Bicolored graph partitioning: or, how to win elections

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Abstract

This study is motivated by an electoral application where we look into the following question: how much biased can the assignment of parliament seats be in a majority system under the effect of vicious gerrymandering when the two competing parties have the same electoral strength? To give a first theoretical answer to this question, we introduce a stylized combinatorial model, where the territory is represented by a rectangular grid graph, the vote outcome by a "balanced" red/blue node bicoloring and a district map by a connected partition whose components all have the same size. We constructively prove the existence in cycles and grid graphs of a balanced bicoloring and of two antagonist "partisan" district maps such that the discrepancy between their number of "red" (or "blue") districts for that bicoloring is extremely large, in fact as large as allowed by color balance.

Keywords: graph partitioning, graph coloring, gerrymandering.

1 Introduction

Not long after the dawn of modern democracies, in which the lawmaking power is delegated by citizens to elected representatives, insidious practices started to creep in, aimed to favor a certain candidate or party through the artful design of the electoral district boundaries. These malpractices, which came to be known under the name of gerrymandering [1], have occurred numerous times throughout the modern history of elections (see [6]) and pose a dangerous threat even nowadays [1]. In order to oppose gerrymandering practices, some districting criteria are commonly adopted: integrity (no unit may be split between two or more

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[1]In 1810 Elbridge Gerry, governor of Massachusetts, enacted a salamander-shaped district so as to enhance the probability of being re-elected. Hence the term "Gerrymander" (a contraction of Gerry+Salamander).

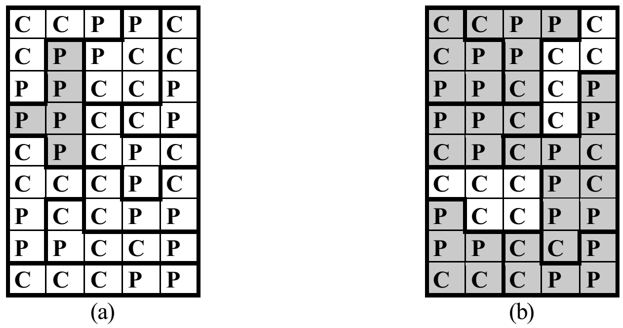


Figure 1: Example by Dixon and Plischke: (a) Party P wins 1 seat and party C wins 8; (b) Party P wins 7 seat and party C wins 2.

districts); contiguity (the units within the same district should be geographically contiguous); population equality (the district populations should be equal or nearly equal, especially in majoritarian systems); compactness (each district should be compact, that is, according to the Oxford Dictionary, "closely and neatly packed together"); conformity to administrative boundaries (the electoral district boundaries should not cross other administrative boundaries, such as those of regions, provinces, local or minority communities).

The aim of the present paper is to give a theoretical answer to the question: "how bad can the outcome of gerrymandering be?" Basically, our answer will be: "as bad as materially possible" (we are going to give a precise meaning to this statement later). Our worst-case analysis will be performed on a stylized combinatorial model of elections, which generalizes the one proposed in Dixon and Plischke's (1950) classical example, showing how gerrymandering can dramatically reverse the election outcome.

We recall here Dixon and Plischke's conceptual model.

Suppose that only two parties P and C compete under a first-past-the-post system and that, as in Figure 1, the territory is divided into elementary units having the same population with an homogeneous electoral behavior, that is, the whole population of an elementary unit votes for the same party. If the district map of Figure 1 (a) is adopted, party C wins in 8 districts out of 9; however, if the alternative district map of Figure 1 (b) is adopted, party C wins only in 2 districts out of 9, so the outcome is drastically reversed.

Notice that in this example the two parties feature nearly equal overall electoral strengths: 24 units vote for party C and 21 units vote for party P. Likewise, in our analysis we shall assume that the total number of votes is equally, or nearly equally, split between the two parties.

A careful look at Figure 1 gives us a clue about an effective strategy for maximizing the number of districts won by either party: the districts should be designed so that every win should be close and every loss should be sweeping.

2 Problem statement and paper outline

In this section we shall consider an idealized graph-theoretic formulation that captures the essence of the artificial example by Dixon and Plischke. Given a territory composed by territorial units, define the following integers:

- n is the number of territorial units;
- p is the number of districts;
- s is the common district size (number of territorial units in each district).

Clearly, the three parameters n, p, s must satisfy the relation $n = ps$.

We model the territory as an undirected graph $G = (V, E)$ with $|V| = n$, where the vertices represent territorial units and the edges represent adjacency between territorial units.

A *connected partition* of G is a partition of its set of vertices V such that each component induces a connected subgraph of G.

A *district design* is a connected partition of the graph into p components or *districts* of the same size. Notice that this definition takes into account the criteria of integrity, contiguity and population equality.

A *vote outcome* is a bicoloring of the vertices that assigns to each vertex either the color blue or the color red: this means that all voters in the corresponding unit vote for the same party, *blue* or *red*, respectively. A vote outcome is *balanced* if the total number n_b of blue vertices and that n_r of red ones satisfy the relation $|n_b - n_r| \leq 1$; that is, $n_b = n_r$ when n is even and, without loss of generality, $n_b = n_r + 1$ when n is odd. A balanced vote outcome corresponds to a situation in which the electoral population is split as equally as possible between two parties.

From now on we shall consider only balanced vote outcomes. We shall also make the following assumptions on the integers n, s, and p:

- s is odd and greater then or equal to 3: this assumption forbids trivial cases and ties between the two parties;
- the relation $n = ps$ holds.

We will denote by Π the set of all district designs and by Ω the set of all possible balanced vote outcomes.

If in a district D the number of blue vertices is greater than the number of red ones, we will say that D is a *blue district*. In a similar way we define a *red district*. Given a district design $\pi \in \Pi$, we will refer to the corresponding partition as a *blue partition* if the number of blue districts in π is greater than the number of red ones. In a similar way we define a *red partition*.

We define an *electoral competition* to be a pair (ω, π) such that $\omega \in \Omega$ and $\pi \in \Pi$. The functions $b(\omega, \pi)$ and $r(\omega, \pi)$, compute the number of blue and red districts, respectively, resulting from the electoral competition (ω, π). Let

$$B(G) = \max_{\omega \in \Omega, \pi \in \Pi} b(\omega, \pi)$$

be the maximum number of blue districts for all the electoral competitions $(\omega, \pi) \in \Omega \times \Pi$. In a similar way we can define $R(G)$.

Under the vote balance condition, whatever the district design, neither party can win in all districts, since the excess of blue votes in the blue districts must be compensated by a surplus of red votes in the red districts. On these grounds, in Sec. 4 we derive the following upper bounds on the maximum number of districts that can be won by either party:

if n is even then

$$b(\omega, \pi), r(\omega, \pi) \leq \lfloor n/(s+1) \rfloor,$$

if n is odd then

$$b(\omega, \pi) \leq \lfloor (n+1)/(s+1) \rfloor$$
$$r(\omega, \pi) \leq \lfloor (n-1)/(s+1) \rfloor$$

For a given bicoloring $\omega \in \Omega$ a partition π will be called *(blue) extremal* w.r.t. ω if the number $b(\omega, \pi)$ of blue districts in π attains its upper bound. Similar concepts can be introduced for the red party. It is not hard to prove that the above upper bounds are sharp. An explicit formula for $B(G)$ and $R(G)$ ensues (see Sec. 5). A more challenging problem consists in finding, for a given $\omega \in \Omega$, the range of all possible values for the number $b(\omega, \pi)$ of blue districts when $\pi \in \Pi$. Having this in mind, we formally introduce the following optimization problem:

$$\mathrm{GAP}(G) = \max_{\omega \in \Omega} (\max_{\pi \in \Pi} b(\omega, \pi) - \min_{\pi \in \Pi} b(\omega, \pi)).$$

For a given graph G the function $\mathrm{GAP}(G)$ is a measure of the maximum bias of an electoral outcome in terms of number of seats in single member majority districts.

Our main results imply that any grid graph has the following property: there exist both a blue extremal partition and a red extremal partition relative to *the same* balanced vote outcome. Graphs having this property, and the corresponding balanced vote outcome, will be called *two faced*. In a two faced graph, we can obtain a simple explicit formula for the gap. In this case, gerrymandering has the ability to reverse, as much as permitted by sheer vote balance, the outcome of an election in terms of parliament seats.

Here is an outline of our paper. After providing the electoral motivation of our study (Sec. 1) and formally defining the graph-theoretic problems under investigation together with the appropriate notation (Sec. 2), in Section 3 we discuss the existence in a graph of a connected partition into equally sized components, both from a theoretical and complexity viewpoint. Section 4 presents some useful arithmetic properties of extremal partitions in an arbitrary graph. Section 5 includes our main results: all cycles and all grid graphs are two faced. In fact, the result for cycles implies that every hamiltonian graph is two faced; in particular, even grid graphs are such (the result for odd grid graphs is trickier to prove). Finally, in Section 6 we exhibit some simple and not so simple examples of graphs that are not two faced.

Some of our results were presented in a previous paper of ours [2], where, however, only the case of even n was dealt with and different constructions ("boas") were employed.

3 Equipartitionable graphs

Let, as before, $n = ps$, s odd.

Definition 1 *A (connected) s-equipartition of G is any partition $\pi = \{C_1, \ldots, C_p\}$ such that, for each $k = 1, \ldots, p$:*

(i) C_k induces a connected subgraph of G;

(ii) $|C_k| = s$.

Notice that in the previous section district designs have been modelled as (connected) s-equipartions of graphs.

Definition 2 *G is s-equipartitionable if there exists some s-equipartition of G.*

In the present section we deal with the question of the existence of (connected) s-equipartitions in a graph. If the graph has n vertices, an obvious necessary condition is that s divides n, that is, $n = ps$ for some positive integer p, which may be interpreted as the number of components of the partition. We assume throughout this section that this easy condition always holds.

Proposition 1 *s-equipartitionable trees can be recognized in linear time.*

Proof. Root the tree T at any vertex. By a bottom-up recursion, count the number ν_i of descendants of each node i (including i itself) in the current tree. Whenever some i for which $\nu_i = s$ is found, delete from the current tree i and all its descendants. If the algorithm at the end returns the empty tree, then T is

s-equipartitionable. On the other hand, if some i is found, such that $\nu_i > s$ and $\nu_j < s$ for each child j of i in the current tree, then T is not s-equipartitionable. Such procedure can be clearly implemented in linear time; it may be viewed as a special case of an algorithm by Kundu and Misra [7], for min-max tree partitioning. □

Proposition 2 *Any s-equipartitionable tree admits a unique s-equipartition.*

Proof. Any s-equipartitionable tree T has an edge e such that one of the two branches T_1 and T_2 resulting from cutting e has exactly s vertices. We claim that e must be cut in every s-equipartition of T: otherwise, any cut would occur within either T_1 or T_2. In the former case, a component with less than s vertices would be found; if no cut falls within T_1 then the component containing e would have more than s vertices. The statement then follows by induction on the number of cuts. □

Theorem 1 *A connected graph is s-equipartitionable if and only if it has some s-equipartitionable spanning tree.*

Proof. The if) is trivial. Let us prove the only if). Let $\pi = \{C_1, \ldots, C_p\}$ be any s-equipartition of G; let T_k be a spanning tree of the subgraph $G(C_k)$ induced by C_k. For each k, declare all edges of T_k to be green. Let $B(\pi)$ be the *block-incidence graph* of G w.r.t. π: that is, the nodes of π are the components C_k, and any two nodes of $B(\pi)$ are adjacent iff the corresponding components of π are adjacent. Let $\mathbf{T}$ be any spanning tree of $B(\pi)$. For each edge (C_h, C_k) in $\mathbf{T}$, select in G one edge between C_h and C_k, and declare also any such edge to be green. The partial graph of G spanned by all green edges is the required s-equipartitionable spanning tree. □

Remark 1 *The above proof is nonconstructive and does not imply that graphs that are s-equipartitionable can be recognized in polynomial time: finding the required spanning tree might be computationally hard.*

Corollary 1 *If s is any positive divisor of n, any graph with a hamiltonian path (in particular, any hamiltonian graph) is s-equipartitionable. Thus, all grid graphs are s-equipartitionable, since they are hamiltonian if n is even and they have a hamiltonian path if n is odd.*

Actually, a stronger result holds:

Theorem 2 *Every grid graph with $n = ps$ vertices admits an s-equipartition in which all the components are grid graphs with the same number of rows and the same number of columns.*

Proof. Let M be the number of rows and N the number of columns of a given grid graph. Since $MN = ps$, there exist four natural numbers M_1, M_2, N_1 and N_2 such that:

$$M = M_1M_2, \ N = N_1N_2, \ M_1N_1 = s, \ M_2N_2 = p.$$

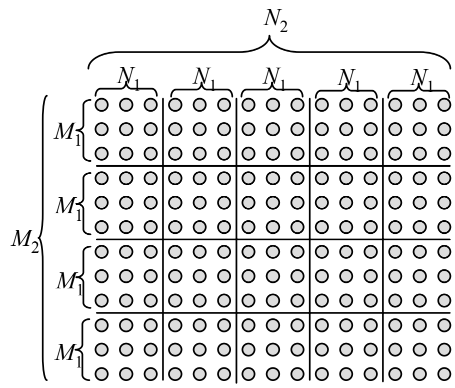


Figure 2: Decomposition of a grid graph with M rows and N columns into p grid subgraphs with M_1 rows and N_1 columns.

As shown in Figure 2, by partitioning the columns of G into N_2 components having N_1 columns each and the rows of G into M_2 components having M_1 columns each, one can decompose G into p grid subgraphs having M_1 rows and N_1 columns each. Notice that, when s is odd, also M_1 and N_1 are odd. □

The above positive results for trees and grid graphs may lead one to hope that the property of being s-equipartitionable can be easily recognized, at least in bipartite graphs. The following negative result, by Dyer and Frieze [4], defeats this hope.

Theorem 3 *Let G be a graph with n vertices, and let s be a positive integer divisor of n. Deciding whether G is s-equipartitionable is NP-complete even when G is bipartite.*

However, when G is sufficiently connected, G turns out to be s-equipartitionable, as Corollary 2 below shows.

Theorem 4 *If G is p-connected and $s_1, \ldots, s_p$ are any p positive integers such that $s_1 + \ldots + s_p = n$, then there always exists a connected partition of G into p components with sizes $s_1, \ldots, s_p$, respectively.*

Proof. See Győri, 1976 [5] and Lovász, 1979 [8]. □

Corollary 2 *Every p-connected graph with $n = ps$ vertices is s-equipartitionable.*

Remark 2 *As far as we know, neither Győri's graph-theoretical proof nor the topological one by Lovász directly provides a polynomial time algorithm for constructing the required s-equipartition.*

Fortunately, in the present paper we need to deal only with paths, cycles, and grid graphs; for these graphs, as mentioned above, the condition $n = ps$ is both necessary and sufficient for the existence of an s-equipartition.

4 Structure and arithmetic properties of extremal partitions in general graphs

We start this section with some upper bounds on the number of blue and red districts in a given district design for a given vote outcome. Given an electoral competition $(\omega, \pi) \in \Omega \times \Pi$, for any district k, $k = 1, ..., p$, let

- b_k be number of blue vertices in district k;
- r_k be number of red vertices in district k;
- $\delta = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$.

Proposition 3 *Given an s-equipartitionable graph G, for any $(\omega, \pi) \in \Omega \times \Pi$ the following inequalities hold:*

$$b(\omega, \pi) \le \lfloor (n + \delta)/(s + 1) \rfloor,$$

$$r(\omega, \pi) \le \lfloor (n - \delta)/(s + 1) \rfloor.$$

Proof. Since ω is balanced, we may assume:

$$\sum_{k=1,\dots,p} (b_k - r_k) = \delta.$$

Hence:

$$\delta = \sum_{k=1,\dots,p} (b_k - r_k) = \sum_{k:b_k > r_k} (b_k - r_k) + \sum_{k:b_k < r_k} (b_k - r_k)$$

$$\ge b(\omega, \pi) - s(p - b(\omega, \pi)) = (s + 1)b(\omega, \pi) - sp.$$

Since $n = ps$ and $b(\omega, \pi)$ is a natural number we obtain:

$$b(\omega, \pi) \le \lfloor (n + \delta)/(s + 1) \rfloor.$$

Similarly

$$-\delta = \sum_{k=1,\dots,p} (r_k - b_k) = \sum_{k:r_k > b_k} (r_k - b_k) + \sum_{k:r_k < b_k} (r_k - b_k)$$

$$\ge r(\omega, \pi) - s(p - r(\omega, \pi)) = (s + 1)r(\omega, \pi) - sp.$$

Then

$$r(\omega, \pi) \le \lfloor (n - \delta)/(s + 1) \rfloor.$$

□

Corollary 3 *If G is s-equipartitionable, then the bounds in Proposition 3 are sharp. Hence $B(G) = \lfloor (n + \delta)/(s + 1) \rfloor$ and $R(G) = \lfloor (n - \delta)/(s + 1) \rfloor$.*

Proof. Let $\pi \in \Pi$ be any district design. It is always possible to color the vertices of the graph G in such a way that $\lfloor (n+\delta)/(s+1) \rfloor$ districts have at least $(s+1)/2$ blue vertices. In fact, in any balanced vote outcome, the number of blue vertices is $(n+\delta)/2$ and:

$$\frac{s+1}{2}\left\lfloor \frac{(n+\delta)}{s+1} \right\rfloor \leq \frac{(n+\delta)}{2}.$$

Since a district with $(s+1)/2$ blue vertices is blue, we obtain a vote outcome with at least $\lfloor (n+\delta)/(s+1) \rfloor$ blue districts. But, by Proposition 3, this is an upper bound for the number of blue districts, hence $B(G) = \lfloor (n+\delta)/(s+1) \rfloor$.

A similar argument can be used to prove that $R(G) = \lfloor (n-\delta)/(s+1) \rfloor$.

□

Corollary 4 *If G is s-equipartitionable, and $p = q(s+1)+r$ with $1 \leq r \leq s+1$ then* [2]*:*

$$B(G) = \begin{cases} qs + r - 1 & \textit{if } r \geq 2 \\ qs + r & \textit{if } r = 1 \end{cases}$$

and

$$R(G) = qs + r - 1.$$

Hence $B(G) = R(G)$, unless $r = 1$, in which case $B(G) = R(G) + 1$.

Proof. From Corollary 3 one has:

$$B(G) = \left\lfloor \frac{n+\delta}{s+1} \right\rfloor = qs + \left\lfloor \frac{rs+\delta}{s+1} \right\rfloor.$$

Since $r - \delta \leq s+1$,

$$\left\lfloor \frac{rs+\delta}{s+1} \right\rfloor = \left\lfloor r - \frac{r-\delta}{s+1} \right\rfloor = \begin{cases} r-1 & \text{if } r \geq 2 \\ r & \text{if } r = 1 \end{cases}$$

hence

$$B(G) = \begin{cases} qs + r - 1 & \text{if } r \geq 2 \\ qs + r & \text{if } r = 1 \end{cases}.$$

Similarly, from Corollary 3 one has:

$$R(G) = \left\lfloor \frac{n-\delta}{s+1} \right\rfloor = qs + \left\lfloor \frac{rs-\delta}{s+1} \right\rfloor.$$

Since $r + \delta \leq s+1$,

$$\left\lfloor \frac{rs-\delta}{s+1} \right\rfloor = \left\lfloor r - \frac{r+\delta}{s+1} \right\rfloor = r - 1$$

hence

$$R(G) = qs + r - 1.$$

□

[2]Notice that q and r might not coincide with the quotient and the remainder, respectively, of the division of p by $s+1$.

Let γ be defined as follows:

$$\gamma = \begin{cases} 0 & \text{if } r \geq 2 \\ 1 & \text{if } r = 1 \end{cases} .$$

We can write:

$$B(G) = qs + r - 1 + \gamma.$$

The following result relates the function GAP(G) to $B(G)$ and $R(G)$.

Proposition 4 *GAP$(G) \leq B(G) + R(G) - p$.*

Proof. Since $b(\omega, \pi) + r(\omega, \pi) = p$, then

$$\text{GAP}(G) = \max_{\omega \in \Omega} \left(\max_{\pi \in \Pi} b(\omega, \pi) + \max_{\pi \in \Pi} r(\omega, \pi) \right) - p \leq \tag{1}$$

$$\max_{\omega \in \Omega} \max_{\pi \in \Pi} b(\omega, \pi) + \max_{\omega \in \Omega} \max_{\pi \in \Pi} r(\omega, \pi) - p = B(G) + R(G) - p.$$

□

Corollary 5 *We have*

$$\text{GAP}(G) = B(G) + R(G) - p \tag{2}$$

if and only if G is two faced (as defined in Section 2).

Proof. Follows from (1). □

Two faced graphs are those for which gerrymandering exhibits its worst case bias. There is an absolute threshold for the maximum number of seats that a party can obtain when the vote outcome is balanced. In two faced graphs, for a suitable balanced vote, both parties can achieve this threshold by artful gerrymandering. By Corollary 4 this threshold is equal for the red and the blue party except when $r = 1$.

The colors of the vertices within districts in an extremal partition follow a well defined scheme. In a blue (red) extremal partition, the blue (red) vertices are distributed among the districts in such a way that each blue (red) district has at least $(s+1)/2$ blue (red) vertices and the number of blue (red) districts is maximum.

Let k_B be the maximum number of blue vertices that belong to red districts in a blue extremal partition and k_R be the maximum number of red vertices that belong to blue districts in a red extremal partition. Proposition 5 yields the actual values of k_B and k_R; as expected, they are smaller than $(s+1)/2$.

Proposition 5 *We have:*

$$k_B = \frac{s - r + 1 + \delta}{2} - \gamma \left(\frac{s+1}{2} \right)$$

$$k_R = \frac{s - r + 1 - \delta}{2}.$$

Proof. Recall that $n = sp = qs(s+1) + rs$. A blue district must contain at least $(s+1)/2$ blue vertices. Since $B(G) = qs + r - 1 + \gamma$, we have:

$$k_B = \frac{qs(s+1)+rs+\delta}{2} - (qs+r-1+\gamma)\left(\frac{s+1}{2}\right) = \frac{s-r+1+\delta}{2} - \gamma\left(\frac{s+1}{2}\right)$$

Similarly, a red district must contain at least $(s+1)/2$ red vertices. Since $R(G) = qs + r - 1$, we have:

$$k_R = \frac{qs(s+1)+rs-\delta}{2} - (qs+r-1)\left(\frac{s+1}{2}\right) = \frac{s-r+1-\delta}{2}$$

□

Given a bicoloring $\omega \in \Omega$ and a partition $\pi \in \Pi$, we say that a district is:

- (*blue*) *edgy* if it contains $(s+1)/2$ blue vertices and $(s-1)/2$ red vertices;
- (*red*) *edgy* if it contains $(s+1)/2$ red vertices and $(s-1)/2$ blue vertices;
- (*blue*) *sweeping* if all its vertices are blue;
- (*red*) *sweeping* if all its vertices are red;
- (*blue*) *quasi sweeping* if it contains k_R red vertices and $s - k_R$ blue vertices;
- (*red*) *quasi sweeping* if it contains k_B blue vertices and $s - k_B$ red vertices.

Notice that quasi sweeping districts could be sweeping.

We say that a blue (red) extremal partition is *blue (red) edgy* if all blue (red) districts are edgy. We will use these extremal partitions in the next section, where we will show that s-equipartitionable cycles and grid graphs are two faced. Table 1 contains useful information related to edgy extremal partitions.

	Blue edgy extr. part.	Red edgy extr. part.
N. of edgy districts	$qs + r - 1 + \gamma$	$qs + r - 1$
N. of sweeping districts	q	q
N. of quasi sweeping districts	$1 - \gamma$	1
k_B , k_R	$\frac{s-r+1+\delta}{2} - \gamma(\frac{s+1}{2})$	$\frac{s-r+1-\delta}{2}$

Table 1: Arithmetic characteristics of edgy extremal partitions.

Remark 3 *If $r = 1$ or $r = s+1$ then $k_B = 0$ and all blue extremal partitions are edgy. Similarly, if $r = s+1$ then $k_R = 0$ and all red extremal partitions are edgy.*

Remark 4 *If $1 \le p \le s+1$, each extremal partition has at most one quasi sweeping district.*

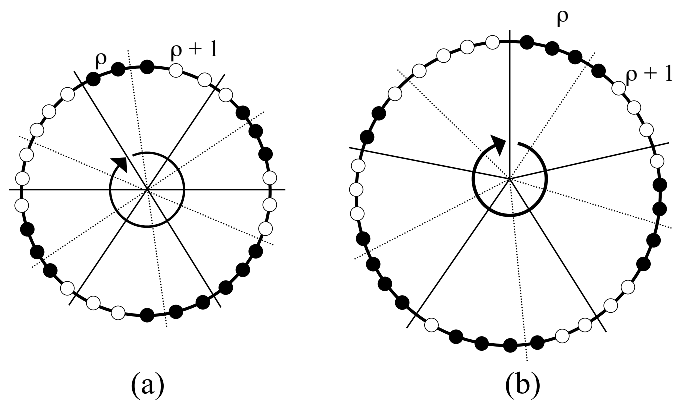


Figure 3: The output of algorithm **CycleBicoloring** for (a) $n = 30$, $s = 5$; (b) $n = 35$, $s = 7$.

5 Two facedness of cycles and grid graphs

5.1 Gerrymandering on cycles

In this section we will show that, under the hypothesis that s is odd, any cycle $H = (V_H, E_H)$ having $n = sp$ vertices is two faced. Since H is a cycle, any partition into p connected components can be obtained by cutting p edges. Moreover any s-equipartition is uniquely determined by one of its cuts and can be obtained from any other s-equipartition by a shifting of all cuts by t edges in the same direction, for a given $t \in \{1, ..., s-1\}$. In the following we will fix a shifting direction, say clockwise, and, given an s-equipartition, we will call a shifting of all cuts by t edges in this direction a *t-rotation*, $t \in \{1, ..., s-1\}$.

We will show that any s-equipartitionable cycle admits a two faced bicoloring of the vertices such that the red extremal partition can be obtained from the blue one by an $(s-1)/2$-rotation. For this reason we will say that $\rho = (s-1)/2$ is the *rotation number*. Let a *block* be a subpath of H and a *t-block* be a block having t vertices. Two blocks A and B are adjacent if in H there exist two adjacent vertices v and u such that $v \in A$ and $u \in B$. Notice that the concatenation of a ρ-block and an adjacent $(\rho+1)$-block is a district. We say that a $(\rho+1)$-block is *blue complementable* if, together with a ρ-block, it can form a blue sweeping or quasi sweeping district of the red extremal partition. Note that a blue complementable $(\rho+1)$-block cannot contain more than k_R red vertices.

We call the attention of the reader on the fact that the case $r = 1$ is inherently different from the case $r \geq 2$, since, by Corollary 4, it is the only case where the number of red districts in a red extremal partition does not match the number of blue districts in a blue extremal partition.

Let us consider the bicolored cycle of Figure 3 (a) where $n = 30$ and $s = 5$.

In this case a blue edgy extremal partition has five blue edgy districts and one red sweeping district and symmetrically a red edgy extremal partition has five red edgy districts and one blue sweeping district. Here and in the following figures we will represent blue vertices in black and red vertices in white. The red edgy extremal partition can be obtained from the blue one by a ρ-rotation.

In Figure 3 (b) we consider a bicolored graph with $n = 35$ and $s = 7$. In this case a blue edgy extremal partition has four blue edgy districts and one red quasi sweeping district with $k_B = 2$ blue vertices and and a red edgy extremal partition has four red edgy districts and one blue quasi sweeping district with $k_R = 1$ red vertices. Also in this case the red extremal partition is obtained from the blue one by a ρ-rotation. In both the examples, the cycle is partitioned into a sequence of ρ-blocks and $(\rho + 1)$-blocks. Each ρ-block, together with one of the adjacent $(\rho + 1)$-blocks, forms a district of the blue extremal partition while, together with the other adjacent $(\rho + 1)$-block, forms a district of the red extremal partition. In the following we will present a bicoloring algorithm where each block is colored taking into account both the blue and the red extremal partitions.

We start by considering the case $3 \leq p = r \leq s + 1$. If $p \geq 2$ an edgy extremal partition has one sweeping or quasi sweeping district and $p - 1$ edgy districts. The cases $p = 1, 2$ are degenerate since a blue extremal partition is also red extremal and viceversa. We present a bicoloring algorithm that visits in a fixed direction the vertices of the cycle and assigns colors alternatively to the vertices of a ρ-block and of a $(\rho + 1)$-block in such a way that each ρ-block forms, together with the next $(\rho + 1)$-block, a district of an edgy blue extremal partition and, at the same time, it forms, with the previous $(\rho + 1)$-block, a district of the red edgy extremal partition.

Algorithm **CycleBicoloring (case $3 \le p \le s+1$)**

pick a vertex and visit H clockwise;
let $h := 0$;
repeat
  the next ρ-block has h red vertices and $\rho - h$ blue vertices so that it forms a red edgy district together with the previous $(\rho+1)$-block (at the beginning the previous $(\rho+1)$-block is the last block generated by the algorithm);
  the next $(\rho+1)$-block has $\rho - h$ red vertices and $h+1$ blue vertices so that it forms a blue edgy district together with the previous ρ-block;
  let $h := h+1$;
until $h = \rho - k_R + 1$, that is, the last $(\rho+1)$-block is blue complementable
let $h := 0$;
repeat
  the next ρ-block has h red vertices and $\rho - h$ blue vertices so that it forms a red edgy district together with the previous $(\rho+1)$-block;
  the next $(\rho+1)$-block has $\rho - h$ red vertices and $h+1$ blue vertices so that it forms a blue edgy district together with the previous ρ-block;
  let $h := h+1$;
until $h = \rho - k_B$, that is, the total number of blue districts $2\rho - k_B - k_R + 1$ is equal to its upper bound $p-1$;
the next ρ-block has k_B blue vertices and $\rho - k_B$ red vertices so that it forms a red edgy district together with the previous $(\rho+1)$-block;
the next $(\rho+1)$-block has $\rho+1$ red vertices so that it forms a blue quasi sweeping district together with the previous ρ-block.

In the above algorithm the first "repeat" cycle colors $\rho - k_R + 1$ blue edgy districts and the second "repeat" cycle colors $\rho - k_B$ blue edgy districts. Remember that $k_B, k_R \le \rho$, so $\rho - k_B, \rho - k_R \ge 0$. The last two colored blocks form a blue quasi sweeping district. Hence the number of colored districts is $2\rho + 2 - k_R - k_B = p$ and the algorithm colors each vertex of the given cycle exactly once.

We now formalize the bicoloring generated by the above algorithm. We will denote $S(h)$ a ρ-block containing h red vertices and $L(h)$ a $(\rho+1)$-block containing h red vertices. A *cobra* $C(i,j)$ is a sequence of blocks defined as follows:

$$C(i,j) = S(i)L(\rho - i)S(i+1)L(\rho - (i+1)) \dots S(j)L(\rho - j)$$

if $0 \le i \le j \le \rho$. We also define $C(i,j)$ to be the empty sequence if $i > j$. Notice that a cobra $C(i,j)$ contains $\max\{j-i+1, 0\}$ blue edgy districts.

A *shifted cobra* $\overline{C}(i,j)$ is the sequence:

$$\overline{C}(i,j) = L(\rho + 1 - i)S(i)L(\rho - i)S(i+1) \dots L(\rho + 1 - j)S(j)$$

if $0 \le i \le j \le \rho$. As for cobras, $\overline{C}(i,j)$ is the empty sequence if $i > j$. A shifted cobra contains $\max\{j-i+1, 0\}$ red edgy districts.

The following relations hold:

$$\begin{aligned}
&C(i,j) = S(i)\overline{C}(i+1,j)L(\rho-j)\\
&C(i,j)S(j+1) = S(i)\overline{C}(i+1,j+1)\\
&L(\rho+1-i)C(i,j) = \overline{C}(i,j)L(\rho-j).
\end{aligned} \tag{3}$$

The first "repeat" cycle of the bicoloring algorithm generates a cobra $C(0,\rho-k_R)$ and the second "repeat" cycle generates a cobra $C(0,\rho-k_B-1)$. Then the sequence given by the algorithm is:

$$C(0,\rho-k_R)C(0,\rho-k_B-1)S(\rho-k_B)L(\rho+1).$$

This sequence admits a unique equipartition that is blue edgy extremal. On the other hand, the sequence:

$$\overline{C}(1,\rho-k_R)L(k_R)S(0)\overline{C}(1,\rho-k_B)L(\rho+1)S(0)$$

admits a unique equipartition that is red edgy extremal. By relations (3), this sequence can be obtained by a ρ-rotation from the previous one. Then the above algorithm provides a bicoloring such that the colored cycle has a blue and a red extremal partition.

Let us consider now the general case. As in Section 4 we write $p = q(s+1)+r$ where $q \geq 0$ and $1 \leq r \leq s+1$. As shown in Corollary 4 a blue extremal partition has $qs+r-1+\gamma$ blue edgy districts and $q+1-\gamma$ red sweeping or quasi sweeping districts containing an overall number k_B of blue vertices. A red extremal partition has $qs+r-1$ red edgy districts and $q+1$ blue sweeping or quasi sweeping districts containing an overall number k_R of red vertices. One can imagine the cycle with $n = ps = qs(s+1)+rs$ vertices partitioned into q paths having $(s+1)s$ vertices each and one more path having rs vertices. In a (blue or red) edgy extremal partition, for any of the first q paths there must be one sweeping district and s edgy districts, while in the last path there must be one quasi sweeping district and $r-1$ edgy districts if $r \geq 2$, or one blue edgy (blue quasi sweeping in a red extremal partition) district if $r = 1$. Hence by applying the bicoloring algorithm to each of the above paths we can show that any s-equipartitionable cycle is two faced. The following algorithm finds a two faced bicoloring for the general case. Here we use the notions of $S(h)$ block, $L(h)$ block and cobra.

Algorithm **CycleBicoloring (general case)**

- pick a vertex and visit H clockwise;
- for q times
  - the next $s(\rho+1)$ vertices are a cobra $C(0,\rho)$;
  - the next $s\rho$ vertices are a cobra $C(0,\rho-1)$;
  - the next s vertices are a sequence $S(\rho)L(\rho+1)$;
- the next $s(\rho-k_R+1)$ vertices are a cobra $C(0,\rho-k_R)$;
- if $r \geq 2$ then
  - the next $s(\rho-k_B)$ vertices are a cobra $C(0,\rho-k_B-1)$;
  - the next s vertices are a sequence $S(\rho-k_B)L(\rho+1)$.

Remark 5 *If $r = 2$ then $k_B = \rho$; hence the cobra $C(0, \rho - k_B - 1)$ is empty and the sequence $C(0, \rho - k_R)C(0, \rho - k_B - 1)S(\rho - k_B)L(\rho + 1)$ contains two districts.*

Lemma 1 *Algorithm CycleBicoloring colors $n = ps$ vertices.*

Proof. The "for" loop colors $qs(s + 1)$ vertices.
Case $r \geq 2$
$k_B, k_R \leq \rho$, so the last two cobras generated by the algorithm contain $2\rho - k_R - k_B + 1 = r - 1$ districts. Then, adding the district $S(\rho - k_B)L(\rho + 1)$, we have $q(s + 1) + r = p$ colored districts, that is $n = ps$ colored vertices.
Case $r = 1$
$k_R = \rho$, so the last cobra generated by the algorithm contain one district. Hence we have $q(s + 1) + r = p$ colored districts, that is, $n = ps$ colored vertices. □

The above algorithm generates the following sequence:

$$\begin{array}{l} q \text{ times: } C(0, \rho)C(0, \rho - 1)S(\rho)L(\rho + 1) \\ C(0, \rho - k_R) \\ C(0, \rho - k_B - 1)S(\rho - k_B)L(\rho + 1), \text{ if } r \geq 2. \end{array} \tag{4}$$

Lemma 2 *The unique s-equipartition of sequence (4) is blue extremal.*

Proof. The q pairs of cobras $C(0, \rho), C(0, \rho - 1)$ contain qs blue edgy districts.
Case $r \geq 2$
The two cobras $C(0, \rho - k_R)$ and $C(0, \rho - k_B - 1)$ contain $2\rho - k_R - k_B + 1 = r - 1$ blue edgy districts.
Case $r = 1$
The cobra $C(0, \rho - k_R)$ contains one edgy district if $r = 1$.

Hence in any case the upper bound on the number of blue edgy districts is attained. □

By relations (3), after a ρ-rotation, the sequence provided by the algorithm is:

$$\begin{array}{l} q \text{ times: } \overline{C}(1, \rho)L(0)S(0)\overline{C}(1, \rho)L(\rho + 1)S(0) \\ \overline{C}(1, \rho - k_R)L(k_R)S(0) \\ \overline{C}(1, \rho - k_B)L(\rho + 1)S(0) \text{ if } r \geq 2. \end{array} \tag{5}$$

Remark 6 *If $r = 1$ then $\overline{C}(1, \rho - k_R)$ is empty; if $r = 2$ then both $\overline{C}(1, \rho - k_R)$ and $\overline{C}(1, \rho - k_B)$ are empty.*

Lemma 3 *The unique s-equipartition of sequence (5) is red extremal.*

Proof. The q sequences $\overline{C}(1,\rho), L(0), S(0), \overline{C}(1,\rho), L(\rho+1), S(0)$ contain qs red edgy districts.
Case $r \geq 2$
The sequence $\overline{C}(1,\rho-k_R)L(k_R)S(0)\overline{C}(1,\rho-k_B)L(\rho+1)S(0)$ has $2\rho - k_R - k_B + 1 = r-1$ red edgy districts. Hence the total number of red edgy districts is $qs + r - 1$.
Case $r = 1$
The cobra $\overline{C}(1,\rho-k_R)$ is empty, then the total number of red edgy districts is qs.

Hence in each case the upper bound on the number of red edgy districts is attained. □

Theorem 5 *Any cycle with $n = ps$ vertices is two faced.*

Proof. Follows from Lemmas 1, 2 and 3. □

Corollary 6 *Any hamiltonian graph with $n = ps$ vertices is two faced.*

Proof. Follows from Theorem 5. □

Corollary 7 *Let $H_{s(s+1)}$ be a cycle with $n = s(s+1)$ vertices, to be partitioned into $p = s+1$ districts, each of size s. Then*

$$lim_{odd\ s\to\infty} \frac{GAP(H_{s(s+1)})}{s+1} = 1.$$

Proof. After Corollary 5 and Theorem 5, one has

$$\frac{\mathrm{GAP}(H_{s(s+1)})}{s+1} = \frac{B(G)+R(G)-s-1}{s+1} = \frac{2s-s-1}{s+1} = \frac{s-1}{s+1}.$$

When s odd $\to \infty$, the thesis follows. □

Corollary 7 is really stunning: it means that, for certain infinite families of cycles, as the number and size of the districts grow, vicious gerrymandering can make the percentages of blue districts and red ones both arbitrarily close to 1 even under the assumptions that the vote outcome is the same and that the blue party and the red one get the same total number of votes.

5.2 Gerrymandering on grid graphs

In this section we will show that any s-equipartitionable grid graph with M rows and N columns, M, $N \geq 2$, is two faced. As shown in Section 3, any grid graph with $n = ps$ vertices is s-equipartitionable. Notice that, if $M = 1$ or $N = 1$, the graph is a path, then it cannot be two faced since it admits a unique s-equipartition.

Theorem 6 *Any s-equipartitionable grid graph with M, $N \geq 2$ and with an even number of vertices is two faced.*

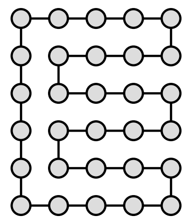


Figure 4: Hamiltonian cycle in a grid graph with an even number of rows.

Proof. If n is even, so that at least one of M and N is even, then it is well known and easy to show that G is hamiltonian (see Figure 4). By Theorem 5, any s-equpartitionable cycle is two faced, and hence it follows that any s-equipartitionable grid graph having an even number of vertices is two faced. $\square$

Let us consider now the case $n = MN$ odd. By Theorem 2, we can decompose G into a grid subgraph $G_s = (V_s, E_s)$ having s vertices and M_s rows and N_s columns and a subgraph $\overline{G}_s$ induced by the vertices in $V - V_s$ (see Figure 5). We can suppose that G_s contains one of the vertices of G having degree 2, that is, one of the vertices on a corner of G. Since M, N and s are odd, M_s and N_s are odd and $M - M_s$ and $N - N_s$ are even. Hence $\overline{G}_s$ is a grid graph with an even number of rows or columns, or can be decomposed into two grid graphs which have an even number of rows, equal to $M - M_s$, and an even number of columns, equal to $N - N_s$, respectively. As shown in the example of Figure 5, $\overline{G}_s$ has a hamiltonian cycle that can be obtained by appropriately joining the hamiltonian cycles of these grid subgraphs. Moreover $\overline{G}_s$ is s-equipartitionable since it contains $n - s$ vertices, hence, by Theorem 5, it is two faced.

Suppose, without loss of generality, that $M_s \leq N_s$ and G_s is the top left corner of G. Consider the unique row of G_s such that all its vertices are adjacent to vertices of $\overline{G}_s$. Since $s \geq 3$, this row contains at least three vertices. Let u and v be two adjacent vertices of this row such that u is the bottom left corner of G_s (see Figure 5). Notice that u is not an articulation vertex of G_s.

Remark 7 *There exists a hamiltonian path of $\overline{G}_s$ having two adjacent vertices $\overline{u}$ and $\overline{v}$ that are adjacent to u and v, respectively.*

The bicoloring of G provided by the following algorithm is two faced.

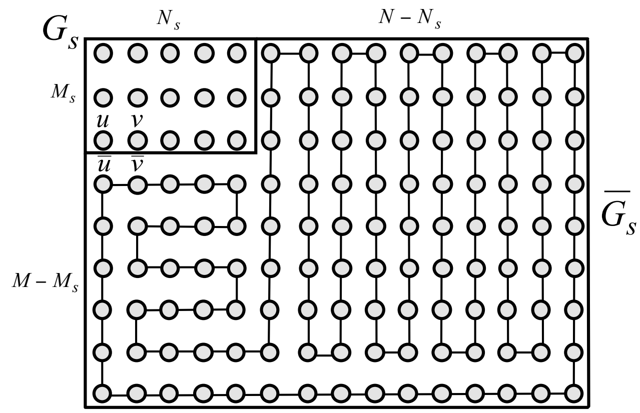


Figure 5: Decomposition of G and hamiltonian cycle of $\overline{G}_s$.

Algorithm **GridBicoloring (case n odd)**

decompose G into G_s and $\overline{G}_s$ (see Figure 6 (a));
let u, v, $\overline{u}$ and $\overline{v}$ be as defined above;
color in red ρ vertices of G_s and in blue $\rho + 1$ vertices of G_s
in such a way that u is blue;
let $\overline{H}$ be a hamiltonian cycle of $\overline{G}_s$ such that
$\overline{u}$ and $\overline{v}$ are adjacent;
color $\overline{H}$ using the algorithm CycleBicoloring in such a way that, if $r \geq 2$,
$\overline{v}$ and $\overline{u}$ belong to the blue quasi sweeping district of the red extremal
partition and $\overline{v}$ is red and is not an articulation vertex of the district.

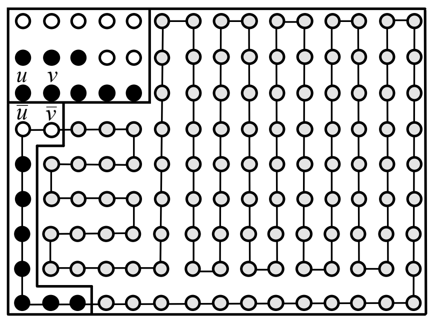
Lemma 4 *The bicoloring provided by Algorithm GridBicoloring is balanced.*

Proof. Since n and s are odd, $n - s$ is even; hence, by construction, $\overline{H}$ has an even number of vertices, $(n-s)/2$ red and $(n-s)/2$ blue. It follows that G has $(n-1)/2$ red vertices and $(n+1)/2$ blue vertices. □

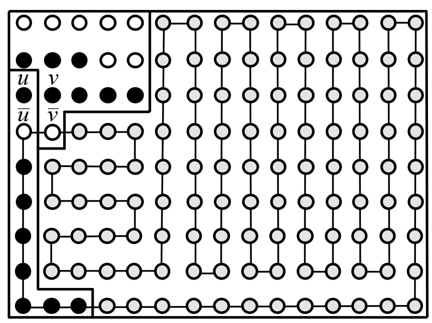
Lemma 5 *Any s-equipartitionable grid graph with an odd number of vertices colored by Algorithm GridBicoloring has a blue extremal partition.*

Proof. Let $\overline{\pi}_B$ be the blue extremal partition of $\overline{H}$, which has qs blue edgy districts if $r = 1$ and $qs + r - 2$ blue edgy districts if $r \geq 2$. Since V_s is blue edgy, the partition $\pi_B = \overline{\pi}_B \cup V_s$ of G has $qs + r$ blue edgy districts if $r = 1$ and $qs + r - 1$ blue edgy districts if $r \geq 3$. Hence π_B is blue extremal. □

Lemma 6 *Any s-equipartitionable grid graph with an odd number of vertices colored by Algorithm GridBicoloring has a red extremal partition.*



(a)



(b)

Figure 6: Construction of a red extremal partition of G.

Proof. Let $\overline{\pi}_R$ be the red extremal partition of $\overline{H}$, which has qs red edgy districts if $r = 1$ and $qs + r - 2$ red edgy districts if $r \geq 3$. If $r = 1$, since V_s is blue edgy, the partition $\pi_R = \overline{\pi}_R \cup V_s$ of G has qs red edgy districts and so is red extremal. If $r \geq 3$, $\overline{\pi}_R$ has a blue quasi sweeping district W with $(s - r + 2)/2$ red vertices, one more than in a blue quasi sweeping district of G. Moreover, V_s has one red vertex less than a red edgy district. Let $W' = W - \{\overline{v}\} \cup \{u\}$ and $V_s^{'} = V_s - \{u\} \cup \{\overline{v}\}$. W' is a red edgy district and V_s is a blue quasi sweeping district of G. The partition $\pi_R = \overline{\pi}_R - W \cup W' \cup V_s^{'}$ of G has $qs + r - 1$ red edgy districts (see Figure 6 (b)). Hence π_R is red extremal. □

Theorem 7 *Any s-equipartitionable grid graph with at least two rows and two columns is two faced.*

Proof. Follows from Theorem 6 and Lemmas 4, 5 and 6. □

In conclusion, we have shown that for all hamiltonian graphs and grid graphs one can construct Dixon-Plischke-like examples where gerrymandering can heavily reverse the electoral result in terms of Parliament seats.

6 Examples of non two faced graphs

In the previous section we have shown that all hamiltonian graphs and all grid graphs are two faced. But do non two faced graphs exist? An immediate example is given by trees since they admit a unique s-equipartition. Looking for more significant examples we notice that both cycles and grid graphs are 2-connected, bipartite and planar. Here is an example of a graph sharing these properties, but not two faced.

Consider the graph G on 18 vertices in Figure 7 and let $s = 3$. It is easy to see that in every connected 3-equipartition of G vertices 1, 2, and 3 must belong to the same component and the same must hold for vertices 10, 11, and 12. Thus, a connected 3-equipartition of G is always given by the two components $\{1, 2, 3\}$, $\{10, 11, 12\}$ and four additional components obtained by splitting each of the two hexagons into two parts. Hence, every pair of connected 3-equipartitions of G has at least two components in common. Since for every possible bicoloring of the vertices of G the common components of the two partitions will be always colored in the same way, no pair of connected 3-equipartitions, one blue extremal and one red extremal, can be found in G w.r.t. $s = 3$.

The graph G shown in Figure 7 is non two faced for $s = 3$, due to the fact that it is not possible to find a pair of connected 3-equipartitions without common components. In our third and last example this is indeed possible but there is a subtler reason for which G is not two faced.
Consider the graph G shown in Figure 8, with $n = 30$ and $s = 5$. Let C be the cycle induced in G by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and P the path given by vertices from 10 to 30.

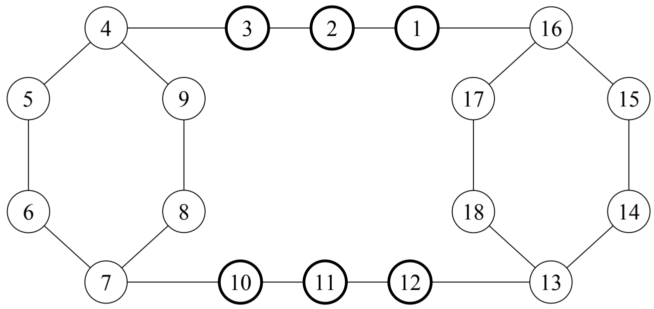


Figure 7: The graph G is non two faced since a pair of connected 3-equipartitions without common components cannot be found in G.

Claim 1 *In every connected* 5*-equipartition of* G *the vertices* 1*,* 29 *and* 30 *never belong to the same component.*

Proof. Suppose that the claim is not true, that is, there exists at least a 5-equipartition of G in which the component containing vertex 1 (and both 29 and 30) contains also a number m, $m = 0, 1, 2$, of vertices of C. In any case, the remaining vertices of C form a path with a number of vertices ranging from 6 to 8. These vertices must belong to at least two components, but only one of them contains vertex 10, implying that the other component is entirely contained in C. Thus, the latter component must contain both vertices 5 and 6, while any other connected component in C cannot contain more than three vertices. This is a contradiction, since we assumed to have a 5-equipartition. □

By the claim, in a connected 5-equipartition of G either:

1) vertex 29 and vertex 30 belong to the same component;

2) vertex 1 and vertex 30 belong to the same component.

A connected 5-equipartition matching condition 1 will be referred to as *partition of type I*, while one that satisfies condition 2) will be called *partition of type II*.

In a partition of type I there is always a component consisting of a path of five vertices in C including vertex 1. Then, a second component is forced to be formed by the remaining four vertices of C together with vertex 10. The other components are uniquely generated by partitioning the path from vertex 11 to vertex 30 into the four consecutive subpaths $\{11, 12, 13, 14, 15\}$, $\{16, 17, 18, 19, 20\}$, $\{21, 22, 23, 24, 25\}$, $\{26, 27, 28, 29, 30\}$.
A partition of type II is characterized by a component given by a path of 4 vertices in C, including vertex 1, attached to vertex 30. The rest of the vertices

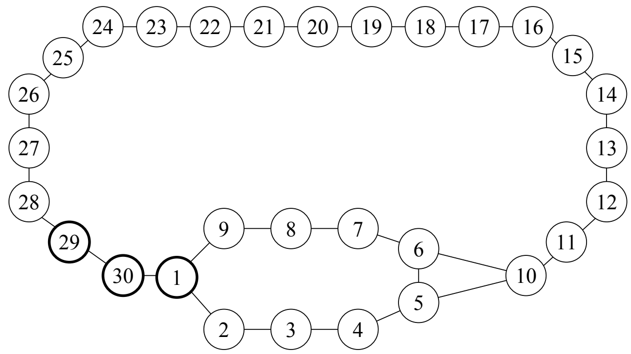


Figure 8: The graph G is non two faced since, given any pair of connected 5-equipartitions in G without common components, no bicoloring of the vertices of G exists such that one partition is blue extremal and the other is red extremal.

in C form another component, while the additional components of the partition are automatically provided by partitioning the path from vertex 10 to vertex 29 into the four consecutive subpaths $\{10, 11, 12, 13, 14\}$, $\{15, 16, 17, 18, 19\}$, $\{20, 21, 22, 23, 24\}$, $\{25, 26, 27, 28, 29\}$.

On the basis of the above results, we know the structure of *all* the possible connected 5-equipartitions of G, namely those of type I and II. We also notice that all the partitions of type I share at least one common component (for example, $\{26, 27, 28, 29, 30\}$), and the same holds for the partitions of type II (for example, they share component $\{25, 26, 27, 28, 29\}$). In addition, along the path from vertex 10 to vertex 30, the components of a partition of type I and those belonging to a partition of type II differ by only one vertex. Then we have the following.

Claim 2 *The graph G shown in Figure 8 is not two faced for $s = 5$.*

Proof. Suppose that the claim is not true, that is, G is two faced. In this case, on the basis of the above considerations, the blue and red extremal partitions (connected 5-equipartitions) must be of different types. Without loss of generality, suppose that the blue one is of type I and the red one is of type II. In the blue partition of type I, let D be the component that contains vertex 1 and D' the one that contains all the vertices of the cycle C that are not included in D; similarly, in the red partition of type II, let F be the component that contains vertex 1 and F' the one that contains all the vertices of the cycle C that are not included in F. Given any two vertices i and j, we denote by $\Delta(i, j)$ the difference between the number of blue vertices and the number of red vertices belonging to the unique path from i to j (in counterclockwise order). Notice that function $\Delta(i,j)$ is additive w.r.t. the concatenation of consecutive paths. It is not possible that one among D and D' is red sweeping and, simultaneously, one among F and F' is blue sweeping, since, in this case, we would have $\Delta(11,30) = 4$ and $\Delta(10,29) = -4$, which is not possible. Thus, at least one of the two sweeping components does not intersect C, but it is entirely contained in path P. Suppose that it is red sweeping, then in P there should be a red component with at least 4 red vertices; on the other hand, suppose that it is blue sweeping, then in P there is a blue component with at least 4 blue vertices. Both cases lead to a contradiction, showing that G is not two faced. □

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