A Darling-Siegert formula relating some Bessel integrals and random walks

A. De Gregorio
Dipartimento di Scienze Statistiche
University of Padua
Via Cesare Battisti 241, 35121 Padua, Italy

E. Orsingher*
Dipartimento di Statistica, Probabilità e Statistiche Applicate
University of Rome "La Sapienza"
p.le Aldo Moro 5, 00185 Rome, Italy

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Abstract

The combinatorial identity

$$\sum_{k=0}^{r}\binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{1}{k+m+1}=\frac{1}{m+1}\binom{2r+2m+1}{r+2m+1},$$

for $m=0,1,...$, emerging in the study of random flights in the space $\mathbb{R}^4$ is examined.

A probabilistic interpretation of this formula based on the first-passage time and the time of first return to zero of symmetric random walks is given. A combinatorial proof of this result is also provided. A detailed analysis of the first-passage time distribution is presented together with its fractional counterpart.

Keywords: Bessel functions, first passage times, maximal distributions, first returns to the origin, Random flights, Stirling's formula.

1 Introduction

In the study of random flights in $\mathbb{R}^4$ (see De Gregorio and Orsingher (2005)) a crucial role is played by the following property of Bessel functions

$$\int_0^a \frac{J_\mu(z)J_\nu(a-z)}{z(a-z)}dz=\left(\frac{1}{\mu}+\frac{1}{\nu}\right)\frac{J_{\mu+\nu}(a)}{a},\quad a>0, \tag{1.1}$$

valid for $Re\ \mu>0, Re\ \nu>0$. For the evaluation of the integral (1.1) the following relationship is needed

$$\sum_{k=0}^{r}\frac{\Gamma(2k+\mu)\Gamma(2(r-k)+\nu)}{k!(r-k)!\Gamma(k+\mu+1)\Gamma(r-k+\nu+1)}=\left(\frac{1}{\mu}+\frac{1}{\nu}\right)\frac{\Gamma(2r+\mu+\nu)}{r!\Gamma(r+\mu+\nu+1)}. \tag{1.2}$$

*Corresponding author: enzo.orsingher@uniroma1.it

From (1.2) a further identity involving Beta functions can easily be inferred, namely that

$$\sum_{k=0}^{r} \binom{r}{k} \left\{ \frac{B(2k+\mu, 2(r-k)+\nu)}{B(k+\mu, r-k+\nu)} \frac{1}{k+\mu} + \frac{B(2(r-k)+\mu, 2k+\nu)}{B(r-k+\mu, k+\nu)} \frac{1}{k+\nu} \right\}$$
$$= \frac{1}{\mu} + \frac{1}{\nu}, \tag{1.3}$$

where $B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$.

For the special case $\mu = \nu$ formula (1.3) yields the unexpected result

$$\sum_{k=0}^{r} \binom{r}{k} \frac{B(2k+\mu, 2(r-k)+\mu)}{B(k+\mu, r-k+\mu)} \frac{\mu}{k+\mu} = 1. \tag{1.4}$$

This suggests that for the special case $\mu = \nu = 1$, the following discrete probability distribution can be extracted from (1.4)

$$\begin{aligned} q_k^r &= \binom{r}{k} \frac{B(2k+1, 2(r-k)+1)}{B(k+1, r-k+1)} \frac{1}{k+1} \\ &= \binom{2k}{k} \frac{1}{k+1} \frac{\binom{2r-2k}{r-k}}{\binom{2r+1}{r+1}} \\ &= \frac{\binom{2k}{k}\binom{2r-2k}{r-k}}{\binom{2r}{r}} \frac{r+1}{(k+1)(2r+1)}, \qquad k = 0, 1, ..., r. \end{aligned} \tag{1.5}$$

We shall show below that this distribution has an interesting interpretation and is connected with a sort of discrete Darling-Siegert relationship (first appeared in Darling and Siegert (1953) for diffusion processes) for the symmetric random walk generated by coin flips.

Formula (1.1) can be found in Gradshteyn-Ryzhik (1980) (formula 6.533.2, page 678) and also in Watson (1922) (page 380, formula 5) where the proof is sketched and the result is attributed to Bateman. Therefore the related formula (1.2) appears as a by-product of Bateman's result (1.1) (also proved by Kapteyn).

For some integer values of μ and ν we are able to show by means of combinatorial arguments that (1.2) holds.

Formula (1.2) is strictly related to the distribution of the first return to the origin (and also to the first passage time) of the symmetric random walk with steps with values ± 1.

Formula (1.2) for $\mu = \nu = 1$ yields

$$\begin{aligned} \sum_{k=0}^{r} \frac{(2k)!(2(r-k))!}{k!(r-k)!(k+1)!(r-k+1)!} &= \sum_{k=0}^{r} \binom{2k}{k} \binom{2(r-k)}{r-k} \frac{1}{(k+1)(r-k+1)} \\ &= \frac{1}{r+2} \sum_{k=0}^{r} \binom{2k}{k} \binom{2(r-k)}{r-k} \left\{ \frac{1}{k+1} + \frac{1}{r-k+1} \right\} \\ &= \frac{2}{r+2} \sum_{k=0}^{r} \binom{2k}{k} \binom{2(r-k)}{r-k} \frac{1}{k+1} \\ &= \frac{2}{r+2} \binom{2r+1}{r+1} = 2 \frac{\Gamma(2r+2)}{r!\Gamma(r+3)}. \end{aligned} \tag{1.6}$$

The crucial point is that the following identity

$$\sum_{k=0}^{r} \binom{2k}{k} \binom{2r-2k}{r-k} \frac{1}{k+1} = \binom{2r+1}{r+1}, \tag{1.7}$$

holds. It will be shown below by means of combinatorial arguments completely independent of the previous results.

If X_k is the symmetric two-valued r.v. (with values ± 1) connected with successive coin flips, then, for $S_k = \sum_{j=1}^{k} X_j, k = 1, 2, ...,$ the first return to zero is defined as

$$T = \inf\{k > 0, S_k = 0\}. \tag{1.8}$$

It is well-known (see Fristedt and Gray (1997) page 180) that

$$P\{T = n+1\} = \begin{cases} \binom{n-1}{\frac{n-1}{2}} \frac{1}{n+1} \frac{1}{2^{n-1}} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}. \tag{1.9}$$

In particular, if $n = 2k + 1$, the distribution (1.9) takes the following form

$$P\{T = 2k+2\} = \binom{2k}{k} \frac{1}{k+1} \frac{1}{2^{2k+1}}, \quad k = 1, 2, ... \tag{1.10}$$

This offers the possibility of the following probabilistic interpretation of the identity (1.7)

$$\sum_{k=0}^{r} P\{T = 2k+2\} P\{S_{2r-2k} = 0\} = P\{S_{2r+2} = 0\}. \tag{1.11}$$

Formula (1.11) is a sort of discrete version of the Darling-Siegert relationship. It can be obtained from (1.7) by dividing both members by $\frac{1}{2^{2r+1}}$ and then by considering (1.10) and the binomial structure of the random walk $S_k, k \geq 1$.

We can also note that (1.5) can be interpreted as the following conditional probability

$$q_k^r = P\{T = 2k+2, S_{2r-2k} = 0 | S_{2r+2} = 0\}. \tag{1.12}$$

Formula (1.7) is a special case of the ensemble of combinatorial identities (which also can be obtained from (1.2) for $\nu = \mu = m + 1$) and which read

$$\sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{k+m+1} = \frac{1}{m+1} \binom{2r+2m+1}{r+2m+1}, \quad m = 0, 1, ... \tag{1.13}$$

We are able to prove explicitly (1.13) by induction in some cases, namely for $m = 0, 1, 2, 3$. In the general case our technique fails because the structure of the decompositions involved seem to escape to a general rule. Formula (1.13) also suggests a Darling-Siegert probabilistic interpretation based on the first-passage time

$$T_{m+1} = \inf\{k > 0, S_k = m+1\}, \tag{1.14}$$

of the random walk generated by coin flips. In this case we have that

$$\sum_{k=0}^{r} P\{T_{m+1} = 2k+m+1\} P\{S_{2r-2k+m} = m\} = Pr\{S_{2r+2m+1} = 2m+1\}. \tag{1.15}$$

For $m = 0$, formula (1.15) coincides with (1.11) as one should expect, because $P\{T_1 = 2k+1\} = P\{T = 2k+2\}$ and $P\{S_{2r+1} = 1\} = P\{S_{2r+2} = 0\}$.

Finally a relationship of the form (1.15) can also be rewritten for all real numbers $\mu \geq 1$ when random walks subject to space and time translation are envisaged.

We also obtain the following relationship, valid for all integers $m \geq 0$ and playing a certain role in our proofs,

$$\begin{aligned} \widehat{S}_{r,m} &= \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m+1}{r-k+m+1} \frac{1}{k+m+1} \qquad (1.16) \\ &= \frac{2r+2m+2}{r+2m+2} \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{k+m+1} = \frac{2r+2m+2}{r+2m+2} S_{r,m}. \end{aligned}$$

An alternative form of formula (1.2) can be given on the basis of the integral (see Watson, page 150)

$$\int_0^{\pi/2} \cos^{\mu+\nu+2m}\theta \cos((\mu-\nu)\theta) d\theta = \frac{\pi\Gamma(\mu+\nu+2m+1)}{2^{\mu+\nu+2m+1}\Gamma(\mu+m+1)\Gamma(\nu+m+1)}, \qquad (1.17)$$

as follows

$$\begin{aligned} &\frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \sin(\mu\theta) \sin\theta \cos^{\mu-1}\theta \sin(\nu\varphi) \sin\varphi \cos^{\nu+1}\varphi \frac{\cos^{2r}\varphi - \cos^{2r}\theta}{\cos^2\varphi - \cos^2\theta} d\theta d\varphi \\ &= \int_0^{\pi/2} \sin((\mu+\nu)\theta) \sin\theta \cos^{\mu+\nu+2r-1}\theta d\theta, \qquad (1.18) \end{aligned}$$

which can be obtained after some simple calculations.

2 The first return to the origin and the Darling-Siegert relationship

The distribution of the first return to the origin can be derived by applying the reflection principle or by counting the number of sample paths not intersecting the zero level up to the final instant. We here use a more formal approach and derive the distribution (1.10) with some details.

Our first step consists in deriving the relationship between the distribution of the r.v. (1.8) and that of the maximum of the random walk $S_j, j = 1, 2, ...$ In fact, for the r.v. $T = \inf\{j > 0, S_j = 0\}$ we have that

$$\begin{aligned} P\{T = n+1\} &= P\{S_1 \neq 0, ..., S_n \neq 0, S_{n+1} = 0\} \qquad (2.1) \\ &= P\{S_1 \neq 0, ..., S_n \neq 0\} - P\{S_1 \neq 0, ..., S_{n+1} \neq 0\} \\ &= P\left\{\max_{1 \leq j \leq n-1} S_j \leq 0\right\} - P\left\{\max_{1 \leq j \leq n} S_j \leq 0\right\}. \end{aligned}$$

The distribution of the maximum of a symmetric random walk is well-known and reads

$$P\left\{\max_{1 \leq j \leq n} S_j \geq N\right\} = 2P\{S_n \geq N\} - P\{S_n = N\}, \; N = -1, 0, 1, ..., n, \; n \geq 1. \qquad (2.2)$$

Therefore from (2.2) we obtain that

$$\begin{aligned} P\left\{\max_{1\le j\le n} S_j \le 0\right\} &= 1 - P\left\{\max_{1\le j\le n} S_j \ge 1\right\} \\ &= 1 - 2P\{S_n \ge 1\} + P\{S_n = 1\} \\ &= P\{S_n = 0\} + 2P\{S_n \ge 1\} - 2P\{S_n \ge 1\} + P\{S_n = 1\} \\ &= P\{S_n = 0\} + P\{S_n = 1\}. \end{aligned} \tag{2.3}$$

By combining (2.1) and (2.3) we get that

$$P\{T = n+1\} = P\{S_{n-1} = 0\} + P\{S_{n-1} = 1\} - P\{S_n = 0\} - P\{S_n = 1\}. \tag{2.4}$$

From (2.4), by means of some calculations we are able to obtain the exact expression of the distribution of the first return to the origin.

For $n = 2k+1$, we have that

$$\begin{aligned} P\{T = 2k+2\} &= P\{S_{2k} = 0\} - P\{S_{2k+1} = 1\} \\ &= \binom{2k}{k}\frac{1}{2^{2k}} - \binom{2k+1}{k+1}\frac{1}{2^{2k+1}} \\ &= \binom{2k}{k}\frac{1}{2^{2k+1}}\frac{1}{k+1}. \end{aligned} \tag{2.5}$$

For $n = 2k$ we have instead that

$$\begin{aligned} P\{T = 2k+1\} &= P\{S_{2k-1} = 1\} - P\{S_{2k} = 0\} \\ &= \binom{2k-1}{k}\frac{1}{2^{2k-1}} - \binom{2k}{k}\frac{1}{2^{2k}} = 0. \end{aligned} \tag{2.6}$$

The distribution of the first-passage time through level b can be obtained by applying the reflection principle and reads

$$P\{T_b = n\} = \binom{n}{\frac{n+b}{2}}\frac{b}{n}\frac{1}{2^n}, \quad n = b, b+1, ..., \tag{2.7}$$

where

$$T_b = \inf\{k > 0; S_k = b\}, \quad b \in \mathbb{N}.$$

A proof of (2.7) can be organized as follows

$$\begin{aligned} P\{T_b = n\} &= P\{\max_{1\le j\le n-1} S_j \le b-1, S_{n-1} = b-1, X_n = 1\} \\ &= \frac{1}{2}\left[P\{S_{n-1} = b-1\} - P\{\max_{1\le j\le n-1} S_j \ge b, S_{n-1} = b-1\}\right]. \end{aligned} \tag{2.8}$$

Now, in view of the symmetry and independence of the steps X_k we have that

$$\begin{aligned} P\{\max_{1\le j\le n-1} S_j \ge b, S_{n-1}=b-1\} &= P\{T_b \le n-1, S_{n-1}=b-1\} \\ &= \sum_{j=1}^{n-1} P\{T_b=j, S_j+(S_{n-1}-S_j)=b-1\} \\ &= \sum_{j=1}^{n-1} P\{T_b=j, S_{n-1}-S_j=-1\} \\ &= \sum_{j=1}^{n-1} P\{T_b=j\}P\{S_{n-1}-S_j=1\} \\ &= \sum_{j=1}^{n-1} P\{T_b=j, S_j+S_{n-1}-S_j=b+1\} \\ &= P\{T_b\le n-1, S_{n-1}=b+1\} \\ &= P\{S_{n-1}=b+1\}, \end{aligned} \tag{2.9}$$

since $(S_{n-1}=b+1)\subset(T_b\le n-1)$. By inserting this result into (2.8) we get that

$$P\{T_b=n\}=\frac{1}{2}\left[P\{S_{n-1}=b-1\}-P\{S_{n-1}=b+1\}\right],$$

and this yields (2.7).

We note that $(T=n+1)=(T_1=n)$ because the first step of the random walk leads either to 1 or to -1 and the first return to zero corresponds to the first passage time through 1 (or -1) for a random walk emanating from the origin. Therefore, in light of (2.7) we have for $n=2k+1$ that

$$\begin{aligned} P\{T=2k+2\} &= P\{T_1=2k+1\} \\ &= \binom{2k+1}{k+1}\frac{1}{2k+1}\frac{1}{2^{2k+1}} \\ &= \binom{2k}{k}\frac{1}{k+1}\frac{1}{2^{2k+1}}, \quad k=0,1,.... \end{aligned} \tag{2.10}$$

Remark 2.1 In order to prove that

$$\sum_{k=0}^{\infty} P\{T=2k+2\}=\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{k+1}\frac{1}{2^{2k+1}}=1, \tag{2.11}$$

we need to show that the following identity holds for all $0\le s\le 1$

$$Es^T=1-\sqrt{1-s^2}=\sum_{k=0}^{\infty}\binom{2k}{k}\frac{s^{2k+2}}{k+1}\frac{1}{2^{2k+1}}. \tag{2.12}$$

Thus the function appearing in (2.12) is the probability generating function of the random time T of first return to zero of the symmetric random walk related to coin flips.

Our proof of (2.12) is based on the binomial expansion of $\frac{1}{\sqrt{1-s^2}}$ and reads

$$\begin{aligned}\frac{s}{\sqrt{1-s^2}} &= s\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-s^2)^k \\ &= s\sum_{k=0}^{\infty}\frac{(2k)!}{k!^2 2^{2k}}(-1)^k(-s^2)^k \\ &= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{s^{2k+1}}{2^{2k}},\end{aligned} \tag{2.13}$$

because

$$\begin{aligned}\binom{-\frac{1}{2}}{k} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!} \\ &= (-1)^k\frac{1\cdot 3\cdots(2k-1)}{k!2^k} = (-1)^k\frac{(2k)!}{k!^2 2^{2k}}.\end{aligned}$$

The relationship (2.12) can be obtained from (2.13) by integration in $(0,s)$. For $s=1$ in (2.12) we then get (2.11). Furthermore

$$\begin{aligned}ET &= \sum_{k=0}^{\infty}(2k+2)\binom{2k}{k}\frac{1}{k+1}\frac{1}{2^{2k+1}} = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}} \\ &= \frac{d}{ds}Es^T\bigg|_{s=1} = \frac{s}{\sqrt{1-s^2}}\bigg|_{s=1} = \infty.\end{aligned} \tag{2.14}$$

The mean value of T diverges because the terms in the series (2.14) decrease as $\frac{1}{\sqrt{\pi k}}$ while the probabilities in (2.11), for large k, are

$$Pr\{T=2k+2\} \sim \frac{1}{\sqrt{\pi k}}\frac{1}{2k+2},$$

as the Stirling's formula readily shows.

In view of all results presented above, the relationship

$$\sum_{k=0}^{r}\binom{2k}{k}\binom{2r-2k}{r-k}\frac{1}{k+1} = \binom{2r+1}{r+1}, \tag{2.15}$$

can be given the following probabilistic interpretation. By multiplying both members of (2.15) by $\frac{1}{2^{2r+1}}$ we get that

$$\sum_{k=0}^{r}\binom{2k}{k}\frac{1}{k+1}\frac{1}{2^{2k+1}}\binom{2r-2k}{r-k}\frac{1}{2^{2r-2k}} = \binom{2r+1}{r+1}\frac{1}{2^{2r+1}} = \binom{2r+2}{r+1}\frac{1}{2^{2r+2}}.$$

Therefore we have that

$$\sum_{k=0}^{r}P\{T=2k+2\}P\{S_{2r-2k}=0\} = P\{S_{2r+2}=0\}, \tag{2.16}$$

which is a sort of discrete Darling-Siegert relationship for the symmetric random walk (see Figure 1).

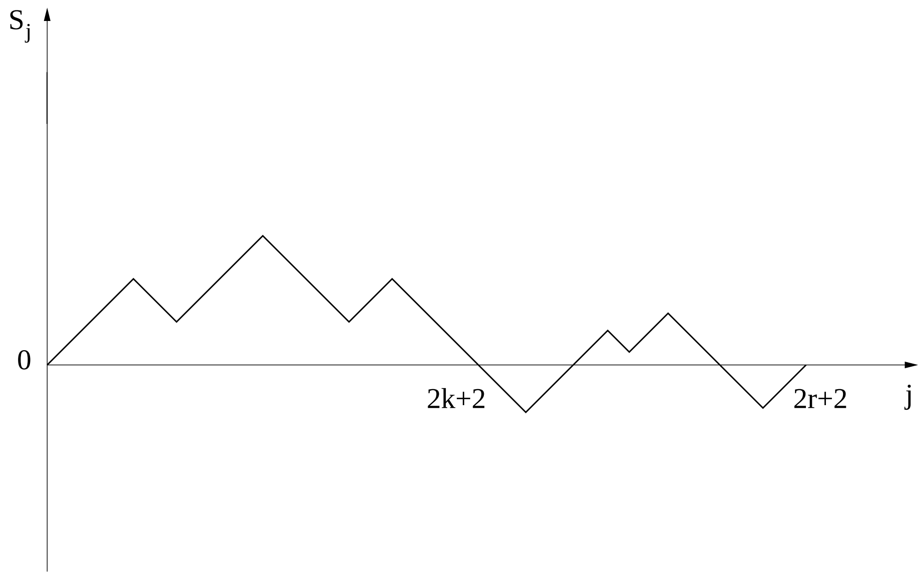

Figure 1: The sample path depicted above attains the origin for the first time at $T = 2k+2$ and then oscillates around zero until the final instant $2r+2$.

3 Extensions

From (1.2) we can also obtain a generalization of the identity (2.15) and write down the Darling-Siegert relationship based on the first-passage time of symmetric random walks.

Formula (1.2) for $\mu = \nu = m+1, m \in \mathbb{N}$ gives the following relationship which extends (2.15)

$$\begin{aligned} \frac{2}{m+1}\frac{\Gamma(2r+2m+2)}{r!\Gamma(r+2m+3)} &= \frac{2}{m+1}\frac{(2r+2m+1)!}{r!(r+2m+2)!} \\ &= \sum_{k=0}^{r} \frac{(2k+m)!(2(r-k)+m)!}{k!(r-k)!(k+m+1)!(r-k+m+1)!} \\ &= \sum_{k=0}^{r} \binom{2k+m}{k+m}\frac{1}{k+m+1}\binom{2r-2k+m}{r-k+m}\frac{1}{r-k+m+1} \\ &= \frac{2}{r+2m+2}\sum_{k=0}^{r}\binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{1}{k+m+1}. \end{aligned} \tag{3.1}$$

Therefore we get from (3.1) the fine combinatorial identity

$$\sum_{k=0}^{r}\binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{1}{k+m+1} = \frac{1}{m+1}\binom{2r+2m+1}{r+2m+1}. \tag{3.2}$$

valid for all $m \in \mathbb{N}$.

In order to give a probabilistic interpretation of (3.2) we set $b = m+1, n = 2k+m+1$

in the distribution (2.7) of the first-passage time, which therefore becomes

$$\begin{aligned} P\{T_{m+1} = 2k+m+1\} &= \binom{2k+m+1}{k+m+1} \frac{m+1}{2k+m+1} \frac{1}{2^{2k+m+1}} \\ &= \binom{2k+m}{k+m} \frac{m+1}{k+m+1} \frac{1}{2^{2k+m+1}}, \end{aligned} \tag{3.3}$$

for $k = 0, 1, ...$

For $m = 0$ the distribution (3.3) coincides with that of the first return to zero because we have that

$$P\{T = 2k+2\} = P\{T_1 = 2k+1\} = \binom{2k}{k} \frac{1}{k+1} \frac{1}{2^{2k+1}}.$$

It should pointed out that T includes the first step from 0 to +1, while T_1 is related to the random walk starting from 1.

By multiplying both members of (3.2) by $\frac{1}{2^{2r+2m+1}}$ we get

$$\begin{aligned} &\sum_{k=0}^{r} \binom{2k+m}{k+m} \frac{1}{k+m+1} \frac{1}{2^{2k+m+1}} \binom{2r-2k+m}{r-k+m} \frac{1}{2^{2r-2k+m}} \\ &= \frac{1}{m+1} \binom{2r+2m+1}{r+2m+1} \frac{1}{2^{2r+2m+1}}. \end{aligned} \tag{3.4}$$

Now, taking into account the distribution (3.3) we have again a Darling-Siegert relationship

$$\sum_{k=0}^{r} P\{T_{m+1} = 2k+m+1\} P\{S_{2r-2k+m} = m\} = P\{S_{2r+2m+1} = 2m+1\}. \tag{3.5}$$

Remark 3.1 For $\mu = m+1$ we can verify directly formula (1.4) of the introduction which can be written as

$$\begin{aligned} &\sum_{k=0}^{r} \binom{r}{k} \frac{B(2k+m+1, 2(r-k)+m+1)}{B(k+m+1, r-k+m+1)} \frac{m+1}{k+m+1} \\ &= \sum_{k=0}^{r} \binom{r}{k} \frac{\Gamma(2k+m+1)\Gamma(2(r-k)+m+1)\Gamma(r+2m+2)}{\Gamma(k+m+1)\Gamma(r-k+m+1)\Gamma(2r+2m+2)} \frac{m+1}{k+m+1} \\ &= \frac{(m+1)\Gamma(r+2m+2)r!}{\Gamma(2r+2m+2)} \sum_{k=0}^{r} \frac{(2k+m)!}{k!(k+m)!} \frac{1}{k+m+1} \frac{(2r-2k+m)!}{(r-k)!(r-k+m)!} \\ &= \frac{(m+1)(r+2m+1)!r!}{(2r+2m+1)!} \sum_{k=0}^{r} \binom{2k+m}{k+m} \frac{1}{k+m+1} \binom{2r-2k+m}{r-k+m} \\ &= \binom{2r+2m+1}{r+2m+1} \frac{(r+2m+1)!r!}{(2r+2m+1)!} = 1. \end{aligned}$$

4 About the fractional version of the first-passage time

Let us consider an extension of formula (3.2) for $m = \mu - 1, \mu \geq 1$, which reads

$$\sum_{k=0}^{r} \frac{\Gamma(2k+\mu)}{\Gamma(k+\mu)\Gamma(k+1)} \frac{1}{k+\mu} \frac{\Gamma(2r-2k+\mu)}{\Gamma(r-k+1)\Gamma(r-k+\mu)} = \frac{1}{\mu} \frac{\Gamma(2r+2\mu)}{\Gamma(r+2\mu)\Gamma(r+1)}. \tag{4.1}$$

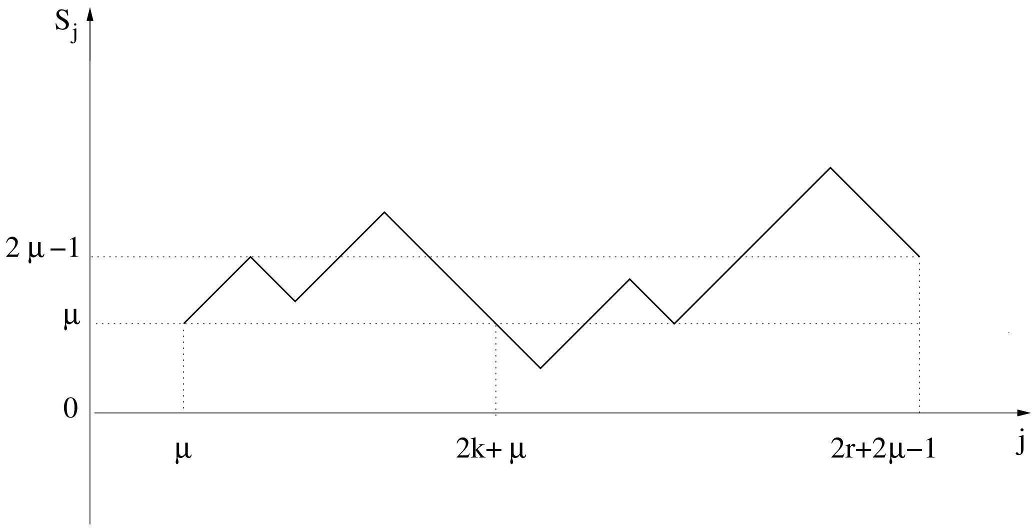

Figure 2: A shifted sample path of the random walk which does not intersect the level μ until the time $2k+\mu$ is represented.

We note also that the distribution (2.7) can be rewritten as

$$\begin{aligned} P\{T_b = n\} &= \binom{n}{\frac{n+b}{2}} \frac{b}{n} \frac{1}{2^n} \\ &= \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+b}{2}+1\right)\Gamma\left(\frac{n-b}{2}+1\right)} \frac{b}{n} \frac{1}{2^n}, \qquad n = b, b+1, ... \end{aligned} \tag{4.2}$$

Since the Gamma function exists for all real values we can extend formally (4.2) by means of the position $n = 2k+\mu, b = \mu$, and this yields

$$\begin{aligned} P\{T_\mu = 2k+\mu\} &= \frac{\Gamma(2k+\mu+1)}{\Gamma(k+\mu+1)\Gamma(k+1)} \frac{\mu}{2k+\mu} \frac{1}{2^{2k+\mu}} \\ &= \frac{\Gamma(2k+\mu)}{\Gamma(k+\mu)(k+\mu)} \frac{\mu}{\Gamma(k+1)} \frac{1}{2^{2k+\mu}}. \end{aligned} \tag{4.3}$$

The expression (4.3) is a genuine probability distribution and represents the law of the first return to μ of a shifted symmetric random walk starting at μ at time μ (see Figure 2). If we, analogously, write the distribution of the random walk $S_n, n \geq 1$, as

$$\begin{aligned} P\{S_n = j\} &= \binom{n}{\frac{n+j}{2}} \frac{1}{2^n} \\ &= \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+j}{2}+1\right)\Gamma\left(\frac{n-j}{2}+1\right)} \frac{1}{2^n}, \end{aligned} \tag{4.4}$$

by means of the position $n = 2r-2k+\mu-1$ and $j = \mu-1$ we get that

$$P\{S_{2r-2k+\mu-1} = \mu-1\} = \frac{\Gamma(2r-2k+\mu)}{\Gamma(r-k+\mu)\Gamma(r-k+1)} \frac{1}{2^{2r-2k+\mu-1}}. \tag{4.5}$$

This can be interpreted as the distribution of the random walk generated by coin tossing, shifted in time and space.

Analogously, we can write that

$$P\{S_{2r+2\mu-1} = 2\mu-1\} = \frac{\Gamma(2r+2\mu)}{\Gamma\left(r+2\mu\right)\Gamma\left(r+1\right)}\frac{1}{2^{2r+2\mu-1}}, \tag{4.6}$$

so that by multiplying both members of (4.2) by $\frac{1}{2^{2r+2\mu-1}}$ we obtain again the following extended Darling-Siegert relationship

$$\sum_{k=0}^{r} P\{T_\mu = 2k+\mu\}P\{S_{2r-2k+\mu-1} = \mu-1\} = P\{S_{2r+2\mu-1} = 2\mu-1\}. \tag{4.7}$$

From (4.7), for $\mu = m+1$, we reobtain the relationship (3.5). Analogously, the identity (2.16) emerges from (4.7) for $\mu = 1$, because $P\{T_1 = 2k+1\} = P\{T = 2k+2\}$ and $P\{S_{2r+1} = 1\} = P\{S_{2r+2} = 0\}$.

5 Combinatorial derivation of formulas

So far we worked on formula (1.2) and on its by products and have given some probabilistic interpretation of (3.2) and of its extension (4.1). We present now some independent proof of these formulas on the base of purely combinatorial arguments.

We have seen above that from (1.2) for $\mu = \nu = m+1$ we can extract the following combinatorial identity (see Section 3 for details)

$$\sum_{k=0}^{r} \binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{1}{k+m+1} = \frac{1}{m+1}\binom{2r+2m+1}{r+2m+1}. \tag{5.1}$$

We now present some general results concerning the sums in the left-hand member of (5.1). Our first Lemma here is a useful tool in proving (5.1).

Lemma 5.1 *If*

$$S_{r,m} = \sum_{k=0}^{r} \binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{1}{k+m+1},$$

and

$$\widehat{S}_{r,m} = \sum_{k=0}^{r} \binom{2k+m}{k+m}\binom{2r-2k+m+1}{r-k+m+1}\frac{1}{k+m+1},$$

then we have that

$$\widehat{S}_{r,m} = \frac{2r+2(m+1)}{r+2(m+1)}S_{r,m}. \tag{5.2}$$

Proof.

$$\begin{aligned}
\widehat{S}_{r,m} &= \sum_{k=0}^{r} \binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\frac{(2r-2k+m+1)}{(r-k+m+1)(k+m+1)} \\
&= \sum_{k=0}^{r} \binom{2k+m}{k+m}\binom{2r-2k+m}{r-k+m}\left\{\frac{2}{k+m+1} - \frac{m+1}{r+2(m+1)}\left\{\frac{1}{k+m+1} + \frac{1}{r-k+m+1}\right\}\right\} \\
&= 2S_{r,m} - \frac{2(m+1)}{r+2(m+1)}S_{r,m} = \frac{2r+2(m+1)}{r+2(m+1)}S_{r,m}.
\end{aligned}$$

□

Another important recurrent relation which permits us to prove (5.1) by induction is presented in the next Lemma.

Lemma 5.2 *We have the following expression*

$$
\begin{aligned}
S_{r+1,m} \quad &= \quad \left\{4 - \frac{2(m+1)}{r+2m+2} + \frac{m-1}{r+m+2}\right\} S_{r,m} \\
&\quad + \frac{m-1}{r+m+2} \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{r-k+1} + \binom{2r+m+2}{r+m+1} \frac{1}{r+m+2}.
\end{aligned} \tag{5.3}
$$

Proof. We note that the difference $S_{r+1,m} - S_{r,m}$ can be developed as follows

$$
\begin{aligned}
& S_{r+1,m} - S_{r,m} \\
&= \sum_{k=0}^{r+1} \binom{2k+m}{k+m} \binom{2r-2k+m+2}{r-k+m+1} \frac{1}{k+m+1} - \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{k+m+1} \\
&= \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{k+m+1} \left\{ \frac{(2r-2k+m+2)(2r-2k+m+1)}{(r-k+m+1)(r-k+1)} - 1 \right\} \\
&\quad + \binom{2r+m+2}{r+m+1} \frac{1}{r+m+2} \\
&= \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{k+m+1} \left\{ 3 + \frac{m-1}{r-k+1} - \frac{m+1}{r-k+m+1} \right\} \\
&\quad + \binom{2r+m+2}{r+m+1} \frac{1}{r+m+2} \\
&= 3S_{r,m} + \frac{m-1}{r+m+2} \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \left\{ \frac{1}{k+m+1} + \frac{1}{r-k+1} \right\} \\
&\quad - \frac{m+1}{r+2m+2} \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \left\{ \frac{1}{k+m+1} + \frac{1}{r-k+m+1} \right\} \\
&\quad + \binom{2r+m+2}{r+m+1} \frac{1}{r+m+2} \\
&= \left\{ 3 - \frac{2(m+1)}{r+2m+2} + \frac{m-1}{r+m+2} \right\} S_{r,m} \\
&\quad + \frac{m-1}{r+m+2} \sum_{k=0}^{r} \binom{2k+m}{k+m} \binom{2r-2k+m}{r-k+m} \frac{1}{r-k+1} + \binom{2r+m+2}{r+m+1} \frac{1}{r+m+2}.
\end{aligned}
$$

This clearly coincides with (5.3). □

The difficulty now stems from the fact that in the sum appearing in (5.3) appears a factor $\frac{1}{r-k+1}$, which does not depend on m. Thus it must be decomposed in such manner that its parts can be expressed in terms of the sums $S_{r,m}$.

We now prove the identity (5.1) for some values of m by exploiting Lemma 5.1 and Lemma 5.2.

Case $m = 0$

We consider first the case $m = 0$ where the relationship (5.3) reduces to

$$\begin{aligned} S_{r+1,0} &= \left(4 - \frac{2}{r+2} - \frac{1}{r+2}\right) S_{r,0} - \frac{1}{r+2} S_{r,0} + \binom{2r+2}{r+1}\frac{1}{r+2} \\ &= \left(4 - \frac{4}{r+2}\right)\binom{2r+1}{r+1} + \binom{2r+2}{r+1}\frac{1}{r+2} \\ &= \frac{(2r+1)!}{(r+1)!r!}\left(\frac{4r+4}{r+2} + \frac{2}{r+2}\right) = \binom{2r+3}{r+2}. \end{aligned} \tag{5.4}$$

Clearly in the reasoning by induction we assumed that

$$S_{r,0} = \binom{2r+1}{r+1},$$

and thus $S_{r+1,0}$ must emerge from this formula with r replaced by $r+1$.

We pass now to the case $m = 3$ because it explains why the procedure by induction can be applied to the general case with value m with great difficulty.

Case $m = 3$

Formula (5.3) of Lemma 5.2 in this case can be written down as

$$\begin{aligned} S_{r+1,3} &= \left(4 - \frac{8}{r+8} + \frac{2}{r+5}\right) S_{r,3} + \frac{1}{r+5}\binom{2r+5}{r+4} \\ &+ \frac{2}{r+5}\sum_{k=0}^{r}\binom{2k+3}{k+3}\binom{2r-2k+3}{r-k+3}\frac{1}{r-k+1}. \end{aligned} \tag{5.5}$$

The sum in (5.5) can be developed in the following manner

$$\begin{aligned} &\sum_{k=0}^{r}\binom{2k+3}{k+3}\binom{2r-2k+3}{r-k+3}\frac{1}{r-k+1} \\ &= \sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+3}{r-k+3}\frac{2k+3}{(k+3)(r-k+1)} \\ &= \sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+3}{r-k+3}\left\{\frac{2r+5}{(r+4)(r-k+1)} - \frac{3}{(r+4)(k+3)}\right\} \\ &= \frac{2r+5}{r+4}\sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+3}{r-k+3}\frac{1}{r-k+1} - \frac{3}{r+4}\frac{2r+6}{r+6}S_{r,2}. \end{aligned} \tag{5.6}$$

where, in the last step, formula (5.2) for $m = 2$ has been applied. The sum (5.6) can be further developed as follows

$$\begin{aligned}
&\sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+3}{r-k+3}\frac{1}{r-k+1} \qquad (5.7)\\
&=\sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+2}{r-k+2}\frac{2r-2k+3}{(r-k+3)(r-k+1)}\\
&=\sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+2}{r-k+2}\left\{\frac{2}{r-k+3}+\frac{1}{2}\left\{\frac{1}{r-k+1}-\frac{1}{r-k+3}\right\}\right\}\\
&=\frac{3}{2}S_{r,2}+\frac{1}{2}\sum_{k=0}^{r}\binom{2k+2}{k+2}\binom{2r-2k+1}{r-k+1}\frac{2}{r-k+2}\\
&=\frac{3}{2}S_{r,2}+\frac{2r+4}{r+4}S_{r,1}.
\end{aligned}$$

In the last step we applied (5.2) for $m=1$ after a change of variable.

By collecting all pieces together and by inserting (5.6) and (5.7) into (5.5) we have that

$$\begin{aligned}
&S_{r+1,3}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)S_{r,3}+\frac{1}{r+5}\binom{2r+5}{r+4}\\
&\quad+\frac{2}{r+5}\left\{-\frac{3}{r+4}\frac{2r+6}{r+6}S_{r,2}+\frac{2r+5}{r+4}\left\{\frac{3}{2}S_{r,2}+\frac{2r+4}{r+4}S_{r,1}\right\}\right\}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}+\frac{1}{r+5}\binom{2r+5}{r+4}\\
&\quad+\frac{2}{r+5}\left\{-\frac{3}{r+4}\frac{2r+6}{r+6}\binom{2r+5}{r+5}\frac{1}{3}+\frac{2r+5}{r+4}\frac{1}{2}\binom{2r+5}{r+5}+\frac{2r+4}{r+4}\frac{2r+5}{r+4}\frac{1}{2}\binom{2r+3}{r+3}\right\}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}\\
&\quad+\frac{1}{r+5}\binom{2r+5}{r+5}\left\{\frac{r+5}{r+1}-\frac{2(2r+6)}{(r+4)(r+6)}+\frac{2r+5}{r+4}+\frac{r+5}{r+4}\right\}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}+\frac{1}{r+5}\binom{2r+5}{r+5}\left\{\frac{(2r+5)(2r+6)}{(r+4)(r+1)}-\frac{2(2r+6)}{(r+4)(r+6)}\right\}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}+\frac{1}{r+5}\binom{2r+5}{r+5}\frac{2r+6}{r+4}\frac{2r^2+15r+28}{(r+1)(r+6)}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}+\frac{1}{r+5}\binom{2r+5}{r+5}\frac{(2r+6)(2r+7)}{(r+1)(r+6)}\\
&=\left(4-\frac{8}{r+8}+\frac{2}{r+5}\right)\binom{2r+7}{r+7}\frac{1}{4}+\frac{r+7}{(r+1)(r+5)}\binom{2r+7}{r+7}\\
&=\left(1-\frac{2}{r+8}+\frac{3}{2(r+1)}\right)\binom{2r+7}{r+7}=\frac{2r^2+17r+36}{2(r+1)(r+8)}\binom{2r+7}{r+7}\\
&=\frac{(2r+9)(r+4)}{2(r+1)(r+8)}\binom{2r+7}{r+7}=\binom{2r+9}{r+8}\frac{1}{4},
\end{aligned}$$

and this concludes the proof that

$$S_{r,3} = \frac{1}{4}\binom{2r+7}{r+7}. \tag{5.8}$$

The procedure of splitting up the sum appearing in (5.3) becomes more and more cumbersome and we have not been able to generalize it to an arbitrary value of m.

With little effort and by applying the same arguments used above we can easily show that (1.13) holds also for $m = 1, 2$.

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