=1}^{\bar{r}} (x_R - x_i)\tilde{w}_i = x_R \bar{r} - \left(\frac{\bar{r}}{\bar{r}-1} - \tilde{B}_{\bar{R}} \right)(\bar{r}-1)\tilde{m}_{\bar{r}} \\ &= x_R \bar{r} \left[1 - \frac{\tilde{m}_{\bar{r}}}{x_R} \left(1 - \frac{\bar{r}-1}{\bar{r}} \tilde{B}_{\bar{R}} \right) \right]. \end{aligned} \tag{A12}$$

Since $\frac{\tilde{m}_{\bar{r}}}{x_R} = 1 - \tilde{I}$, (A12) reduces to

$$\sum_{i=1}^{\bar{r}} \tilde{g}_i \tilde{w}_i = x_R \bar{r} \left[1 - (1-\tilde{I}) \left(1 - \frac{\bar{r}-1}{\bar{r}} \tilde{B}_{\bar{R}} \right) \right], \tag{A13}$$

which coincides with (20) by setting $\tilde{K}(x_R, x) = \frac{\tilde{p}}{x_R \bar{r} n}$.

□

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